JOURNAL OF
PURE AND
APPLIED ALGEBRA

Journal of Pure and Applied Algebra 163 (2001) 119–146
www.elsevier.com/locate/jpaa

On comparing the cohomology of algebraic groups, finite Chevalley groups and Frobenius kernels

Christopher P. Bendel[a], Daniel K. Nakano[b,*,1], Cornelius Pillen[c]

[a] *Department of Mathematics, Statistics and Computer Science, University of Wisconsin, Stout, Menomonie, WI 54751, USA*
[b] *Department of Mathematics and Statistics, Utah State University, Lund Hall, Logan, UT 84322-3900, USA*
[c] *Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA*

Received 28 April 2000
Communicated by E.M. Friedlander

Abstract

Let G be a semisimple simply connected algebraic group defined and split over the field $\mathbb{F}_p$ with p elements, $G(\mathbb{F}_q)$ be the finite Chevalley group consisting of the $\mathbb{F}_q$-rational points of G where $q = p^r$, and G_r be the rth Frobenius kernel of G. This paper investigates relationships between the extension theories of G, $G(\mathbb{F}_q)$, and G_r over the algebraic closure of $\mathbb{F}_p$. First, some qualitative results relating extensions over $G(\mathbb{F}_q)$ and G_r are presented. Then certain extensions over $G(\mathbb{F}_q)$ and G_r are explicitly identified in terms of extensions over G. © 2001 Elsevier Science B.V. All rights reserved.

MSC: Primary 20C; 20G; secondary 20J06; 20G10

1. Introduction

1.1. Let G be a semisimple simply connected algebraic group defined and split over the field $\mathbb{F}_p$ with p elements. Let k denote the algebraic closure of $\mathbb{F}_p$. Let $F : G \to G^{(1)}$ be the Frobenius map. For a fixed $r \geqslant 1$, let G_r be the kernel of the rth iteration of the Frobenius map. Furthermore, let $G(\mathbb{F}_q)$ where $q = p^r$ be the finite Chevalley

---

[*] Corresponding author. Tel.: +1-435-797-3934; fax: +1-435-797-1822.

E-mail addresses: bendelc@uwstout.edu (C.P. Bendel), nakano@sunfs.math.usu.edu (D.K. Nakano), pillen@mathstat.usouthal.edu (C. Pillen).

[1] Research was supported in part by NSF grant DMS-9800960.

PII: S0022-4049(01)00024-X

group consisting of the $\mathbb{F}_q$-rational points of G. For over 30 years, there has been a considerable amount of effort in relating the representation theory of reductive algebraic groups, Frobenius kernels and finite groups of Lie type in the defining characteristic, for example [5]. In this paper we will be primarily interested in the following three questions:

(1.1.1) Can all extensions between G_r-modules be found via an extension theory for G?
(1.1.2) Can all extensions between $G(\mathbb{F}_q)$-modules be found via an extension theory for G?
(1.1.3) Given two rational G-modules M and N, is there a relationship between $\mathrm{Ext}^{\bullet}_{G_r}(M,N)$ and $\mathrm{Ext}^{\bullet}_{G(\mathbb{F}_q)}(M,N)$?

Humphreys [9] and Andersen [1, p. 388] first posed question (1.1.2) for Ext^1 between simple $G(\mathbb{F}_q)$-modules. Andersen was able to answer this question affirmatively for simple $G(\mathbb{F}_q)$-modules in several generic cases (see [1, Theorem 3.2]).

The methods employed in this paper will be functorial and different from the ideas used in the past. In order to answer questions (1.1.1) and (1.1.2) we compare the category of $G(\mathbb{F}_q)$-modules and G_r-modules with truncated categories of G-modules which contain enough projective objects. This will be accomplished by constructing two Grothendieck spectral sequences. These truncated categories of G-modules are highest weight categories, and are thus equivalent to module categories for finite-dimensional quasi-hereditary algebras. Our procedure will also demonstrate why it is easier to provide an answer to (1.1.1) as opposed to (1.1.2) for simple modules (see the Corollary given in 6.2). The idea of comparing the cohomology of finite groups with extensions of modules over certain finite-dimensional algebras was also used earlier in work of Doty, Erdmann and the second author [7]. In that paper, it was shown that one can construct a spectral sequence involving extensions for the Schur algebra $S(n,d)$ $(n \geqslant d)$ to extensions for the symmetric group on d-letters Σ_d. Results comparing the cohomology in these two categories involving specific classes of modules are given in [14]. Applications involving computations of cohomology are provided in [8,17]. Question (1.1.3) is the most intractable question of the three. We will show that ideas from [15] can be used to provide a vanishing criterion of cohomology for finite Chevalley groups in terms of vanishing of Lie algebra cohomology. These results only work in the case when $r = 1$, but have the advantage that there are no strong restrictions on the size of the prime.

1.2. Now we will describe the contents of the paper in greater detail. The results in Sections 2 and 3 provide qualitative results involving the relationship between the cohomology of $G(\mathbb{F}_q)$ and G_r. In Section 2, we give a criterion for projectivity of modules over $G(\mathbb{F}_q)$. A criterion for G_r was given earlier in [3, Theorem (4.2.1)]. This criterion involves induced modules, Weyl modules for G and principal series modules for $G(\mathbb{F}_q)$. In the following section, we investigate sufficient conditions to insure the vanishing of $\mathrm{Ext}^n_{G(\mathbb{F}_p)}(M,N)$ for $M, N \in \mathrm{mod}(G)$. This is given in terms of the vanishing of certain weight spaces in the cohomology for U_1 where U is the unipotent radical of the Borel subgroup B of G.

The remainder of the paper is devoted to providing answers to questions (1.1.1) and (1.1.2). In Section 4, we first define a saturated set of weights π_s in $X(T)_+$ and let $\mathscr{C}_s$ be the full subcategory of G-modules whose composition factors all have highest weight in π_s. We proceed by constructing two functors $\mathscr{G}_s(-)$ and $\mathscr{H}_s(-)$ from the category of $G(\mathbb{F}_q)$-modules and the category of G_r-modules, respectively, into $\mathscr{C}_s$. For $M \in \mathscr{C}_s$ and $N_1 \in \mathrm{mod}(G(\mathbb{F}_q))$ and $N_2 \in \mathrm{mod}(G_r)$ two spectral sequences are obtained:

$$E_2^{i,j} = \mathrm{Ext}^i_G(M, R^j\mathscr{G}_s(N_1)) \Rightarrow \mathrm{Ext}^{i+j}_{G(\mathbb{F}_q)}(M, N_1), \tag{1.1}$$

$$E_2^{i,j} = \mathrm{Ext}^i_G(M, R^j\mathscr{H}_s(N_2)) \Rightarrow \mathrm{Ext}^{i+j}_{G_r}(M, N_2). \tag{1.2}$$

The higher right derived functors $R^j\mathscr{G}_s$ and $R^j\mathscr{H}_s$ are of particular interest since these provide us with vital information about the connection between these categories. The composition factors of these functors when applied to a module can be found by calculating extensions (in $G(\mathbb{F}_q)$ or G_r) between the module and the appropriate projective cover in $\mathscr{C}_s$. The constructions in Section 4 allow us to prove stability results in Section 5. In particular, the Theorem of 5.2, the Corollary of 5.3, and the Theorem of 5.5 by and large provide an answer to questions (1.1.1) and (1.1.2) given in Section 1. In Section 6 we look at the image of $\mathscr{H}_s$ on a simple module and show in the Theorem of 6.1 that this is a semisimple G-module for large enough primes p. As a byproduct one obtains a formula in the Corollary of 6.2 which allows one to compute Ext^1 between two simple modules in $\mathrm{mod}(G_r)$ by knowing Ext^1 between simples in $\mathrm{mod}(G)$. We demonstrate in Section 7 that the situation for finite Chevalley groups is not as straightforward as in the Frobenius kernel case by looking at $\mathscr{G}_s$ on simple modules. Our analysis shows that one can recover the isomorphisms given in [2] (see the Corollary given in 7.1) in a more functorial manner via our approach. The Theorem given in 7.6 generalizes Andersen's results for the generic case [2, Theorem 3.2] allowing for a larger class of modules and extensions of higher degree. Furthermore, in the Theorem given in 7.5 we are able to give a complete description of the cohomology of the finite group $G(\mathbb{F}_q)$ and all its simple modules in terms of cohomology for the algebraic group with simple modules in the truncated category $\mathscr{C}_s$, provided the prime is large enough.

1.3. Notation. Unless otherwise stated G will always denote a semisimple simply connected algebraic group defined and split over the finite field $\mathbb{F}_p$ with p elements for a prime p. Here k denotes the algebraic closure of $\mathbb{F}_p$. Let Φ be a root system associated to G with respect to a maximal split torus T. Moreover, let Φ^+ (resp. Φ^-) be positive (resp. negative) roots and Δ be a base consisting of simple roots. Let B be a Borel subgroup containing T corresponding to the positive roots and U be the unipotent radical of B.

The Euclidean space associated with Φ will be denoted by $\mathbb{E}$ and the inner product on $\mathbb{E}$ will be denoted by $\langle , \rangle$. The Weyl group W is the group generated by reflections associated to the root system Φ and the affine Weyl group W_p is the group generated by W and translations by elements in $p\mathbb{Z}\Phi$. Let $X(T)$ be the integral weight lattice obtained from Φ. The set $X(T)$ has a partial ordering defined as follows. If $\lambda, \mu \in X(T)$ then $\lambda \geqslant \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$. If $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is the coroot corresponding to $\alpha \in \Phi$, then the set of dominant integral weights is defined by

$$X(T)_+ = \{\lambda \in X(T)\colon 0 \leqslant \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta\}.$$

Furthermore, the set of p^r-restricted weights is

$$X_r(T) = \{\lambda \in X(T)\colon 0 \leqslant \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in \Delta\}.$$

The affine Weyl group W_p acts on $X(T)$ via the "dot action" given by $w\cdot\lambda = w(\lambda+\rho)-\rho$ where $w \in W$ ($w \in W_p$), $\lambda \in X(T)$, and ρ is the half sum of positive roots.

Let H be an affine algebraic group scheme over k and let $H_r = \ker F^r$. Here $F : H \to H^{(1)}$ is the Frobenius map and F^r is the rth iteration of the Frobenius map. We note that there is a categorical equivalence between restricted Lie(H)-modules and H_1-modules. For any group scheme H, let mod(H) be the category of finite-dimensional rational H-modules. If $r \geqslant 1$ and $q = p^r$, let $H(\mathbb{F}_q)$ be the finite group obtained by taking the $\mathbb{F}_q$ rational points of H.

For a reductive algebraic group G the simple modules will be denoted by $L(\lambda)$ and the induced modules by $H^0(\lambda) = \operatorname{ind}_B^G \lambda$, where $\lambda \in X(T)_+$. For the infinitesimal group scheme G_r, the simple modules will be denoted by $L_r(\lambda)$. If $\lambda \in X_r(T)$, then $L_r(\lambda) \cong L(\lambda)$. The induced and coinduced modules are given by $Z_r'(\lambda) = \operatorname{ind}_{B_r}^{G_r} \lambda$ and $Z_r(\lambda) = \operatorname{coind}_{B_r}^{G_r} \lambda$ for $\lambda \in X_r(T)$. If $\lambda \in X_r(T)$, the injective hull of $L_r(\lambda)$ in mod(G_r) is $Q_r(\lambda)$. This is also the projective cover of $L_r(\lambda)$ as a G_r-module [11, II. 11.5(4)]. For the finite group $G(\mathbb{F}_q)$, the simple modules are $L(\lambda)$ for $\lambda \in X_r(T)$. Moreover, for each $\lambda \in X(T(\mathbb{F}_q))$, the induced module is given by $M_r(\lambda) = \operatorname{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \lambda$.

2. Projectivity results

2.1. It is not known how to construct the simple G_r-modules nor how one can compute the dimensions of these modules. So, for practical purposes it does not make sense to test a module for projectivity by showing that extensions between the module and the simple modules are zero. There are other canonical modules like Weyl modules, induced modules and Verma modules whose construction is well-known and whose character can be determined. In [3, Theorem (4.2.1)], it was shown that one can test projectivity for a module over G_r by examining the extensions between the module and a family of canonical modules. The result is presented below.

Theorem. *Let G be a reductive algebraic k-group scheme and Λ be a set of representatives for $X(T)/p^rX(T)$. Moreover, let $M \in$ mod(G). The following statements are equivalent*:

(a) *M is projective in* mod(G_r);

(b) $\mathrm{Ext}^i_{G_r}(M, \bigoplus_{\lambda\in\Lambda} L_r(\lambda)) = 0$ *for all* $i > 0$;
(c) $\mathrm{Ext}^i_{G_r}(M, \bigoplus_{\lambda\in\Lambda} H^0(\lambda)) = 0$ *for all* $i > 0$;
(d) (i) $\mathrm{Ext}^i_{G_r}(M, \bigoplus_{\lambda\in\Lambda} Z'_r(\lambda)) = 0$ *for all* $i > 0$
  (ii) $R^i \operatorname{ind}_B^G(\mathrm{Hom}_{B_r}(M, \bigoplus_{\lambda\in\Lambda} \lambda)^{(-r)}) = 0$ *for all* $i > 0$;
(e) $\mathrm{Ext}^i_{G_r}(\bigoplus_{\lambda\in\Lambda} L_r(\lambda), M) = 0$ *for all* $i > 0$;
(f) $\mathrm{Ext}^i_{G_r}(\bigoplus_{\lambda\in\Lambda} V(\lambda), M) = 0$ *for all* $i > 0$;
(g) (i) $\mathrm{Ext}^i_{G_r}(\bigoplus_{\lambda\in\Lambda} Z_r(\lambda), M) = 0$ *for all* $i > 0$
  (ii) $R^i \operatorname{ind}_B^G \left(\mathrm{Hom}_{B_r^+}(\bigoplus_{\lambda\in\Lambda} \lambda, M)^{(-r)}\right) = 0$ *for all* $i > 0$.
Moreover, statements (a)–(c), (e), *and* (f) *are equivalent for* $M \in \mathrm{mod}(G_r)$.

2.2. In general, the simple $G(\mathbb{F}_q)$-modules have also not been determined. The following result shows that projectivity over $G(\mathbb{F}_q)$ can also be checked by using principal series modules, Weyl modules or induced modules.

Theorem. *Let* Λ *be a set of representatives for* $X(T)/p^r X(T)$, *and* $\Gamma = X(T(\mathbb{F}_q))$. *Moreover, let* $M \in \mathrm{mod}(G(\mathbb{F}_q))$. *The following statements are equivalent*:
(a) M *is projective in* $\mathrm{mod}(G(\mathbb{F}_q))$;
(b) $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, \bigoplus_{\lambda\in\Lambda} L_r(\lambda)) = 0$ *for all* $i > 0$;
(c) $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, \bigoplus_{\lambda\in\Lambda} H^0(\lambda)) = 0$ *for all* $i > 0$;
(d) $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(\bigoplus_{\lambda\in\Lambda} V(\lambda), M) = 0$ *for all* $i > 0$;
(e) $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, \bigoplus_{\lambda\in\Gamma} M_r(\lambda)) = 0$ *for all* $i > 0$.

Proof. If $M \in \mathrm{mod}(G(\mathbb{F}_q))$ then (a) $\Leftrightarrow$ (b). Furthermore, (a) implies (c)–(e). In order to finish the proof we show that (e) $\Rightarrow$ (a), (c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (b).

(e) $\Rightarrow$ (a): From our assumption and Frobenius reciprocity we have

$$0 = \mathrm{Ext}^i_{G(\mathbb{F}_q)}\left(M, \bigoplus_{\lambda\in\Gamma} M_r(\lambda)\right) \cong \mathrm{Ext}^i_{B(\mathbb{F}_q)}\left(M, \bigoplus_{\lambda\in\Gamma} \lambda\right).$$

Therefore, M is projective in $\mathrm{mod}(B(\mathbb{F}_q))$, and thus projective in $\mathrm{mod}(U(\mathbb{F}_q))$. Since $U(\mathbb{F}_q)$ is the p-Sylow subgroup in $G(\mathbb{F}_q)$, it follows that M is projective in $\mathrm{mod}(G(\mathbb{F}_q))$.

(c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (b): Define the following order relation on $X(T)$. Let $\mu \leqslant_{\mathbb{Q}} \lambda$ on $X(T)$ if and only if $\lambda - \mu$ is a non-negative rational linear combination of simple roots. Moreover, let

$$C_{\mathbb{Z}} = \{\lambda \in X(T)\colon 0 < \langle \lambda + \rho, \beta^\vee \rangle \leqslant p - 1 \text{ for all } \beta \in \Phi^+\}.$$

We will prove the statement by using induction on the order relation. Assume first that condition (c) holds. If $\lambda \in C_{\mathbb{Z}}$ then $H^0(\lambda) = L_r(\lambda)$, so $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\lambda)) = 0$ for all $i > 0$ for all $\lambda \in C_{\mathbb{Z}}$. Now for an arbitrary $\lambda \in \Lambda$ assume that $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\mu)) = 0$ for all $i > 0$ and $\mu <_{\mathbb{Q}} \lambda$ with $\mu \in \Lambda$. There exists a short exact sequence of G-modules given by

$$0 \to L_r(\lambda) \to H^0(\lambda) \to N \to 0.$$

All composition factors $L(\sigma)$ of N satisfy $\sigma < \lambda$. If $\sigma \in \Lambda$ then by induction $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\sigma)) = 0$ for all $i > 0$. On the other hand, if $\sigma \notin \Lambda$ then write $\sigma = \sigma_0 + p^r \sigma_1$

where $\sigma_0 \in \Lambda$. By using Steinberg's tensor product theorem, it follows that

$$\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\sigma)) = \mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\sigma_0) \otimes L_r(\sigma_1)).$$

But $\sigma <_{\mathbb{Q}} \lambda$ so all the composition factors of $L_r(\sigma_0) \otimes L_r(\sigma_1)$ have high weights which are less than λ. Consequently, $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\sigma_0) \otimes L_r(\sigma_1)) = 0$, and $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, N) = 0$ for all $i > 0$.

Consider the long exact sequence in cohomology:

$$\begin{aligned} 0 &\to \mathrm{Hom}_{G(\mathbb{F}_q)}(M, L_r(\lambda)) \to \mathrm{Hom}_{G(\mathbb{F}_q)}(M, H^0(\lambda)) \to \mathrm{Hom}_{G(\mathbb{F}_q)}(M, N) \\ &\to \mathrm{Ext}^1_{G(\mathbb{F}_q)}(M, L_r(\lambda)) \to \mathrm{Ext}^1_{G(\mathbb{F}_q)}(M, H^0(\lambda)) \to \mathrm{Ext}^1_{G(\mathbb{F}_q)}(M, N) \\ &\to \mathrm{Ext}^2_{G(\mathbb{F}_q)}(M, L_r(\lambda)) \to \cdots . \end{aligned}$$

We have $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\lambda)) = 0$ for all $i \geqslant 2$. Therefore, by Bendel and Nakano [3, Proposition 2.4.2(c)] one has $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M, L_r(\lambda)) = 0$ for all $i > 0$. To show that (d) $\Rightarrow$ (b), one can use an analogous inductive argument. □

3. Vanishing of cohomology for $G(\mathbb{F}_p)$ and G_1

3.1. Let M be a rational G-module. In [15] it was shown that if M is projective over G_1 then M is projective over $G(\mathbb{F}_p)$. The converse to this statement also holds if one assumes that the composition factors of M have high weights which are not too large (i.e. M is in the p-bounded category). A natural generalization to this question is the following.

(3.1.1) Let M and N be rational G-modules. Under what conditions does $\mathrm{Ext}^i_{G_1}(M, N) = 0$ imply that $\mathrm{Ext}^i_{G(\mathbb{F}_p)}(M, N) = 0$ for all $i > 0$?

Observe if $N = \bigoplus_{\lambda \in \Lambda} L_1(\lambda)$ or $\bigoplus_{\lambda \in \Lambda} H^0(\lambda)$ then this holds by Lin and Nakano [15, Corollary 3.5], Bendel and Nakano [3, Theorem (4.2.1)] and the Theorem given in 2.2. In order to demonstrate the subtlety of this problem, we present the following example. Let $G = \mathrm{SL}_2(k)$ with $p \geqslant 7$. There are three blocks for $k\,\mathrm{SL}_2(\mathbb{F}_p)$ and $(p+1)/2$ blocks for G_1. Let $L_1(\lambda)$ be in the principal block for $\mathrm{SL}_2(\mathbb{F}_p)$, but not in the principal block for G_1. Then for some $i > 0$, $\mathrm{H}^i(\mathrm{SL}_2(\mathbb{F}_p), L_1(\lambda)) \neq 0$, but $\mathrm{H}^i(G_1, L_1(\lambda)) = 0$ for all $i \geqslant 0$.

3.2. Let M and N be B-modules. According to [15, Theorem 3.2], there exists a spectral sequence

$$E_1^{i,j} = \mathrm{Ext}^{i+j}_{U_1}(M, N)_i \Rightarrow \mathrm{Ext}^{i+j}_{U(\mathbb{F}_p)}(M, N). \tag{3.1}$$

The following result demonstrates that the $T(\mathbb{F}_p)$ invariants on the U_1-cohomology are directly related to the calculation of the $B(\mathbb{F}_p)$ and the $G(\mathbb{F}_p)$ cohomology.

Theorem. *Let* $n \geqslant 0$.

(a) *If* $M, N \in \mathrm{mod}(B)$ *and* $\mathrm{Ext}^n_{U_1}(M, N)^{T(\mathbb{F}_p)} = 0$ *then* $\mathrm{Ext}^n_{B(\mathbb{F}_p)}(M, N) = 0$.

(b) *If* $M, N \in \mathrm{mod}(G)$ *and* $\mathrm{Ext}^n_{U_1}(M, N)^{T(\mathbb{F}_p)} = 0$ *then* $\mathrm{Ext}^n_{G(\mathbb{F}_p)}(M, N) = 0$.

Proof. (a) First observe that $B(\mathbb{F}_p) = T(\mathbb{F}_p) \ltimes U(\mathbb{F}_p)$ and modules over $kT(\mathbb{F}_p)$ are semisimple. Therefore, the Lyndon–Hochschild–Serre spectral sequence yields for $n \geqslant 0$:

$$\mathrm{Ext}^n_{B(\mathbb{F}_p)}(M,N) \cong \mathrm{Ext}^n_{U(\mathbb{F}_p)}(M,N)^{T(\mathbb{F}_p)}. \tag{3.2}$$

The spectral sequence (3.1) for $M,N \in \mathrm{mod}(B)$ has differentials which are $T(\mathbb{F}_p)$-homomorphisms. The fixed point functor $(-)^{T(\mathbb{F}_p)}$ is exact. This means that we can obtain another spectral sequence from (3.1)

$$E_1^{i,j} = \mathrm{Ext}^{i+j}_{U_1}(M,N)_i^{T(\mathbb{F}_p)} \Rightarrow \mathrm{Ext}^{i+j}_{B(\mathbb{F}_p)}(M,N). \tag{3.3}$$

Next, observe that

$$\mathrm{Ext}^n_{U_1}(M,N)^{T(\mathbb{F}_p)} = \bigoplus_{i \leqslant 0} \bigoplus_{j=n-i} \mathrm{Ext}^{i+j}_{U_1}(M,N)_i^{T(\mathbb{F}_p)}. \tag{3.4}$$

Consequently, if $\mathrm{Ext}^n_{U_1}(M,N)^{T(\mathbb{F}_p)} = 0$ then $\mathrm{Ext}^{i+j}_{U_1}(M,N)_i^{T(\mathbb{F}_p)} = 0$ for all i,j such that $i+j=n$, thus $\mathrm{Ext}^n_{B(\mathbb{F}_p)}(M,N) = 0$.

(b) Since $U(\mathbb{F}_p)$ is the p-Sylow subgroup of $G(\mathbb{F}_p)$ the restriction map res: $\mathrm{Ext}^n_{G(\mathbb{F}_p)}(M,N) \to \mathrm{Ext}^n_{B(\mathbb{F}_p)}(M,N)$ is injective for all $n \geqslant 0$. The result now follows from part (a). □

4. Spectral sequences

4.1. Let G be an algebraic k-group scheme and H be a closed subgroup scheme of G. Moreover, let $M \in \mathrm{mod}(G)$ and $N \in \mathrm{mod}(H)$. There exists a first quadrant spectral sequence [11, I. 4.5 Proposition]:

$$E_2^{i,j} = \mathrm{Ext}^i_G(M, R^j \,\mathrm{ind}_H^G(N)) \Rightarrow \mathrm{Ext}^{i+j}_H(M,N). \tag{4.1}$$

When G is as defined in 1.3 and $H = G(\mathbb{F}_q)$ the induction functor $\mathrm{ind}^G_{G(\mathbb{F}_q)}$—is exact because H is a finite algebraic k-group. In this case the spectral sequence (4.1) collapses and yields the following isomorphism:

(4.1.1) For $i \geqslant 0$, $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(M,N) \cong \mathrm{Ext}^i_G(M, \mathrm{ind}^G_{G(\mathbb{F}_q)} N)$.

Similarly, if G is reductive and $H = G_r$ then G/G_r is affine. Therefore, the induction functor $\mathrm{ind}^G_{G_r}$—is exact. The spectral sequence (4.1) collapses in this case and yields the following isomorphism:

(4.1.2) For $i \geqslant 0$, $\mathrm{Ext}^i_{G_r}(M,N) \cong \mathrm{Ext}^i_G(M, \mathrm{ind}^G_{G_r} N)$.

Isomorphisms (4.1.1) and (4.1.2) provide affirmative answers to questions (1.1.1) and (1.1.2). Unfortunately, for practical purposes this answer is not satisfactory because the modules $\mathrm{ind}^G_{G(\mathbb{F}_q)} N$ and $\mathrm{ind}^G_{G_r} N$ are, in general, infinite-dimensional and difficult to compute. We will need to modify this construction to provide a better solution to computing cohomology for $G(\mathbb{F}_q)$ and G_r via cohomology for G.

4.2. Let us review some general properties about coalgebras and comodules. The conventions and results will follow those proved in [6, Section 1]. Let C be any coalgebra, and $\mathrm{Mod}(C)$ be the category of right comodules for C. Now suppose that B is a subcoalgebra of C and let $\mathrm{Mod}_B(C)$ be the full subcategory of $\mathrm{Mod}(C)$ whose objects belong to B. If $M \in \mathrm{Mod}(C)$, then let M_B be the unique maximal C-subcomodule of M belonging to B. The functor $M \mapsto M_B$ is a functor from $\mathrm{Mod}(C)$ to $\mathrm{Mod}_B(C)$. Since the image of the structure map $M_B \to M_B \otimes C$ is contained in $M_B \otimes B$, one can regard M_B as a B-comodule. The functor $\mathscr{F}_B : M \mapsto M_B$, from $\mathrm{Mod}(C)$ to $\mathrm{Mod}(B)$, is left exact and takes injectives to injectives.

Let π be a set of simple C-comodules and $M \in \mathrm{Mod}(C)$. Let $\mathcal{O}_\pi(M)$ be the unique maximal C-subcomodule such that all the composition factors of which lie in π. Set $C(\pi) = \mathcal{O}_\pi(C)$. Then $C(\pi)$ is a subcoalgebra of C. Furthermore, $\mathcal{O}_\pi(M)$ is the same as $\mathscr{F}_{C(\pi)}(M)$ if we regard $\mathscr{F}_{C(\pi)}(M)$ in $\mathrm{Mod}(C)$ by inflation.

Let G be an algebraic k-group scheme and H be a closed subgroup scheme of G. Set $C = k[G]$ and let π be a set of simple C-comodules (or equivalently simple rational G-modules). Consider the functor $\mathscr{T} = \mathscr{F}_{C(\pi)} \circ \mathrm{ind}_H^G$. This is a functor from $\mathrm{Mod}(k[H])$ to $\mathrm{Mod}(C(\pi))$. We can now construct the following spectral sequence.

Theorem. *Let G be an algebraic k-group scheme with H a closed subgroup scheme of G. Let $M \in \mathrm{Mod}(C(\pi))$ and $N \in \mathrm{Mod}(H)$. Then there exists a first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{C(\pi)}(M, R^j\mathscr{T}(N)) \Rightarrow \mathrm{Ext}^{i+j}_H(M,N).$$

Proof. The functor $\mathscr{S} = \mathrm{Hom}_{C(\pi)}(M,-)$ is left exact. Moreover, the functor $\mathscr{T} : \mathrm{Mod}(H) \to \mathrm{Mod}(C(\pi))$ takes injectives to injectives because $\mathscr{F}_{C(\pi)}$ and ind_H^G– have this property. Furthermore, note that since $M \in \mathrm{Mod}(C(\pi))$ we have

$$\mathscr{S} \circ \mathscr{T}(-) = \mathrm{Hom}_{C(\pi)}(M, \mathscr{T}(-)) = \mathrm{Hom}_C(M, \mathrm{ind}_H^G(-)) = \mathrm{Hom}_H(M,-).$$

The result now follows by Jantzen [11, I. Proposition 4.1]. □

4.3. Let G be as defined in 1.3 and let π be a subset of $X(T)_+$. In the context of the previous section, we will abuse notation and also let π denote the set of simple rational G-modules having highest weight in π. Since $\mathcal{O}_\pi$ is equivalent to $\mathscr{F}_{C(\pi)}$, it follows that $\mathrm{Mod}(C(\pi))$ is equivalent to the full subcategory of G-modules whose composition factors have highest weight in π. Now assume that π is saturated. That is, if $\lambda \in \pi$ and $\mu \in X(T)_+$ such that $\mu \leqslant \lambda$ then $\mu \in \pi$. In this case, Donkin proved the following isomorphism [6, (2.1f) Theorem]:

(4.3.1) For $M,N \in \mathrm{Mod}(C(\pi))$, $\mathrm{Ext}^n_G(M,N) \cong \mathrm{Ext}^n_{C(\pi)}(M,N)$ for $n \geqslant 0$.

For a fixed $r \geqslant 1$, let $q = p^r$ and for all $0 < s \leqslant p-1$ let

$$\pi_s = \{\lambda \in X(T)_+ : \langle \lambda + \rho, \alpha_0^\vee \rangle < 2sp^r \langle \rho, \alpha_0^\vee \rangle\},$$

where α_0 denotes the highest short root. Observe that if $\mu \leqslant \lambda$ then $\lambda - \mu = \sum_{\alpha \in \Delta} c_\alpha \alpha$ where $c_\alpha \geqslant 0$. Then

$$\langle (\lambda + \rho) - (\mu + \rho), \alpha_0^\vee \rangle = \left\langle \sum_{\alpha \in \Delta} c_\alpha \alpha, \alpha_0^\vee \right\rangle \geqslant 0.$$

Therefore, $\langle \lambda + \rho, \alpha_0^\vee \rangle \geqslant \langle \mu + \rho, \alpha_0^\vee \rangle$. This computation shows that π_s is saturated. Furthermore, let $\mathscr{C}_s$ be the full subcategory of G-modules all of whose composition factors have highest weights in π_s. For $s = 1$, the category $\mathscr{C}_1$ essentially coincides with the p^r-bounded category as defined in [11, p. 360]. Since the category $\mathscr{C}_s$ is equivalent to the category $\mathrm{Mod}(C(\pi_s))$ (as noted more generally above), the isomorphism in (4.3.1) holds for modules in $\mathscr{C}_s$.

4.4. Let $\mathscr{G}_s = \mathscr{F}_{C(\pi_s)} \circ \mathrm{ind}_{G(\mathbb{F}_q)}^G$ and $\mathscr{H}_s = \mathscr{F}_{C(\pi_s)} \circ \mathrm{ind}_{G_r}^G$. The Theorem given in 4.2 (with $H = G(\mathbb{F}_q)$ or $H = G_r$ and $\pi = \pi_s$) can be combined with the isomorphism in (4.3.1) to construct the following spectral sequences.

Theorem. *Let $M \in \mathscr{C}_s$.*
(a) *For $N \in \mathrm{mod}(G(\mathbb{F}_q))$ there exists a spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}_G^i(M, R^j\mathscr{G}_s(N)) \Rightarrow \mathrm{Ext}_{G(\mathbb{F}_q)}^{i+j}(M, N).$$

(b) *For $N \in \mathrm{mod}(G_r)$ there exists a spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}_G^i(M, R^j\mathscr{H}_s(N)) \Rightarrow \mathrm{Ext}_{G_r}^{i+j}(M, N).$$

4.5. Given $\lambda \in \pi_s$ with corresponding simple G-module $L(\lambda)$, let $\mathscr{P}(\lambda)$ denote the projective cover of $L(\lambda)$ in the category $\mathscr{C}_s$. Projective covers exist in the category $\mathscr{C}_s$ since it is equivalent to the category $\mathrm{Mod}(C(\pi_s))$ and $C(\pi_s)$ is a finite-dimensional coalgebra (see the discussion in 4.2). The following result describes the composition factors of the higher right derived functors of $\mathscr{G}_s$ and $\mathscr{H}_s$.

Theorem. *For $j \geqslant 0$:*
(a) *If $M \in \mathrm{mod}(G(\mathbb{F}_q))$ then*

$$[R^j\mathscr{G}_s(M) : L(\lambda)] = \dim_k \mathrm{Ext}_{G(\mathbb{F}_q)}^j(\mathscr{P}(\lambda), M).$$

(b) *If $N \in \mathrm{mod}(G_r)$ then*

$$[R^j\mathscr{H}_s(N) : L(\lambda)] = \dim_k \mathrm{Ext}_{G_r}^j(\mathscr{P}(\lambda), N).$$

Proof. We will prove part (a). The proof for part (b) is completely analogous. Since $R^j\mathscr{G}_s(M)$ is a module in $\mathscr{C}_s$, by definition of $\mathscr{P}(\lambda)$, we have

$$[R^j\mathscr{G}_s(M) : L(\lambda)] = \dim_k \mathrm{Hom}_{\mathscr{C}_s}(\mathscr{P}(\lambda), R^j\mathscr{G}_s(M)). \tag{4.2}$$

From the Theorem given in 4.4 there is a spectral sequence

$$E_2^{i,j} = \mathrm{Ext}_{\mathscr{C}_s}^i(\mathscr{P}(\lambda), R^j\mathscr{G}_s(M)) \Rightarrow \mathrm{Ext}_{G(\mathbb{F}_q)}^{i+j}(\mathscr{P}(\lambda), M).$$

Since $\mathscr{P}(\lambda)$ is projective in $\mathscr{C}_s$, the spectral sequence collapses to

$$E_2^{0,j} = \operatorname{Hom}_{\mathscr{C}_s}(\mathscr{P}(\lambda), R^j\mathscr{G}_s(M)) \cong \operatorname{Ext}^j_{G(\mathbb{F}_q)}(\mathscr{P}(\lambda), M),$$

from which the result follows. □

4.6. For a p^r-restricted weight $\lambda \in X_r(T)$, if $p \geqslant 2(h-1)$, the G_r structure on $Q_r(\lambda)$ lifts to a G-structure [12]. Moreover, when considered as a G-module, $Q_r(\lambda)$ is p^r-bounded and can be identified as the injective hull (and projective cover) of $L(\lambda)$ in the p^r-bounded category $\mathscr{C}_1$ [11, p. 360]. In other words, for a restricted weight λ, the module $\mathscr{P}(\lambda)$ may be identified with $Q_r(\lambda)$.

The following proposition, which is an adaptation of Lemma 2.2 of [2], gives us information about $\mathscr{P}(\lambda)$ for general $\lambda \in \pi_s$. For convenience, let $\pi_0 = X_r(T)$. Note that any weight $\lambda \in \pi_s$ may be expressed as $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r(T)$, that is with λ_0 being p^r-restricted. Further, for $s \geqslant 1$, λ_1 must satisfy $\langle \lambda_1, \beta^\vee \rangle < 2s(h-1)$ (or $\langle \lambda_1 + \rho, \beta^\vee \rangle < (2s+1)(h-1)$) for β short and $\beta \in \Phi \cap X(T)_+$. Hence if $p \geqslant (2s+1)(h-1)$, the weight λ_1 lies in the bottom alcove $\bar{C}_{\mathbb{Z}}$.

Proposition. *Assume $s \geqslant 1$, $M \in \mathscr{C}_s$, and one of the following two conditions holds*:
(i) $\lambda = \lambda_0 + p^r\lambda_1 \in \pi_s$ *and* $p \geqslant (2s+1)(h-1)$;
(ii) $\lambda = \lambda_0 + p^r\lambda_1 \in \pi_{s-1}$ *and* $p \geqslant 2s(h-1)$.
Then the following hold:
(a) $\operatorname{Ext}^1_G(Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}, M) = 0$.
(b) *The projective module $\mathscr{P}(\lambda)$ in $\mathscr{C}_s$ is a quotient of $Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$.*

Proof. Let $S = Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$. For part (a), it suffices to show that $\operatorname{Ext}^1_G(S, L(\mu)) = 0$ for all composition factors $L(\mu)$ of M. Since M is in $\mathscr{C}_s$, any such μ are in π_s, and so may be expressed as $\mu = \mu_0 + p^r\mu_1$ with $\mu_0 \in X_r(T)$. The short exact sequence of group schemes

$$1 \to G_r \to G \to G/G_r \to 1$$

gives rise to the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_{G/G_r}(k, \operatorname{Ext}^j_{G_r}(S, L(\mu))) \Rightarrow \operatorname{Ext}^{i+j}_G(S, L(\mu)).$$

Since $Q_r(\lambda_0)$ is projective over G_r, S is also. Hence, the spectral sequence collapses and gives an isomorphism

$$\operatorname{Ext}^i_G(S, L(\mu)) \cong \operatorname{Ext}^i_{G/G_r}(k, \operatorname{Hom}_G(S, L(\mu))$$

for all i. In particular, we have

$$\begin{aligned}
\operatorname{Ext}^1_G(S, L(\mu)) &\cong \operatorname{Ext}^1_{G/G_r}(k, \operatorname{Hom}_{G_r}(S, L(\mu))) \\
&= \operatorname{Ext}^1_{G/G_r}(k, \operatorname{Hom}_{G_r}(Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}, L(\mu_0) \otimes L(\mu_1)^{(r)})) \\
&= \operatorname{Ext}^1_{G/G_r}(k, \operatorname{Hom}_{G_r}(Q_r(\lambda_0), L(\mu_0)) \otimes (L(\lambda_1)^{(r)})^* \otimes L(\mu_1)^{(r)}) \\
&= \begin{cases} \operatorname{Ext}^1_{G/G_r}(k, k \otimes (L(\lambda_1)^{(r)})^* \otimes L(\mu_1)^{(r)}) & \text{if } \mu_0 = \lambda_0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \operatorname{Ext}^1_{G/G_r}(L(\lambda_1)^{(r)}, L(\mu_1)^{(r)}) & \text{if } \mu_0 = \lambda_0 \\ 0 & \text{otherwise} \end{cases} \\
&= 0
\end{aligned}
$$

by the Linkage Principle (cf. [11, II. 6.17]). More precisely, under condition (i), as both λ and μ lie in π_s and $p \geqslant (2s+1)(h-1)$, both λ_1 and μ_1 lie in the bottom alcove $\bar{C}_{\mathbb{Z}}$. And so the claim follows from the Linkage Principle.

On the other hand, under condition (ii), λ lies in π_{s-1} while μ lies in π_s. Now, the condition on p implies that $\langle \lambda_1 + \rho, \beta^\vee \rangle < (2(s-1)+1)(h-1) \leqslant p-(h-1)$ and $\langle \mu_1 + \rho, \beta^\vee \rangle < (2s+1)(h-1) \leqslant p+(h-1)$ for short $\beta \in \Phi \cap X(T)_+$. In other words, λ_1 lies in the bottom alcove and moreover, more than $h-1$ beneath the upper wall, while μ either lies in the bottom alcove or no more than $h-1$ above the upper wall. Hence, the desired vanishing again follows from the Linkage Principle.

For part (b), by definition of $\mathscr{P}(\lambda)$, there exists a surjection $\mathscr{P}(\lambda) \to L(\lambda)$ of G-modules. On the other hand, since there is a surjection $Q_r(\lambda_0) \to L(\lambda_0)$, there is also a surjection (of G-modules) $S = Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)} \to L(\lambda_0) \otimes L(\lambda_1)^{(r)} \cong L(\lambda)$. To see that the map $S \to L(\lambda)$ lifts to $S \to \mathscr{P}(\lambda)$, consider the short exact sequence

$$0 \to N \to \mathscr{P}(\lambda) \to L(\lambda) \to 0$$

and the corresponding long exact sequence of Ext-groups:

$$0 \to \operatorname{Hom}_G(S,N) \to \operatorname{Hom}_G(S,\mathscr{P}(\lambda)) \to \operatorname{Hom}_G(S,L(\lambda)) \to \operatorname{Ext}^1_G(S,N) \to \cdots .$$

By part (a), $\operatorname{Ext}^1_G(S,N) = 0$ and so the map $\operatorname{Hom}_G(S,\mathscr{P}(\lambda)) \to \operatorname{Hom}_G(S,L(\lambda))$ is a surjection, giving the desired lifting. □

4.7. The preceding proposition allows us to obtain some information about the higher right derived functors of $\mathscr{G}_s$ and $\mathscr{H}_s$.

Corollary. *Let* $s \geqslant 1$ *and* $p \geqslant 2s(h-1)$ *with* $M \in \operatorname{mod}(G(\mathbb{F}_q))$ *and* $N \in \operatorname{mod}(G_r)$. *If* $j > 0$ *then*
(a) $R^j\mathscr{G}_s(M)$ *contains no composition factors with high weight in* π_{s-1};
(b) $R^j\mathscr{H}_s(N)$ *contains no composition factors with high weight in* π_{s-1}.

Proof. Let $\lambda = \lambda_0 + p^r\lambda_1$ be a weight in π_{s-1}. By the Proposition given in 4.6(b), the projective cover $\mathscr{P}(\lambda)$ in $\mathscr{C}_s$ is a quotient of $Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$. But, the highest weight of $Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$ is $2(p^r-1)\rho + w_0\lambda_0 + p^r\lambda_1$. This weight is in π_s and so $Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$ lies in $\mathscr{C}_s$. By the projectivity of $\mathscr{P}(\lambda)$ in $\mathscr{C}_s$, $\mathscr{P}(\lambda)$ must be a G-summand of $Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$. The latter module is projective as a G_r-module and $G(\mathbb{F}_q)$-module. The assertion now follows by the Theorem given in 4.5. □

For $s \geqslant 1$ and $\lambda \in \pi_{s-1}$, the proof in fact shows that the module $\mathscr{P}(\lambda)$ in $\mathscr{C}_s$ may be identified with $Q_r(\lambda_0) \otimes L(\lambda_1)^{(r)}$ since the G-head of each module is $L(\lambda)$.

5. Stability of extensions

5.1. The first result of this section is a vanishing result involving the extensions of certain G-modules with the higher right derived functors of $\mathscr{G}_s$ and $\mathscr{H}_s$.

Proposition. *Let* $s \geqslant 1$, $M \in \operatorname{mod}(G(\mathbb{F}_q))$ *and* $N \in \operatorname{mod}(G_r)$ *and let* $L \in \mathscr{C}_s$ *be such that* L *has only* p^r*-restricted composition factors in its head. If* $p \geqslant 2s(h-1)$ *then*
(a) $\operatorname{Ext}^i_G(L, R^j\mathscr{G}_s(M)) = 0$;
(b) $\operatorname{Ext}^i_G(L, R^j\mathscr{H}_s(M)) = 0$
for $0 \leqslant i \leqslant s-1$ *and* $j > 0$.

Proof. (a) Let $\mathscr{P}_0$ be the projective cover of L in $\mathscr{C}_s$ and R_1 be the kernel of the map $\mathscr{P}_0 \to L$. Inductively, set $\mathscr{P}_i$ to be the projective cover of R_i and R_{i+1} to be the kernel of $\mathscr{P}_i \to R_i$. With this procedure we have constructed a projective resolution of L in $\mathscr{C}_s$:

$$\cdots \to \mathscr{P}_2 \to \mathscr{P}_1 \to \mathscr{P}_0 \to L \to 0.$$

Since the head of L contains only p^r-restricted composition factors, it follows by the Proposition given in 4.6(b) that the composition factors of $\mathscr{P}_0$ (and R_1) are in π_1. An inductive argument shows that the composition factors of $\mathscr{P}_i$ are in π_{i+1} and the composition factors in the head of $\mathscr{P}_i$ are in π_i for $0 \leqslant i \leqslant s-1$.

Therefore, by the Corollary given in 4.7, $\operatorname{Hom}_G(\mathscr{P}_i, R^j\mathscr{G}_s(M)) = 0$ for $j > 0$ and $0 \leqslant i \leqslant s-1$. Consequently, $\operatorname{Ext}^i_G(L, R^j\mathscr{G}_s(M)) = 0$ for $j > 0$ and $0 \leqslant i \leqslant s-1$. Part (b) follows by a similar argument. □

5.2. With the preceding results, one can provide positive answers to questions (1.1.1) and (1.1.2) by identifying certain extensions over $G(\mathbb{F}_q)$ or G_r with extensions between finite-dimensional G-modules.

Theorem. *Let* $s \geqslant 1$, $M \in \operatorname{mod}(G(\mathbb{F}_q))$ *and* $N \in \operatorname{mod}(G_r)$ *and let* $L \in \mathscr{C}_s$ *be such that* L *has only* p^r*-restricted composition factors in its head* (*as G-module*). *If* $p \geqslant 2s(h-1)$ *then there are the following isomorphisms and embeddings*:
(a) $\operatorname{Ext}^i_{G(\mathbb{F}_q)}(L, M) \cong \operatorname{Ext}^i_G(L, \mathscr{G}_s(M))$ *for all* $0 \leqslant i \leqslant s$;
(b) $\operatorname{Ext}^{s+1}_G(L, \mathscr{G}_s(M)) \hookrightarrow \operatorname{Ext}^{s+1}_{G(\mathbb{F}_q)}(L, M)$;
(c) $\operatorname{Ext}^i_{G_r}(L, N) \cong \operatorname{Ext}^i_G(L, \mathscr{H}_s(N))$ *for all* $0 \leqslant i \leqslant s$;
(d) $\operatorname{Ext}^{s+1}_G(L, \mathscr{H}_s(N)) \hookrightarrow \operatorname{Ext}^{s+1}_{G_r}(L, N)$.

Proof. The proofs of the two cases are analogous and we prove parts (a) and (b). From the Theorem given in 4.4 there is a spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_G(L, R^j\mathscr{G}_s(M)) \Rightarrow \operatorname{Ext}^{i+j}_{G(\mathbb{F}_q)}(L, M).$$

By the Propostion given in 5.1, we have $\operatorname{Ext}^i_G(L, R^j\mathscr{G}_s(M)) = 0$ for $j > 0$ and $0 \leqslant i \leqslant s-1$. The spectral sequence has differential with bidegree $(r, 1-r)$. Consequently, $\operatorname{Ext}^i_{G(\mathbb{F}_q)}(L, M) \cong \operatorname{Ext}^i_G(L, \mathscr{G}_s(M))$ for $0 \leqslant i \leqslant s$. Furthermore, since $E_2^{i,j} = 0$ for $j > 0$

and $0 \leqslant i \leqslant s-1$, the image of the differentials are zero in $E_2^{s+1,0}$, thus $E_2^{s+1,0} \hookrightarrow \operatorname{Ext}^{s+1}_{G(\mathbb{F}_q)}(L,M)$. $\square$

5.3. The succeeding corollaries are immediate applications of the theorem to determining extensions between simple modules and Weyl modules.

Corollary. *Let $p \geqslant 2(h-1)$ and $\lambda,\mu \in X_r(T)$. Then there exists the following isomorphisms and embeddings*:

(i) $\operatorname{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda),L(\mu)) \cong \operatorname{Ext}^1_G(L(\lambda),\mathscr{G}_1(L(\mu)))$;
(ii) $\operatorname{Ext}^2_G(L(\lambda),\mathscr{G}_1(L(\mu))) \hookrightarrow \operatorname{Ext}^2_{G(\mathbb{F}_q)}(L(\lambda),L(\mu))$;
(iii) $\operatorname{Ext}^1_{G(\mathbb{F}_q)}(V(\lambda),L(\mu)) \cong \operatorname{Ext}^1_G(V(\lambda),\mathscr{G}_1(L(\mu)))$;
(iv) $\operatorname{Ext}^2_G(V(\lambda),\mathscr{G}_1(L(\mu))) \hookrightarrow \operatorname{Ext}^2_{G(\mathbb{F}_q)}(V(\lambda),L(\mu))$;
(v) $\operatorname{Ext}^1_{G_r}(L(\lambda),L(\mu)) \cong \operatorname{Ext}^1_G(L(\lambda),\mathscr{H}_1(L(\mu)))$;
(vi) $\operatorname{Ext}^2_G(L(\lambda),\mathscr{H}_1(L(\mu))) \hookrightarrow \operatorname{Ext}^2_{G_r}(L(\lambda),L(\mu))$;
(vii) $\operatorname{Ext}^1_{G_r}(V(\lambda),L(\mu)) \cong \operatorname{Ext}^1_G(V(\lambda),\mathscr{H}_1(L(\mu)))$;
(viii) $\operatorname{Ext}^2_G(V(\lambda),\mathscr{H}_1(L(\mu))) \hookrightarrow \operatorname{Ext}^2_{G_r}(V(\lambda),L(\mu))$.

5.4. The following examples demonstrate that in order to study higher extensions between simple modules it is not sufficient to truncate the induction functor from the finite group (or the first Frobenius kernel) to the algebraic group at the level of twice the Steinberg weight. The methods for computing the functors $\mathscr{G}_1(-)$ and $\mathscr{H}_1(-)$ will be described in Sections 6 and 7.

Example. Let $G=\mathrm{SL}_2(k)$ and $G(\mathbb{F}_q)=\mathrm{SL}_2(\mathbb{F}_p)$ with $p \geqslant 7$. By a direct computation we have

$$\mathscr{G}_1(L(p-2)) \cong L(p-2) \oplus L(2p-3).$$

The simple module $L(p-4)$ is not in the same G-block as $L(p-2)$ or $L(2p-3)$, thus $\operatorname{Ext}^2_G(L(p-4),\mathscr{G}_1(L(p-2)))=0$. On the other hand, the structure of the projective indecomposable modules are known for $G(\mathbb{F}_q)$=$\mathrm{SL}_2(\mathbb{F}_p)$. One can construct the minimal projective resolution of the simple module $L(p-4)$ to show that $\operatorname{Ext}^2_{G(\mathbb{F}_p)}(L(p-4),L(p-2)) \neq 0$. Consequently, $\operatorname{Ext}^2_{G(\mathbb{F}_p)}(L(p-4),L(p-2)) \neq \operatorname{Ext}^2_G(L(p-4),\mathscr{G}_1(L(p-2)))$.

Example. Let $G=\mathrm{SL}_2(k)$ and $G_1=(\mathrm{SL}_2)_1$ with $p \geqslant 7$. We have

$$\mathscr{H}_1(L(p-2)) \cong L(p-2) \oplus L(2p-2) \oplus L(2p-2).$$

We have $\operatorname{Ext}^j_{G_1}(L(p-2),L(p-2))=0$ for j odd and $\operatorname{Ext}^2_{G_1}(L(p-2),L(p-2)) \cong L(2)^{(1)}$ as a G-module. There exists a spectral sequence

$$E_2^{i,j}=\operatorname{Ext}^i_{G/G_1}(k,\operatorname{Ext}^j_{G_1}(L(p-2),L(2p-2)) \Rightarrow \operatorname{Ext}^{i+j}_G(L(p-2),L(2p-2)).$$

Observe by the Steinberg tensor product theorem that

$$\operatorname{Ext}^j_{G_1}(L(p-2),L(2p-2)) \cong \operatorname{Ext}^j_{G_1}(L(p-2),L(p-2)) \otimes L(1)^{(1)}.$$

Therefore, $E_2^{1,1}=0$, $E_2^{2,0}=\mathrm{Ext}^2_{G/G_1}(k,L(1)^{(1)})=0$ and $E_2^{0,2}=\mathrm{Hom}_{G/G_1}(k,L(2)^{(1)}\otimes L(1)^{(1)})=0$. This shows that $\mathrm{Ext}^2_G(L(p-2),L(2p-2))=0$. A similar spectral sequence argument can be used to show that $\mathrm{Ext}^2_G(L(p-2),L(p-2))=0$. It follows that $\mathrm{Ext}^2_G(L(p-2),\mathscr{H}_1(L(p-2)))=0$, and $\mathrm{Ext}^2_{G_1}(L(p-2),L(p-2))\neq \mathrm{Ext}^2_G(L(p-2),\mathscr{H}_1(L(p-2)))$.

5.5. Small primes. The results in 5.3 rely on the fact that the prime p is not too small. In this section we obtain a slightly weaker version of the Corollary given in 5.3 for arbitrary primes.

Theorem. *Let $\lambda,\mu\in X_r(T)$ with $\mu\not>\lambda$. Then*
(a) $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda),L(\mu))\cong \mathrm{Ext}^1_G(L(\lambda),\mathscr{G}_1(L(\mu)))$;
(b) $\mathrm{Ext}^1_{G_r}(L(\lambda),L(\mu))\cong \mathrm{Ext}^1_G(L(\lambda),\mathscr{H}_1(L(\mu)))$.

Proof. (a) From [19, Lemma 1.4] one obtains the following statement. Let $\lambda\in X_r(T)$ and let M be a $G(\mathbb{F}_q)$-module that contains only simple composition factors whose p^r-restricted highest weights γ satisfy $\gamma\leqslant (p^r-1)\rho+w_0\lambda$ where w_0 is the long element in the Weyl group. Then the $G(\mathbb{F}_q)$-head of $\mathrm{St}_r\otimes M$ contains only simple modules $L(\mu)$ whose p^r-restricted highest weights satisfy $\mu\geqslant\lambda$. In particular it follows that for $\mu\not\geqslant\lambda$

$$\begin{aligned}&\mathrm{Hom}_G(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\mu))\\ &\quad\subseteq \mathrm{Hom}_{G(\mathbb{F}_q)}(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\mu))=0.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathrm{Hom}_G(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\lambda))\\ &\quad\cong \mathrm{Hom}_{G(\mathbb{F}_q)}(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\lambda))\cong k.\end{aligned}$$

Therefore, we have the following short exact sequence

$$0\to R\to \mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda)\to L(\lambda)\to 0. \tag{5.1}$$

Note that this is a short exact sequence in the category $\mathscr{C}_1$ because $\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda)\in\mathscr{C}_1$. By using the fact that the Steinberg module is projective over $G(\mathbb{F}_q)$ we obtain the following exact sequence:

$$\begin{aligned}&0\to \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda),L(\mu))\to \mathrm{Hom}_{G(\mathbb{F}_q)}(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\mu))\\ &\quad\to \mathrm{Hom}_{G(\mathbb{F}_q)}(R,L(\mu))\to \mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda),L(\mu))\to 0.\end{aligned}$$

If $\mu\not>\lambda$ then by our previous observations we have

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda),L(\mu))\cong \mathrm{Hom}_{G(\mathbb{F}_q)}(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\mu)).$$

Consequently,

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(R,L(\mu))\cong \mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda),L(\mu)). \tag{5.2}$$

From the short exact sequence (5.1) one obtains the following long exact sequence:

$$\begin{aligned}0 &\to \mathrm{Hom}_G(L(\lambda),\mathscr{G}_1(L(\mu))) \to \mathrm{Hom}_G(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),\mathscr{G}_1(L(\mu)))\\ &\to \mathrm{Hom}_G(R,\mathscr{G}_1(L(\mu))) \to \mathrm{Ext}^1_G(L(\lambda),\mathscr{G}_1(L(\mu)))\\ &\to \mathrm{Ext}^1_G(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),\mathscr{G}_1(L(\mu))) \to \cdots\ .\end{aligned}$$

Observe that by adjointness we have for $\mu \not\geq \lambda$

$$\begin{aligned}&0=\mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda),L(\mu)) \cong \mathrm{Hom}_G(L(\lambda),\mathscr{G}_1(L(\mu))),\\ &0=\mathrm{Hom}_{G(\mathbb{F}_q)}(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\mu))\\ &\quad\cong \mathrm{Hom}_G(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),\mathscr{G}_1(L(\mu))).\end{aligned}$$

Similarly, for $\mu=\lambda$ it follows that

$$\begin{aligned}&\mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda),L(\lambda)) \cong \mathrm{Hom}_G(L(\lambda),\mathscr{G}_1(L(\lambda))) \cong k,\\ &\mathrm{Hom}_{G(\mathbb{F}_q)}(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),L(\lambda))\\ &\quad\cong \mathrm{Hom}_G(\mathrm{St}_r\otimes L((p^r-1)\rho+w_0\lambda),\mathscr{G}_1(L(\lambda))) \cong k.\end{aligned}$$

The long exact sequence and these isomorphisms now define an injective map

$$\mathrm{Hom}_G(R,\mathscr{G}_1(L(\mu))) \to \mathrm{Ext}^1_G(L(\lambda),\mathscr{G}_1(L(\mu))).$$

But, $R\in\mathscr{C}_1$ so by adjointness we have

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(R,L(\mu)) \cong \mathrm{Hom}_G(R,\mathscr{G}_1(L(\mu))).$$

It follows by (5.2) that there exists an injective map

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda),L(\mu)) \to \mathrm{Ext}^1_G(L(\lambda),\mathscr{G}_1(L(\mu))).$$

The five term exact sequence of the spectral sequence in 4.4 makes the base map $\mathrm{E}^{1,0}\to\mathrm{E}^1$ an injective map in the other direction. Therefore, one obtains the desired isomorphism

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda),L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda),\mathscr{G}_1(L(\mu))).$$

A similar argument can be used to prove part (b). □

6. Cohomology for Frobenius kernels

6.1. In the previous section, extensions of simple modules over $G(\mathbb{F}_q)$ and G_r were identified with certain extensions over G. These G extensions involve the “induction” functors $\mathscr{G}_s$ and $\mathscr{H}_s$. In this and the following section, we study $\mathscr{G}_s(L(\mu))$ and $\mathscr{H}_s(L(\mu))$ for $\mu\in X_r(T)$, in order to improve this identification to involve only simple modules. We begin by showing that the module $\mathscr{H}_s(L(\mu))$ is semisimple for $\mu\in X_r(T)$.

Theorem. *Let* $p \geqslant (2s+1)(h-1)$ *and* $\mu \in X_r(T)$. *Then* $\mathscr{H}_s(L(\mu))$ *is semisimple. Moreover*,

$$\mathscr{H}_s(L(\mu)) \cong \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_G(L(\nu), \mathscr{H}_s(L(\mu))) \otimes L(\nu) \cong \bigoplus_{\nu\in\pi_s} L(\nu)^{\dim_k \mathrm{Hom}_{G_r}(L(\nu),L(\mu))}.$$

Proof. First, consider the socle of $\mathscr{H}_s(L(\mu))$ over G (or equivalently in the category $\mathscr{C}_s$):

$$\begin{aligned}\mathrm{soc}_G \mathscr{H}_s(L(\mu)) &= \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_G(L(\nu), \mathscr{H}_s(L(\mu))) \otimes L(\nu)\\ &= \bigoplus_{\nu\in\pi_s} L(\nu)^{\dim_k \mathrm{Hom}_{G_r}(L(\nu),L(\mu))}\end{aligned}$$

by adjointness. So, if $\mathscr{H}_s(L(\mu))$ is semisimple, then it has the claimed form.

The Theorem given in 4.5 can be used to obtain information about the composition factors of $\mathscr{H}_s(L(\mu))$. Specifically, we have

$$\begin{aligned}&[\mathscr{H}_s(L(\mu)) : L(\nu)]\\ &\quad = \dim_k \mathrm{Hom}_{G_r}(\mathscr{P}(\nu), L(\mu))\\ &\quad \leqslant \dim_k \mathrm{Hom}_{G_r}(Q_r(\nu_0) \otimes L(\nu_1)^{(r)}, L(\mu)) \quad \text{(by the Proposition of 4.6(b))}.\end{aligned}$$

As a G_r-module, $L(\nu_1)^{(r)}$ is trivial and so $Q_r(\nu_0) \otimes L(\nu_1)^{(r)} = \oplus Q_r(\nu_0)$. Further, since ν_0 and μ are both p^r-restricted weights, $\mathrm{Hom}_{G_r}(Q_r(\nu_0), L(\mu)) = \mathrm{Hom}_{G_r}(L(\nu_0), L(\mu))$, and this will be non-zero if and only if $\nu_0 = \mu$. Hence, the only possible composition factors of $\mathscr{H}_s(L(\mu))$ are those $L(\nu)$ with $\nu = \mu + p^r\sigma$. As noted in 4.6, since ν is in π_s and μ is p^r-restricted, σ must lie in the bottom alcove.

To show that $\mathscr{H}_s(L(\mu))$ is semisimple, it now suffices to show that $\mathrm{Ext}^1_G(L(\nu_1), L(\nu_2)) = 0$ for any such weights $\nu_1 = \mu + p^r\sigma_1$ and $\nu_2 = \mu + p^r\sigma_2$. The argument is similar to that of the Proposition given in 4.6. The short exact sequence of group schemes

$$1 \to G_r \to G \to G/G_r \to 1$$

gives rise to the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_r}(k, \mathrm{Ext}^j_{G_r}(L(\nu_1), L(\nu_2))) \Rightarrow \mathrm{Ext}^{i+j}_G(L(\nu_1), L(\nu_2)).$$

The beginning of the 5-term exact sequence is

$$0 \to E^{1,0} \to E^1 \to E^{0,1} = \mathrm{Hom}_{G/G_r}(k, \mathrm{Ext}^1_{G_r}(L(\nu_1), L(\nu_2))) \to \dots .$$

However,

$$\begin{aligned}\mathrm{Ext}^1_{G_r}(L(\nu_1), L(\nu_2)) &= \mathrm{Ext}^1_{G_r}(L(\mu) \otimes L(\sigma_1)^{(r)}, L(\mu) \otimes L(\sigma_2)^{(r)})\\ &= \mathrm{Ext}^1_{G_r}\Big(\bigoplus L(\mu), \bigoplus L(\mu)\Big) = 0,\end{aligned}$$

since there are no self-extensions over G_r (cf. [11, II. 12.9]). Hence there is an isomorphism

$$\mathrm{Ext}^1_G(L(\nu_1), L(\nu_2)) = E^1 \cong E^{1,0} = \mathrm{Ext}^1_{G/G_r}(k, \mathrm{Hom}_{G_r}(L(\nu_1), L(\nu_2))).$$

Finally, we have

$$\begin{aligned}
&\mathrm{Ext}^1_{G/G_r}(k, \mathrm{Hom}_{G_r}(L(\nu_1), L(\nu_2)))\\
&\quad= \mathrm{Ext}^1_{G/G_r}(k, \mathrm{Hom}_{G_r}(L(\mu)\otimes L(\sigma_1)^{(r)}, L(\mu)\otimes L(\sigma_2)^{(r)}))\\
&\quad= \mathrm{Ext}^1_{G/G_r}(k, \mathrm{Hom}_{G_r}(L(\mu), L(\mu))\otimes (L(\sigma_1)^{(r)})^*\otimes L(\sigma_2)^{(r)})\\
&\quad= \mathrm{Ext}^1_{G/G_r}(k, k\otimes (L(\sigma_1)^{(r)})^*\otimes L(\sigma_2)^{(r)})\\
&\quad= \mathrm{Ext}^1_{G/G_r}(L(\sigma_1)^{(r)}, L(\sigma_2)^{(r)}) = 0
\end{aligned}$$

by the Linkage Principle since σ_1 and σ_2 lie in the same alcove. □

6.2. The following result shows that one can completely determine the extensions of simple modules in $\mathrm{mod}(G_r)$ by knowing the extension theory of simple modules in $\mathrm{mod}(G)$.

Corollary. *Let* $p \geqslant (2s+1)(h-1)$ *and* $\lambda, \mu \in X_r(T)$. *Moreover, let* $L \in \mathscr{C}_s$ *be such that* L *has only* p^r*-restricted composition factors in its head. Then*

(a) *For* $0 \leqslant i \leqslant s$ *there exists the following isomorphism of* k*-vector spaces*:

$$\mathrm{Ext}^i_{G_r}(L, L(\mu)) \cong \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_{G_r}(L(\nu), L(\mu))\otimes \mathrm{Ext}^i_G(L, L(\nu)).$$

(b) *In particular for* $0 \leqslant i \leqslant s$, *we have*

$$\mathrm{Ext}^i_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_{G_r}(L(\nu), L(\mu))\otimes \mathrm{Ext}^i_G(L(\lambda), L(\nu)).$$

(c) *For* $p \geqslant 3(h-1)$, *we have*

$$\mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu\in\pi_1} \mathrm{Hom}_{G_r}(L(\nu), L(\mu))\otimes \mathrm{Ext}^1_G(L(\lambda), L(\nu)),$$

where π_1 *is the set of all* p^r*-bounded weights.*

Proof. The result follows by combining the isomorphism

$$\mathrm{Ext}^i_{G_r}(L, L(\mu)) \cong \mathrm{Ext}^i_G(L, \mathscr{H}_s(L(\mu)))$$

for $0 \leqslant i \leqslant s$ of the Theorem given in 5.2 with the preceding identification of $\mathscr{H}_s(L(\mu))$ given in the Theorem of 6.1. □

Observe that the restriction on the prime indicates that the direct sums appearing here may be taken over those $\nu \in \pi_s$ with $\nu = \mu + p^r\sigma$ and $\sigma \in \bar{C}_{\mathbb{Z}}$.

7. Cohomology for finite Chevalley groups

7.1. In this section, we use the module $\mathscr{G}_s(L(\mu))$ to recover and extend the results of [2] on extensions of $G(\mathbb{F}_q)$-modules. The hypotheses of these results depend on the type of G. This is because when G is not of type A_1, $\langle \alpha, \alpha_0^\vee \rangle \leqslant 1$ for all simple roots α, while $\langle \alpha, \alpha_0^\vee \rangle = 2$ for the simple root $\alpha = \alpha_0$ in type A_1. First, consider the embedding $L(\lambda) \to \mathcal{G}_s(L(\lambda))$ of G-modules, which corresponds to the identity map under the isomorphism $\mathrm{Hom}_G(L(\lambda), \mathcal{G}_s(L(\lambda))) \cong \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda), L(\lambda))$.

Proposition. *Let* $s \geqslant 1$ *and* $p \geqslant (2s+1)(h-1)$. *Suppose* $\lambda, \mu \in X_r(T)$ *satisfy* $\langle \lambda + \mu, \alpha_0^\vee \rangle < p^r - \varepsilon p^{r-1} - 1$ *where* $\varepsilon = 2$ *if* G *is of type* A_1 *and* $\varepsilon = 1$ *in all other cases. Then*

$$\mathrm{Ext}^1_G(L(\lambda), \mathcal{G}_s(L(\mu))/L(\mu)) = 0.$$

Proof. From the Theorem given in 4.5, $L(\mu)$ is the only restricted composition factor of $\mathcal{G}_s(L(\mu))$ and moreover appears only once. Hence, the composition factors of $\mathcal{G}_s(L(\mu))/L(\mu)$ are not restricted. Let $L(v)$ be a non-restricted composition factor of $\mathcal{G}_s(L(\mu))$ and write $v = v_0 + p^r v_1$ as usual. Since v_0 is restricted and v is not, v_1 is necessarily non-zero. According to the Theorem of 4.5 and the Proposition of 4.6(b), we have

$$\begin{aligned}[\mathcal{G}_s(L(\mu)) : L(v)] &= \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(\mathcal{P}(v), L(\mu)) \\ &\leqslant \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0) \otimes L(v_1), L(\mu)) \\ &= \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0), L(\mu) \otimes L(-w_0 v_1)).\end{aligned}$$

So for $L(v)$ to be a composition factor, we must have $\mathrm{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0), L(\omega)) \neq 0$ for some composition factor $L(\omega)$ of $L(\mu) \otimes L(-w_0 v_1)$. By Jantzen [13] or Chastkofsky [4] (see also [2, 1(2)]), $Q_r(v_0)$ decomposes as a direct sum $Q_r(v_0) \cong \bigoplus_\sigma U_r(\sigma)$ (of projective indecomposables for $G(\mathbb{F}_q)$) where each σ satisfies $\langle \sigma, \alpha_0^\vee \rangle \geqslant p^r - 1 + \langle v_0, \alpha_0^\vee \rangle$. Hence, there is at least one ω with $\omega = v_0$ or $\langle \omega, \alpha_0^\vee \rangle \geqslant p^r - 1 + \langle v_0, \alpha_0^\vee \rangle$.

On the other hand, by Lemma 2.5 of [2], if $\mathrm{Ext}^1_G(L(\lambda), L(v)) \neq 0$, then $\langle \lambda + v_0, \alpha_0^\vee \rangle \geqslant p^r \langle v_1, \alpha_0^\vee \rangle - \varepsilon p^{r-1}$. Hence, for such v and any weight ω of $L(\mu) \otimes L(-w_0 v_1)$, we have

$$\begin{aligned}\langle \omega, \alpha_0^\vee \rangle &\leqslant \langle \mu + v_1, \alpha_0^\vee \rangle \\ &< p^r - \varepsilon p^{r-1} - 1 - \langle \lambda + \mu, \alpha_0^\vee \rangle + \langle \mu + v_1, \alpha_0^\vee \rangle \quad \text{(by original assumption)} \\ &\leqslant (p^r - 1)\langle v_1, \alpha_0^\vee \rangle - \varepsilon p^{r-1} - \langle \lambda + \mu, \alpha_0^\vee \rangle + \langle \mu + v_1, \alpha_0^\vee \rangle \\ &= p^r \langle v_1, \alpha_0^\vee \rangle - \varepsilon p^{r-1} - \langle \lambda, \alpha_0^\vee \rangle \\ &\leqslant \langle v_0, \alpha_0^\vee \rangle \quad \text{(by Lemma 2.5 of [2])}.\end{aligned}$$

Hence, $\mathrm{Ext}^1_G(L(\lambda), L(v)) = 0$ for all composition factors $L(v)$ of $\mathcal{G}_s(L(\mu))/L(\mu)$. □

Corollary. *Assume* $p \geqslant 3(h-1)$. Let $\lambda, \mu \in X_r(T)$ *satisfy* $\langle \lambda + \mu, \alpha_0^\vee \rangle < p^r - \varepsilon p^{r-1} - 1$ *where* $\varepsilon = 2$ *if* G *is of type* A_1 *and* $\varepsilon = 1$ *in all other cases. Then*
(a) (*Theorem* 2.8 *of* [2]) $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu))$.
(b) *Moreover, there is an embedding* $\mathrm{Ext}^2_G(L(\lambda), L(\mu)) \hookrightarrow \mathrm{Ext}^2_{G(\mathbb{F}_q)}(L(\lambda), L(\mu))$.

Proof. For (a), consider the the short exact sequence

$$0 \to L(\mu) \to \mathcal{G}_1(L(\mu)) \to \mathcal{G}_1(L(\mu))/L(\mu) \to 0.$$

This short exact sequence induces a long exact sequence

$$\begin{aligned}\cdots &\to \mathrm{Hom}_G(L(\lambda), \mathscr{G}_1(L(\mu))/L(\mu)) \to \mathrm{Ext}^1_G(L(\lambda), L(\mu))\\ &\to \mathrm{Ext}^1_G(L(\lambda), \mathscr{G}_1(L(\mu))) \to \mathrm{Ext}^1_G(L(\lambda), \mathscr{G}_1(L(\mu))/L(\mu)) \to \cdots .\end{aligned}$$

The first term is zero since $\mathscr{G}_1(L(\mu))/L(\mu)$ has no restricted composition factors, and the last term is zero by the Proposition given in 7.1. Hence, there is an isomorphism

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), \mathscr{G}_1(L(\mu)))$$

and the claimed isomorphism follows from the Corollary given in 5.3.

For (b), continuing the above long exact sequence, we have

$$\begin{aligned}\cdots &\to \mathrm{Ext}^1_G(L(\lambda), \mathscr{G}_1(L(\mu))/L(\mu))\\ &\to \mathrm{Ext}^2_G(L(\lambda), L(\mu)) \to \mathrm{Ext}^2_G(L(\lambda), \mathscr{G}_1(L(\mu)) \to \cdots .\end{aligned}$$

Again, since the first term is zero by the Proposition, we have an embedding

$$\mathrm{Ext}^2_G(L(\lambda), L(\mu)) \hookrightarrow \mathrm{Ext}^2_G(L(\lambda), \mathscr{G}_1(L(\mu))).$$

The result follows by combining this with the embedding

$$\mathrm{Ext}^2_G(L(\lambda), \mathscr{G}_1(L(\mu))) \hookrightarrow \mathrm{Ext}^2_{G(\mathbb{F}_q)}(L(\lambda), L(\mu))$$

of the Corollary given in 5.3. □

7.2. More generally, semisimplicity of $\mathscr{G}_s(L(\mu))$ can be used to identify $G(\mathbb{F}_q)$-extensions with G-extensions.

Theorem. *Let $s \geqslant 1$ and $\mu \in X_r(T)$. Then $\mathscr{G}_s(L(\mu))$ is semisimple if and only if*

$$\mathscr{G}_s(L(\mu)) \cong \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\nu), L(\mu)) \otimes L(\nu).$$

Proof. Consider the socle of $\mathscr{G}_s(L(\mu))$ over G (or equivalently in the category $\mathscr{C}_s$):

$$\begin{aligned}\mathrm{soc}_G\, \mathscr{G}_s(L(\mu)) &= \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_G(L(\nu), \mathscr{G}_s(L(\mu))) \otimes L(\nu)\\ &= \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\nu), L(\mu)) \otimes L(\nu).\end{aligned}$$

So $\mathscr{G}_s(L(\mu))$ is semisimple if and only if it has the claimed form. □

The previous result together with the Theorem of 5.2 yields the following corollary.

Corollary. *Let $s \geqslant 1$, $p \geqslant 2s(h-1)$, $\mu \in X_r(T)$, and let $L \in \mathscr{C}_s$ be such that L has only p^r-restricted composition factors in its head (as G-module). Assume that $\mathscr{G}_s(L(\mu))$ is semisimple. Then for $0 \leqslant i \leqslant s$ there exists the following isomorphism of vector spaces:*

$$\mathrm{Ext}^i_{G(\mathbb{F}_q)}(L, L(\mu)) \cong \bigoplus_{\nu\in\pi_s} \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\nu), L(\mu)) \otimes \mathrm{Ext}^i_G(L, L(\nu)).$$

In particular, for $p \geqslant 2(h-1)$, *we have*

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L, L(\mu)) \cong \bigoplus_{\nu\in\pi_1} \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\nu), L(\mu)) \otimes \mathrm{Ext}^1_G(L, L(\nu)),$$

where π_1 *is the set of all* p^r*-bounded weights.*

7.3. We will now refine the result given in the Corollary of 7.2 by identifying homomorphisms over $G(\mathbb{F}_q)$ with homomorphisms over G.

Lemma. *Let* $p \geqslant 2(h-1)$, *and* $\lambda, \mu, \nu \in X_r(T)$ *with* $\langle \nu, \alpha_0^\vee\rangle < p^r - 1$. *Then*

$$\mathrm{Hom}_G(L(\lambda), L(\mu)\otimes L(\nu)) \cong \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)\otimes L(\nu)).$$

Proof. First, it is shown that the G-socle of $L(\mu)\otimes L(\nu)$ contains only simple modules with p^r-restricted highest weights. Assume that

$$0 \neq \mathrm{Hom}_G(L(\gamma_0)\otimes L(\gamma_1)^{(r)}, L(\mu)\otimes L(\nu)),$$

where $\gamma_0 \in X_r(T)$. Without loss of generality, we may assume that $\langle\gamma_0, \alpha_0^\vee\rangle \geqslant \langle\mu, \alpha_0^\vee\rangle$ otherwise replace γ_0 by $-w_0\mu$ and μ by $-w_0\gamma_0$. For the simple module $L(\gamma_0 + p^r\gamma_1)$ to appear as a composition factor in $L(\mu)\otimes L(\nu)$ it is necessary that

$$\langle\gamma_0 + p^r\gamma_1, \alpha_0^\vee\rangle \leqslant \langle\mu+\nu, \alpha_0^\vee\rangle \leqslant \langle\gamma_0+\nu, \alpha_0^\vee\rangle,$$

which implies that $p^r\langle\gamma_1, \alpha_0^\vee\rangle \leqslant \langle\nu, \alpha_0^\vee\rangle < p^r - 1$ and forces $\gamma_1 = 0$.

The above argument shows that $\mathrm{soc}_G L(\mu)\otimes L(\nu) \cong \bigoplus_i L(\sigma_i)$ where all σ_i are p^r-restricted and satisfy $\langle\mu-\nu, \alpha_0^\vee\rangle \leqslant \langle\sigma_i, \alpha_0^\vee\rangle \leqslant \langle\mu+\nu, \alpha_0^\vee\rangle$. Moreover, the G-socle of $L(\mu)\otimes L(\nu)$ is contained in the $G(\mathbb{F}_q)$-socle. Using the injectivity of $Q_r(\gamma)$ in the p^r-bounded category one can embed $L(\mu)\otimes L(\nu)$ in $\bigoplus_i Q_r(\sigma_i)$. The $G(\mathbb{F}_q)$-socle of $L(\mu)\otimes L(\nu)$ is therefore contained in the $G(\mathbb{F}_q)$-socle of $\bigoplus_i Q_r(\sigma_i)$.

Now assume that $\dim_k \mathrm{Hom}_G(L(\lambda), L(\mu)\otimes L(\nu)) < \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)\otimes L(\nu))$. Without loss of generality, we may assume this time that $\langle\mu, \alpha_0^\vee\rangle \geqslant \langle\lambda, \alpha_0^\vee\rangle$. It follows that $\dim_k \mathrm{Hom}_G(L(\lambda), \bigoplus_i Q_r(\sigma_i)) < \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda), \bigoplus_i Q_r(\sigma_i))$. That implies that, for at least one σ_i, $\dim_k \mathrm{Hom}_G(L(\lambda), Q_r(\sigma_i)) < \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(L(\lambda), Q_r(\sigma_i))$.

The module $Q_r(\sigma_i)$ are also injective as $G(\mathbb{F}_q)$-modules. A formula by Jantzen [13] and Chastkofsky [4] tells us how the $Q_r(\sigma_i)$ split into indecomposable injective $G(\mathbb{F}_q)$-modules, denoted here by $U_r(\xi)$. The module $Q_r(\sigma_i)$ contains $U_r(\sigma_i)$ exactly once, while any $U_r(\xi)$ with $\xi \neq \sigma_i$ that appears as a summand forces $\langle\sigma_i, \alpha_0^\vee\rangle + p^r - 1 \leqslant \langle\xi, \alpha_0^\vee\rangle$ (see also [2]). Clearly $\lambda \neq \sigma_i$. Therefore we obtain the following sequence of inequalities:

$$\begin{aligned}\langle\mu, \alpha_0^\vee\rangle - \langle\nu, \alpha_0^\vee\rangle + p^r - 1 &= \langle\mu-\nu, \alpha_0^\vee\rangle + p^r - 1\\ &\leqslant \langle\sigma_i, \alpha_0^\vee\rangle + p^r - 1 \leqslant \langle\lambda, \alpha_0^\vee\rangle \leqslant \langle\mu, \alpha_0^\vee\rangle.\end{aligned}$$

Now $p^r - 1 \leqslant \langle\nu, \alpha_0^\vee\rangle < p^r - 1$ gives us the desired contradiction. □

For large enough primes one obtains the following result:

Theorem. *Let* $s \geqslant 1$, $p^r > 2s(h-1)$, *and* $\mu \in X_r(T)$ *and assume that* $\mathscr{G}_s(L(\mu))$ *is semisimple. For any weight* $v \in \pi_s$ *we set* $v = v_0 + p^r v_1$ with $v_0 \in X_r(T)$. *Then*

$$\mathscr{G}_s(L(\mu)) \cong \bigoplus_{v\in\pi_s} \operatorname{Hom}_G(L(v_0)\otimes L(v_1), L(\mu)) \otimes L(v).$$

Proof. According to the Theorem given in 7.2 we have

$$\mathscr{G}_s(L(\mu)) \cong \bigoplus_{v\in\pi_s} \operatorname{Hom}_{G(\mathbb{F}_q)}(L(v), L(\mu)) \otimes L(v).$$

For any weight $v \in \pi_s$ we set $v = v_0 + p^r v_1$ with $v_0 \in X_r(T)$. Then $L(v) \cong L(v_0)\otimes L(v_1)$ as $G(\mathbb{F}_q)$-modules and so

$$\mathscr{G}_s(L(\mu)) \cong \bigoplus_{v\in\pi_s} \operatorname{Hom}_{G(\mathbb{F}_q)}(L(v_0)\otimes L(v_1), L(\mu)) \otimes L(v).$$

Moreover, $p^r\langle v_1, \alpha_0^\vee\rangle \leqslant \langle v, \alpha_0^\vee\rangle < 2sp^r(h-1)$ implies that $\langle v_1, \alpha_0^\vee\rangle < 2s(h-1) \leqslant p^r - 1$. Therefore, the theorem now follows by applying the preceding Lemma to $\operatorname{Hom}_{G(\mathbb{F}_q)}(L(v_0), L(\mu)\otimes L(-w_0v_1))$. □

Corollary. *Let* $s \geqslant 1$, $p \geqslant 2s(h-1)$, $\mu \in X_r(T)$, *and let* $L \in \mathscr{C}_s$ *be such that* L *has only* p^r*-restricted composition factors in its head* (*as G-module*). *We assume that* $\mathscr{G}_s(L(\mu))$ *is semisimple. For any weight* $v \in \pi_s$ *we set* $v = v_0 + p^r v_1$ with $v_0 \in X_r(T)$. *Then for* $0 \leqslant i \leqslant s$,

$$\operatorname{Ext}^i_{G(\mathbb{F}_q)}(L, L(\mu)) \cong \bigoplus_{v\in\pi_s} \operatorname{Hom}_G(L(v_0)\otimes L(v_1), L(\mu)) \otimes \operatorname{Ext}^i_G(L, L(v)).$$

In particular, *for* $p \geqslant 2(h-1)$, *we have*

$$\operatorname{Ext}^1_{G(\mathbb{F}_q)}(L, L(\mu)) \cong \bigoplus_{v\in\pi_1} \operatorname{Hom}_G(L(v_0)\otimes L(v_1), L(\mu)) \otimes \operatorname{Ext}^1_G(L, L(v)),$$

where π_1 *is the set of all* p^r*-bounded weights.*

7.4. We now identify some μ for which $\mathscr{G}_s(L(\mu))$ is indeed semisimple. Let $X[s] = \{v \in X(T)_+ \mid \langle v, \alpha_0^\vee\rangle < 2s(h-1)\}$.

Theorem. *Let* $s \geqslant 1$, $p \geqslant (2s+1)(h-1)$. *Then* $\mathscr{G}_s(k)$ *is semisimple. Moreover*,

$$\mathscr{G}_s(k) \cong \bigoplus_{v\in X[s]} L((p^r - w_0)v).$$

Proof. The Theorem of 4.5 can be used to obtain information about the composition factors of $\mathscr{G}_s(k)$. Specifically, we have

$$\begin{aligned}[\mathscr{G}_s(k) : L(v)] &= \dim_k \operatorname{Hom}_{G(\mathbb{F}_q)}(\mathscr{P}(v), k)\\ &\leqslant \dim_k \operatorname{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0)\otimes L(v_1)^{(r)}, k) \text{ (by the Proposition of 4.6(b)).}\end{aligned}$$

As a $G(\mathbb{F}_q)$-module, $L(v_1)^{(r)}$ is isomorphic to $L(v_1)$ and so

$$[\mathscr{G}_s(k) : L(v)] \leqslant \dim_k \operatorname{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0), L(-w_0v_1)).$$

Let us assume that $\dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0), L(-w_0v_1)) = 0$ unless $v_0 = -w_0v_1$. Then, for any $v \in \pi_s$, we have

$$\begin{aligned}[\mathscr{G}_s(k) : L(v)] &\leqslant \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0), L(-w_0v_1))\\ &= \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(L(v_0), L(-w_0v_1))\\ &= \dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(L(v), k)\\ &= \dim_k \mathrm{Hom}_G(L(v), \mathscr{G}_s(k)).\end{aligned}$$

Under our assumption it then follows from the Theorem given in 7.2 that $\mathscr{G}_s(k)$ is semisimple. Moreover, all composition factors have highest weight of the form $v = v_0 + p^r v_1 = -w_0v_1 + p^r v_1$ and appear with multiplicity one. An easy weight computation shows that $v = -w_0v_1 + p^r v_1 \in \pi_s$ if and only if $\langle v_1, \alpha_0^\vee\rangle < 2s(h-1)$.

The module $Q_r(v_0)$ is projective over $G(\mathbb{F}_q)$ and $\dim_k \mathrm{Hom}_{G(\mathbb{F}_q)}(Q_r(v_0), L(-w_0v_1))$ equals the multiplicity of the indecomposable projective $G(\mathbb{F}_q)$-module $U_r(-w_0v_1)$ as a summand of $Q_r(v_0)$. It remains to be shown that this multiplicity $[Q_r(v_0), U_r(-w_0v_1)] = 0$ unless $v_0 = -w_0v_1$. As pointed out in the proof of the Lemma given in 7.3, $[Q_r(v_0), U_r(-w_0v_1)] \neq 0$ and $v_0 \neq -w_0v_1$ imply that $p^r - 1 \leqslant \langle v_0, \alpha_0^\vee\rangle + p^r - 1 \leqslant \langle -w_0v_1, \alpha_0^\vee\rangle \leqslant \langle v_1, \alpha_0^\vee\rangle$. But this is impossible because $\langle v_1, \alpha_0^\vee\rangle < 2s(h-1)$. □

We denote by T_γ^δ the translation functor as defined in [11, II. 7].

Corollary. *Let* $s \geqslant 1$, $p \geqslant (2s+1)(h-1)$, *and let* $\lambda \in X_r(T)$ *with* $0 \leqslant i \leqslant s$.
(a) *If* λ *is* p*-regular then*

$$\mathrm{H}^i(G(\mathbb{F}_q), L(\lambda)) \cong \bigoplus_{v \in X[s]} \mathrm{H}^i(G, T_v^0 L(\lambda + p^r v)).$$

(b) *If* λ *is* p*-singular then*

$$\mathrm{H}^i(G(\mathbb{F}_q), L(\lambda)) \cong 0.$$

Proof. From the preceding theorem one can see that all the composition factors of $\mathscr{G}_s(k)$ have p-regular highest weight. Therefore, there can be no non-zero cohomology for simple modules with p-singular highest weight. We may assume that λ is p-regular.

$$\begin{aligned}&\mathrm{H}^i(G(\mathbb{F}_q), L(\lambda))\\ &\quad\cong \mathrm{Ext}^i_{G(\mathbb{F}_q)}(k, L(\lambda))\\ &\quad\cong \mathrm{Ext}^i_{G(\mathbb{F}_q)}(L(-w_0\lambda), k)\\ &\quad\cong \mathrm{Ext}^i_G(L(-w_0\lambda), \mathscr{G}_s(k)) \quad \text{(by the Theorem given in 5.2)}\\ &\quad\cong \bigoplus_{v \in X[s]} \mathrm{Ext}^i_G(L(-w_0\lambda), L((p^r - w_0)v)) \quad \text{(by the preceding Theorem)}\\ &\quad\cong \bigoplus_{v \in X[s]} \mathrm{Ext}^i_G(L(v), L(\lambda + p^r v)) \quad \text{([11, I. 4.4])}\\ &\quad\cong \bigoplus_{v \in X[s]} \mathrm{Ext}^i_G(T_0^v(k), L(\lambda + p^r v))\end{aligned}$$

$$\cong \bigoplus_{\nu\in X[s]} \operatorname{Ext}^i_G(k, T^0_\nu L(\lambda + p^r\nu)) \quad ([11,\ \text{II. } 7.6])$$
$$\cong \bigoplus_{\nu\in X[s]} \mathrm{H}^i(G, T^0_\nu L(\lambda + p^r\nu)). \qquad \square$$

For $i = 1$ we can do slightly better. It follows from [2, Lemma 2.5] that $0 \neq \mathrm{H}^1(G, T^0_\nu L(\lambda + p^r\nu))$ forces $\langle \nu, \alpha_0^\vee\rangle \leqslant h - 1$. Therefore for $p \geqslant 3(h - 1)$ one obtains

$$\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) \cong \bigoplus_{\{\nu\in X(T)_+ \mid \langle \nu,\alpha_0^\vee\rangle \leqslant h-1\}} \mathrm{H}^1(G, T^0_\nu L(\lambda + p^r\nu)).$$

7.5. For large primes we can now prove the existence of a one-to-one correspondence between the ith cohomology for certain simple G-modules and the ith cohomology for the simple $G(\mathbb{F}_q)$-modules. Following [10] we define $\tilde{W}_p$ to be the affine group generated by the ordinary Weyl group and the translations by elements of $pX(T)$. For large p any p-regular weight has trivial stabilizer in $\tilde{W}_p$ [10, Lemma 5].

Theorem. *Let* $s \geqslant 1$, $p \geqslant (4s + 1)(h - 1)$, $\nu \in X_r(T)$ *with* $\langle \nu, \alpha_0^\vee\rangle < 2s(h - 1)$, *and* $u \in \tilde{W}_p$ *such that* $u \cdot \nu$ *is* p^r*-restricted. Then for* $0 \leqslant i \leqslant s$,

$$\mathrm{H}^i(G(\mathbb{F}_q), L(u \cdot \nu)) \cong \mathrm{H}^i(G, L(u \cdot 0 + p^r\nu)).$$

Moreover, all the non-zero ith cohomology for simple modules of the finite group can be obtained in this fashion.

Proof. Assume that $0 \neq \mathrm{H}^i(G(\mathbb{F}_q), L(\lambda))$. By the previous corollary there exists a $\nu \in X(T)_+$ with $\langle \nu, \alpha_0^\vee\rangle < 2s(h-1)$ and $0 \neq \mathrm{H}^i(G, T^0_\nu L(\lambda + p^r\nu))$. Let γ be the highest weight of the simple module $T^0_\nu L(\lambda + p^r\nu)$. By the Linkage Principle there exists a unique $u \in \tilde{W}_p$ such that $\gamma = u \cdot 0 + p^r\nu$. 0 and ν are inside the same alcove. Therefore, $T^\nu_0 T^0_\nu L(\lambda + p^r\nu) = L(\lambda + p^r\nu)$. Now clearly $\lambda = u \cdot \nu$. It remains to be shown that there exists no other weight $\tilde{\nu} \in X(T)_+$ with $\langle \tilde{\nu}, \alpha_0^\vee\rangle < 2s(h-1)$ and $0 \neq \mathrm{H}^i(G, T^0_{\tilde{\nu}} L(\lambda + p^r\tilde{\nu}))$ such that $\lambda = \upsilon \cdot \tilde{\nu}$ for some $\upsilon \in \tilde{W}_p$.

Suppose that $u \cdot \nu = \lambda = \upsilon \cdot \tilde{\nu}$. Then $\tilde{\nu}$ is contained in the $\tilde{W}_p$-orbit of ν. Both ν and $\tilde{\nu}$ are p-regular and inside the lowest alcove. By Jantzen [10, Lemma 1] there exists a unique element w of the Weyl group and a unique weight ρ_w that is the sum of some distinct fundamental weights such that $\tilde{\nu} = w \cdot \nu + p\rho_w$. If $\rho_w \neq 0$, then there exists a simple root β such that $\langle \tilde{\nu}, \beta^\vee\rangle - \langle w(\nu + \rho), \beta^\vee\rangle + 1 = p$. But the left-hand side is less than or equal to $2s(h - 1) - 1 + (2s + 1)(h - 1) - 1 + 1 = (4s + 1)(h - 1) - 1 < p$. Therefore, $\rho_w = 0$. Both ν and $\tilde{\nu}$ are p-regular dominant weights. Thus, w is the identity. Hence, $u = \upsilon$ and $\nu = \tilde{\nu}$. $\square$

For groups whose root lattice is identical to the weight lattice the condition on p can be weakened to $p \geqslant (2s + 1)(h - 1)$. The following example illustrates how one can use the preceding Theorem to compute the $G(\mathbb{F}_q)$-cohomology via the G-cohomology.

Example. Let G be of type G_2 and $q = p > 13$. The simple roots are denoted by α_1 and α_2, where α_1 is short. Let ω_1 and ω_2 denote the corresponding fundamental weights. Write s_β for the reflection belonging to a positive root β. A pair (r,s) will denote the weight $r\omega_1 + s\omega_2$.

It follows from the remark after the Corollary given in 7.4 that $\mathrm{H}^1(G, L(u\cdot 0 + pv)) \neq 0$ is only possible for $\langle v, \alpha_0^\vee\rangle \leqslant h - 1$. This forces all weights $u \cdot 0 + pv$ that are of interest to be p-bounded. From [16, Table 1] we obtain that $\mathrm{H}^1(G, L(u \cdot 0 + pv)) = 0$ for all p-bounded dominant weights $u \cdot 0 + pv$, unless $u \cdot 0 + pv$ is one of the following list:

$$\{(p-5,0),(3,p-2),(p+4,p-4),(3,2p-2),(2p-2,1)\}.$$

For these weights $\dim_k \mathrm{H}^1(G, L(u \cdot 0 + pv)) = 1$.

The first two weights in the list are p-restricted and will appear in the corresponding list of $G(\mathbb{F}_p)$ unchanged. The first non-restricted weight is

$$(p+4,p-4) = p(1,0) + (4,p-4) = p(1,0) + (s_{\alpha_1+\alpha_2}\cdot(0,0) + p(0,1)).$$

Here $u \in \tilde{W}_p$ consists of the reflection $s_{\alpha_1+\alpha_2}$ followed by the translation by $p(0,1)$ and $v = (1,0)$. Under the preceding Theorem the weight corresponding to $(p+4, p-4) = u\cdot(0,0) + p(1,0)$ is

$$u\cdot(1,0) = s_{\alpha_1+\alpha_2}\cdot(1,0) + p(0,1) = (6,p-5).$$

Similarly, one obtains from

$$(3,2p-2) = p(0,1) + ((s_{\alpha_2}\cdot(0,0) + p(0,1)),$$

the corresponding weight

$$(6,p-3) = s_{\alpha_2}\cdot(0,1) + p(0,1)$$

and from

$$(2p-2,1) = p(1,0) + (s_{\alpha_1}\cdot(0,0) + p(1,0)),$$

the corresponding weight

$$(p-3,2) = s_{\alpha_1}\cdot(1,0) + p(1,0).$$

We conclude that $\mathrm{H}^1(G(\mathbb{F}_p), L(\lambda)) = 0$ for all p-restricted weights λ, unless λ is one of the following list:

$$\{(p-5,0),(3,p-2),(6,p-5),(6,p-3),(p-3,2)\}.$$

For these weights $\dim_k \mathrm{H}^1(G(\mathbb{F}_p), L(\lambda)) = 1$.

For a complete list of Ext^1 between simple modules for type G_2 and $q = p \geqslant 13$ see [20].

7.6. The following theorem will give a sufficient condition for the semisimplicity of $\mathcal{G}_s(L(\mu))$. As a consequence we are able to prove a stronger version of

[2, Theorem 3.2]. Following Andersen, we call a weight, μ, m-deep if it has distance at least m from the nearest alcove wall (i.e., if p divides $\langle \mu+\rho, \alpha_0^\vee\rangle + c$ for some $c \in \mathbb{Z}$, then $|c| \geqslant m$).

Theorem. *Let* $s \geqslant 1$, $p \geqslant (2s+1)(h-1)$, *and let* $\mu \in X_r(T)$ *be* $2s(h-1)$*-deep. Then* $\mathcal{G}_s(L(\mu))$ *is semisimple. Moreover, for any* $L \in \mathcal{C}_s$ *that has only* p^r*-restricted composition factors in its head* (*as G-module*), *one obtains for* $0 \leqslant i \leqslant s$ *the following isomorphism of vector spaces*:

$$\operatorname{Ext}^i_{G(\mathbb{F}_q)}(L, L(\mu)) \cong \bigoplus_{\nu\in\pi_s} \operatorname{Hom}_G(L(\nu_0)\otimes L(\nu_1), L(\mu)) \otimes \operatorname{Ext}^i_G(L, L(\nu)).$$

Proof. We will show that $\mathcal{G}_s(L(\mu))$ is semisimple. Arguing as in the proof of the Theorem given in 7.4 we have for any weight $\nu = \nu_0 + p^r\nu_1 \in \pi_s$:

$$[\mathcal{G}_s(L(\mu)) : L(\nu)] \leqslant \dim_k \operatorname{Hom}_{G(\mathbb{F}_q)}(Q_r(\nu_0), L(\mu)\otimes L(-w_0\nu_1)).$$

$\nu \in \pi_s$ and μ has distance at least $2s(h-1)$ from the closest alcove wall. Hence, the tensor product $L(\mu)\otimes L(-w_0\nu_1)$ is semisimple (both as a G-module and a $G(\mathbb{F}_q)$-module) and all summands have p^r-restricted highest weight. Let $[Q_r(\nu_0) : U_r(\gamma)]$ denote the multiplicity of the projective indecomposable $G(\mathbb{F}_q)$-module $U_r(\gamma)$ as a summand of $Q_r(\nu_0)$. Then

$$[\mathcal{G}_s(L(\mu)) : L(\nu)] \leqslant \sum_{\gamma\in X(T)_+} [Q_r(\nu_0) : U_r(\gamma)] \cdot \dim_k \operatorname{Hom}_G(L(\gamma), L(\mu)\otimes L(-w_0\nu_1)).$$

On the other hand, the multiplicity of $L(\nu)$ in the G-socle of $\mathcal{G}_s(L(\mu))$ is given by

$$\dim_k \operatorname{Hom}_{G(\mathbb{F}_q)}(L(\nu_0), L(\mu)\otimes L(-w_0\nu_1)) = \dim_k \operatorname{Hom}_G(L(\nu_0), L(\mu)\otimes L(-w_0\nu_1)).$$

Therefore it is sufficient to prove the following:

$$[Q_r(\nu_0) : U_r(\gamma)] \cdot \dim_k \operatorname{Hom}_G(L(\gamma), L(\mu)\otimes L(-w_0\nu_1)) \text{ is zero unless } \gamma = \nu_0.$$

Let us assume that there exists $\gamma \neq \nu_0$ with

$$[Q_r(\nu_0) : U_r(\gamma)] \cdot \dim_k \operatorname{Hom}_G(L(\gamma), L(\mu)\otimes L(-w_0\nu_1)) \neq 0.$$

By Jantzen [13] or Chastkofsky [4] there exists a dominant weight σ with $\langle \sigma, \alpha_0^\vee\rangle \leqslant h-1$ such that $[L(\gamma)\otimes L(\sigma) : L(\nu_0 + p^r\sigma)]_G \neq 0$. One obtains immediately:

$$\langle \gamma+\rho, \alpha_0^\vee\rangle \geqslant \langle \gamma+\sigma, \alpha_0^\vee\rangle \geqslant p^r\langle \sigma, \alpha_0^\vee\rangle. \tag{7.1}$$

At the same time

$$\begin{aligned}&[L(\gamma)\otimes L(\sigma) : L(\nu_0 + p^r\sigma)]_G\\ &\quad = \dim_k \operatorname{Hom}_G(L(\gamma)\otimes L(\sigma), Q_r(\nu_0)\otimes L(\sigma)^{(r)})\\ &\quad = \dim_k \operatorname{Hom}_G(L(\nu_0 + p^r(-w_0\sigma)), Q_r(\nu_0)\otimes L(-w_0\sigma)).\end{aligned}$$

Here we were using the fact that $Q_r(v_0) \otimes L(\sigma)^{(r)}$ is the injective hull of $L(v_0 + p^r\sigma)$ in the category $\mathscr{C}_s$ (see the remark after the Corollary given in 4.7). Should v_0 have distance greater than $\langle \sigma, \alpha_0^\vee \rangle$ from the nearest alcove wall then an easy dimension argument shows that the G-socle of $Q_r(v_0) \otimes L(-w_0\sigma)$ contains only p^r-restricted weights (see [18, Lemma 5.1(b)]). This is impossible. Thus, the weights v_0 and v have distance at most $\langle \sigma, \alpha_0^\vee \rangle$ from the closest alcove wall. Our assumption implies that $L(v)$ is a composition factor of $\mathscr{G}_s(L(\mu))$. Its linkage class has to be represented in the socle of $\mathscr{G}_s(L(\mu))$. All weights in a linkage class have the same distance from the closest wall. Therefore, there exists a weight $\kappa = \kappa_0 + p^r\kappa_1$ in π_s with $\dim_k \operatorname{Hom}_G(L(\kappa_0), L(\mu) \otimes L(-w_0\kappa_1)) \neq 0$. κ_0 has distance at most $\langle \sigma, \alpha_0^\vee \rangle$ from the nearest alcove wall while μ has distance at least $2s(h-1)$ from the closest wall. Therefore,

$$\langle \kappa_1, \alpha_0^\vee \rangle \geqslant 2s(h-1) - \langle \sigma, \alpha_0^\vee \rangle. \tag{7.2}$$

Of course, the fact $\kappa \in \pi_s$ gives the upper bound $\langle \kappa_1, \alpha_0^\vee \rangle < 2s(h-1)$. The distance of μ from the alcove walls forces the weights μ, γ, and κ_0 to be p-regular and lie inside the same alcove. Therefore, there exists an integer b with

$$(b-1)p^r < \langle \xi + \rho, \alpha_0^\vee \rangle < bp^r \quad \text{for all } \xi \in \{\mu, \gamma, \kappa_0\}.$$

Now (7.1) implies that $\langle \sigma, \alpha_0^\vee \rangle < b$. Together with (7.2) one obtains that $\langle \kappa_1, \alpha_0^\vee \rangle > 2s(h-1) - b$, or equivalently $\langle \kappa_1, \alpha_0^\vee \rangle \geqslant 2s(h-1) - b + 1$.

But now

$$\begin{aligned} &p^r(b-1) + p^r(2s(h-1) - b + 1) \\ &\quad < \langle \kappa_0 + \rho + p^r\kappa_1, \alpha_0^\vee \rangle = \langle \kappa + \rho, \alpha_0^\vee \rangle < 2sp^r(h-1) \end{aligned}$$

gives the desired contradiction. □

7.7. By the tensor identity we have $\operatorname{ind}_{G(\mathbb{F}_q)}^G L(\mu) \cong L(\mu) \otimes \operatorname{ind}_{G(\mathbb{F}_q)}^G k$. The truncation of $L(\mu) \otimes \mathscr{G}_s(k)$ in $\mathscr{C}_s$ is therefore contained in $\mathscr{G}_s(L(\mu))$. For $\mathscr{G}_s(L(\mu))$ to be semisimple it is therefore necessary that the truncation of $L(\mu) \otimes \mathscr{G}_s(k)$ be semisimple. For $p \geqslant (2s+1)(h-1)$, it follows from the Theorem given in 7.4 that

$$\mathscr{G}_s(k) \cong \bigoplus_{v \in X[s]} L((p^r - w_0)v).$$

The truncation of $L(\mu) \otimes \mathscr{G}_s(k)$ will therefore contain the tensor products of the form $L(\mu) \otimes L(-w_0 v)$ where $\mu + p^r v \in \pi_s$. Define b such that $(b-1)p^r < \langle \mu + \rho, \alpha_0^\vee \rangle < bp^r$. It becomes necessary for μ to have distance at least $2s(h-1) - b$ from any upper alcove wall. Otherwise, the tensor product would result in a translation across an upper wall and destroy the semisimplicity of $\mathscr{G}_s(L(\mu))$.

One can actually show that if $\mathscr{G}_s(L(\mu))$ satisfies the conditions of the previous theorem then $\mathscr{G}_s(L(\mu))$ is a submodule of the semisimple module $L(\mu) \otimes \mathscr{G}_s(k)$.

Example. Let $G = \mathsf{SL}(2,k)$, $G(\mathbb{F}_q) = \mathsf{SL}(2,q)$ and $p \geqslant 3$. Then

(a) $\mathscr{G}_1(k) \cong k \oplus L(q+1)$.

(b) If $\mu < q-2$, then $\mathscr{G}_1(L(\mu)) \cong L(\mu) \otimes \mathscr{G}_1(k) \cong L(\mu) \oplus (L(\mu) \otimes L(q+1))$.

(c) $\mathscr{G}_1(L(q-2)) \cong L(q-2) \oplus L(2q-3)$.

(d) $\mathscr{G}_1(\mathrm{St}_r) \cong \mathrm{St}_r \oplus H^0(2q-2)$.

Notice that for $\mathsf{SL}(2,k)$ the module $\mathscr{G}_1(L(\mu))$ is semisimple if and only if μ is p-regular.

Acknowledgements

We would like to acknowledge Jim Humphreys for useful comments and suggestions on an earlier version of this paper. The first author and the third author thank the Department of Mathematics and Statistics at Utah State University for its hospitality during the preparation of this paper. The third author would like to thank the Mathematics Department of the University of Oregon where the author spent his sabbatical while this paper was written.

References

[1] H.H. Andersen, Extensions of modules for algebraic groups, Amer. J. Math. 106 (1984) 498–504.
[2] H.H. Andersen, Extensions of simple modules for finite Chevalley groups, J. Algebra 111 (1987) 388–403.
[3] C.P. Bendel, D.K. Nakano, Complexes and vanishing of cohomology for group schemes, J. Algebra 214 (1999) 668–713.
[4] L. Chastkofsky, Characters of projective indecomposable modules for finite Chevalley groups, Proc. Symp. Pure Math. 37 (1980) 359–362.
[5] E.T. Cline, B.J. Parshall, L.L. Scott, W. van der Kallen, Rational and generic cohomology, Invent. Math. 39 (1977) 143–163.
[6] S. Donkin, On Schur algebras and related algebras I, J. Algebra 104 (1986) 310–328.
[7] S.R. Doty, K. Erdmann, D.K. Nakano, Extensions of modules over Schur algebras, symmetric groups and Hecke algebras, preprint, 1998.
[8] S.R. Doty, D.K. Nakano, Relating the cohomology of the general linear group and symmetric groups, Proceedings of the Conference on Modular Representation Theory, 1998, to appear.
[9] J.E. Humphreys, Non-zero Ext^1 for Chevalley groups (via algebraic groups), J. London Math. Soc. (2), 31 (1985) 463–467.
[10] J.C. Jantzen, Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren, J. Algebra 49 (1977) 441–469.
[11] J.C. Jantzen, Representations of Algebraic Groups, Academic Press, Orlando, 1987.
[12] J.C. Jantzen, Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne, J. Reine Angew. Math. 317 (1980) 157–199.
[13] J.C. Jantzen, Zur Reduktion modulo p der Charaktere von Deligne und Lusztig, J. Algebra 70 (1981) 452–474.
[14] A.S. Kleshchev, D.K. Nakano, On comparing the cohomology of general linear groups and symmetric groups, Pacific J. Math., to appear.
[15] Z. Lin, D.K. Nakano, Complexity for modules over finite Chevalley groups and Lie algebras, Invent. Math. 138 (1999) 85–101.
[16] J. Liu, J. Ye, Extensions of simple modules for the algebraic group of type G_2, Comm. Algebra 21 (6) (1993) 1909–1946.

[17] D.K. Nakano, Some recent developments in the representation theory of general linear and symmetric groups, in: Representations and Quantizations, Proceedings of the International Conference on Representation Theory, Shanghai, China, Springer, Berlin, 2000, pp. 357–374.

[18] C. Pillen, Reduction modulo p of some Deligne–Lusztig characters, Arch. Math. 61 (1993) 421–433.

[19] C. Pillen, Loewy series for principal series representations of finite Chevalley groups, J. Algebra 189 (1997) 101–124.

[20] J. Ye, Extensions of simple modules for $G_2(p)$, Comm. Algebra 22 (8) (1994) 2771–2802.