

Self-extensions for finite symplectic groups via algebraic groups

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Dedicated to James E. Humphreys on the occasion of his 65th birthday

ABSTRACT. For large primes it was proved in [BNP1, BNP3] that a finite group of Lie type does not admit self-extensions, i.e. non-trivial extensions of a simple module with itself, unless the group is one of the symplectic groups $Sp_{2n}(\mathbb{F}_p)$, $n \geq 1$. In this paper it is shown that self-extensions indeed exist for these groups for all ranks and odd primes. The method of proof is based on ideas due to James Humphreys and Henning Andersen. Some of the results in this paper assume the Lusztig Conjecture.

1. Introduction

1.1. Let G be a connected simply connected almost simple algebraic group defined and split over the field \mathbb{F}_p with p elements, and k be the algebraic closure of \mathbb{F}_p . Let $G(\mathbb{F}_q)$ be the finite Chevalley group consisting of the \mathbb{F}_q -rational points of G where $q = p^r$ for a positive integer r . Moreover, let G_r be the r th Frobenius kernel.

In [Hum1] J.E. Humphreys constructed examples of self-extensions, i.e. non-trivial extensions of a simple module with itself, for the finite symplectic groups $Sp_4(\mathbb{F}_p)$ with p odd. In the same paper Humphreys conjectured that the root systems of type C_n , $n \geq 1$, might be exceptional for the existence of self-extensions. Humphreys' conjecture was motivated by a theorem of H.H. Andersen [And1] that says that Frobenius kernels do not admit self-extensions of simple modules unless the underlying root system is of type C_n ($n \geq 1$) and the prime is two. It is well-known that the algebraic group does not admit self-extensions.

In [BNP2] the following generalization of Andersen's result was given: Given a pair of simple G -modules with p -restricted weights λ and μ that are "close", i.e. $\langle \lambda - \mu, \alpha^\vee \rangle < p/3$ for any root α , then $\text{Ext}_{G_1}^1(L(\lambda), L(\mu)) = \text{Ext}_G^1(L(\lambda), L(\mu))$, unless G is of type C_n . However, the existence of "close" pairs of weights with $\text{Ext}_{G_1}^1(L(\lambda), L(\mu)) \neq \text{Ext}_G^1(L(\lambda), L(\mu))$ for type C_n has not been established for $n > 2$.

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In [Hum1] such pairs of weights also appear in the construction of self-extensions for $Sp_4(\mathbb{F}_p)$. The results in [BNP1, BNP3] imply that such pairs of weights with non-vanishing G_1 -extensions are necessary in order to find self-extensions for finite groups of Lie type. In particular, for large primes it was shown that self-extensions can only exist for the finite symplectic groups $Sp_{2n}(\mathbb{F}_p)$, thus confirming Humphreys' conjecture. The purpose of this paper is to prove the existence of self-extensions for symplectic groups of arbitrary rank.

The construction involves extensions between certain pairs of simple modules for an algebraic group G of type C_n of arbitrary rank. One of these modules has restricted highest weight while the other one is non-restricted. The restrictions of these extensions to the finite group $G(\mathbb{F}_p)$ then contain the desired self-extensions as submodules, while the restrictions to the first Frobenius kernels result in the aforementioned "close" G_1 -extensions (Proposition 2.4 and Corollary 5.1).

One family of self-extensions described in this paper (Corollary 4.3(A)) was discovered by Tiep and Zalesskii in [TZ] via reduction modulo p of certain ordinary representations for $Sp_{2n}(\mathbb{F}_p)$. Producing these extensions via the algebraic group $Sp_{2n}(k)$ yields additional new examples of self-extensions for $Sp_{2n}(\mathbb{F}_p)$, especially for large primes (Corollary 4.3(B)).

The nicest and most comprehensive results are obtained when assuming the Lusztig Conjecture. In Section 5 it is shown that all simple $Sp_{2n}(\mathbb{F}_p)$ -modules whose highest weights are p -regular and adjacent to the hyperplane $H_{\alpha_n, p/2} = \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha_n^\vee \rangle = p/2\}$ admit self-extensions. Recall that Andersen's theorem [And1] says that $\text{Ext}_{G_1}(L(\lambda), L(\lambda)) = 0$, unless $p = 2$, G of type C_n , and $\lambda \in H_{\alpha_n, p/2}$.

1.2. Notation: G will always denote a connected simply connected almost simple algebraic group that is defined and split over the field \mathbb{F}_p with p elements. k denotes the algebraic closure of \mathbb{F}_p and G_r is the r th Frobenius kernel of G . The conventions in the paper will follow the ones used in [Jan1]. Let T be a maximal torus in G and Φ the associated root system. The positive roots are denoted by Φ^+ and the negative roots by Φ^- . Let B be a Borel subgroup containing T and corresponding to the negative roots. $X(T)$ denotes the weight lattice, $X(T)_+$ the dominant weights, and $X_r(T)$ the p^r -restricted weights. For a weight $\gamma \in X(T)_+$, $H^0(\gamma)$, $V(\gamma)$, and $L(\gamma)$ denote the induced module, the Weyl module, and the simple module, respectively.

With the exception of Sections 2.1 and 3.1 we will always assume that G is of type C_n . We follow [Bou, p.254] and denote the short simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i < n$, while $\alpha_n = 2\epsilon_n$ is the unique long simple root. The fundamental weights are $\omega_i = \sum_{k=1}^i \epsilon_k$, with ω_1 being the unique minuscule weight. For convenience we will frequently switch between the $\{\epsilon_i\}$, $\{\omega_i\}$, and $\{\alpha_i\}$ bases. The highest short root is $\alpha_0 = \epsilon_1 + \epsilon_2$ and the longest element of the Weyl group W is -1 . The simple modules are therefore self-dual and $H^0(\gamma)$ and $V(\gamma)$ are dual to each other.

The following orders appear: $\lambda \leq \mu$, if $\mu - \lambda$ is a sum of positive roots, $\mu \leq_{\mathbb{Q}} \lambda$, if $\lambda - \mu$ is a linear combination of positive roots with non-negative rational coefficients, and $\lambda \uparrow \mu$ if there exist sequences of weights $\mu_1, \mu_2, \dots, \mu_m$ and reflections s_1, s_2, \dots, s_{m+1} such that $\lambda \leq s_1 \cdot \lambda = \mu_1 \leq s_2 \cdot \mu_1 = \mu_2 \leq \dots \leq s_m \cdot \mu_{m-1} = \mu_m \leq s_{m+1} \cdot \mu_m = \mu$. (see [Jan1, II.6.4.(1)]).

2. Special G -extensions for algebraic groups of type C_n

It is well-known that $\text{Ext}_G^1(V(\mu), H^0(\lambda)) = 0$ for any pair of weights λ, μ . It follows from [And1] that $\text{Ext}_{G_1}^1(V(\lambda), H^0(\lambda)) = 0$ unless G is of type C_n , $p = 2$, and λ is contained in the hyperplane $H_{\alpha_n, p/2} = \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha_n^\vee \rangle = \frac{p}{2}\}$. In [BNP2, Prop. 5.2(a)] a generalization of Andersen's result for odd primes was found. It was shown that $\text{Ext}_{G_1}^1(V(\mu), H^0(\lambda))$ vanishes for a pair of restricted weights λ and μ that are "close", i.e. $\langle \mu - \lambda, \alpha^\vee \rangle < p/3$ for any root α , unless G is of type C_n and the weights are reflections of each other across the hyperplane $H_{\alpha_n, p/2}$. In this section we show that extensions for such "close" pairs of weights indeed exist for type C_n .

2.1. The following lemma is well-known. It is included for the benefit of the reader. We will make repeated use of it in later arguments.

LEMMA . *Let $i > 0$ be a positive integer, α a simple root, V a rational G -module, and $\gamma \in X(T)$ with $-p \leq \langle \gamma, \alpha^\vee \rangle \leq -1$. Then*

$$\text{Ext}_B^i(V, \gamma) \cong \begin{cases} 0 & \text{if } \langle \gamma, \alpha^\vee \rangle = -1 \\ \text{Ext}_B^{i-1}(V, s_\alpha \cdot \gamma) & \text{else.} \end{cases}$$

PROOF. Here $P(\alpha)$ denotes the parabolic subgroup corresponding to the root α . We apply the spectral sequence [Jan1, I.4.5]

$$\text{Ext}_{P(\alpha)}^i(V, R^j \text{ind}_B^{P(\alpha)} \gamma) \Rightarrow \text{Ext}_B^{i+j}(V, \gamma).$$

If $\langle \gamma, \alpha^\vee \rangle = -1$ then $R^j \text{ind}_B^{P(\alpha)} \gamma = 0$ for all $j \geq 0$ [Jan1, II.5.2(b)], which forces $\text{Ext}_B^i(V, \gamma) = 0$ for all $i > 0$.

Otherwise it follows from [Jan1, II.5.2(d)] that

$$\text{Ext}_B^i(V, \gamma) \cong \text{Ext}_{P(\alpha)}^{i-1}(V, R^1 \text{ind}_B^{P(\alpha)} \gamma).$$

Now $-p \leq \langle \gamma, \alpha^\vee \rangle \leq -2$ implies that $0 \leq \langle s_\alpha \cdot \gamma, \alpha^\vee \rangle \leq p-2$. It follows from [Jan1, II.5.3(b)] that $R^1 \text{ind}_B^{P(\alpha)} \gamma \cong \text{ind}_B^{P(\alpha)}(s_\alpha \cdot \gamma)$. Finally, [Jan1, II.4.7(1)] yields

$$\text{Ext}_B^i(V, \gamma) \cong \text{Ext}_{P(\alpha)}^{i-1}(V, \text{ind}_B^{P(\alpha)}(s_\alpha \cdot \gamma)) \cong \text{Ext}_B^{i-1}(V, s_\alpha \cdot \gamma).$$

□

2.2. The G -module $L(\omega_1) \cong H^0(\omega_1)$ is multiplicity free with dimension $2n$. The weights are expressed most conveniently in the form $\pm \epsilon_i$ with $i = 1, \dots, n$.

LEMMA . *Let G be of type C_n , p odd, and $\gamma \in X_1(T)$ such that $\langle \gamma, \alpha_n^\vee \rangle = (p-2 \pm 1)/2$ and $s_{\alpha_n} \cdot (\gamma - p\epsilon_n) \in X_1(T)$. Then the following hold:*

(a) $\text{Ext}_B^1(V(s_{\alpha_n} \cdot (\gamma - p\epsilon_n)), \gamma - p\epsilon_i)$ is isomorphic to

$$\text{Ext}_B^1(L(s_{\alpha_n} \cdot (\gamma - p\epsilon_n)), \gamma - p\epsilon_i) \cong \begin{cases} k & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

(b) If $i < n$ and $\mu \in X(T)_+$ with $\mu \leq \gamma$, then

$$\text{Ext}_B^1(L(\mu), \gamma - p\epsilon_i) \cong \begin{cases} k & \text{if } \langle \gamma, \alpha_i^\vee \rangle \leq p-2, p-1 \leq \langle \gamma, \alpha_i^\vee \rangle + \langle \gamma, \alpha_{i+1}^\vee \rangle, \\ & \text{and } \mu = s_{\alpha_i} \cdot (\gamma - p\epsilon_i), \\ 0 & \text{else.} \end{cases}$$

(c) If $i < n$, then $\text{Ext}_B^2(V(s_{\alpha_n} \cdot (\gamma - p\epsilon_n)), \gamma - p\epsilon_i) = 0$.

$$\begin{aligned}
& \text{(d) If } i < n, \text{ then} \\
& \text{Ext}_B^2(L(s_{\alpha_n} \cdot (\gamma - p\epsilon_n)), \gamma - p\epsilon_i) \\
& \cong \begin{cases} \text{Ext}_G^1(L(s_{\alpha_n} \cdot (\gamma - p\epsilon_n)), H^0(s_{\alpha_i} \cdot (\gamma - p\epsilon_i))) & \text{if } \langle \gamma, \alpha_i^\vee \rangle \leq p-2, \text{ and} \\ & p-1 \leq \langle \gamma, \alpha_i^\vee \rangle + \langle \gamma, \alpha_{i+1}^\vee \rangle, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

PROOF. Let V be a homomorphic image of $V(\mu)$ with $\mu \in X(T)_+$ and $\mu \leq \gamma$. Since γ is restricted one has $-p \leq \langle \gamma - p\epsilon_i, \alpha_i^\vee \rangle \leq -1$ and it follows from Lemma 2.1 that

$$\text{Ext}_B^1(V, \gamma - p\epsilon_i) \cong \begin{cases} 0 & \text{if } \langle \gamma, \alpha_i^\vee \rangle = p-1 \\ \text{Hom}_B(V, s_{\alpha_i} \cdot (\gamma - p\epsilon_i)) & \text{else.} \end{cases}$$

Since V is a highest weight module, the homomorphism group vanishes unless $\mu = s_{\alpha_i} \cdot (\gamma - p\epsilon_i)$ and μ is dominant. Part (a) follows.

Next, assume $i < n$ and $\mu = s_{\alpha_i} \cdot (\gamma - p\epsilon_i)$. Then $\langle s_{\alpha_i} \cdot (\gamma - p\epsilon_i), \alpha_j^\vee \rangle = \langle \gamma - p\epsilon_i + (p-1 - \langle \gamma, \alpha_i^\vee \rangle) \alpha_i, \alpha_j^\vee \rangle \geq 0$ for all $j \neq i+1$, while $\langle s_{\alpha_i} \cdot (\gamma - p\epsilon_i), \alpha_{i+1}^\vee \rangle = \langle \gamma - p\epsilon_i + (p-1 - \langle \gamma, \alpha_i^\vee \rangle) \alpha_i, \alpha_{i+1}^\vee \rangle = \langle \gamma, \alpha_i^\vee \rangle + \langle \gamma, \alpha_{i+1}^\vee \rangle + 1 - p$. If $\langle \gamma, \alpha_i^\vee \rangle + \langle \gamma, \alpha_{i+1}^\vee \rangle < p-1$ then $s_{\alpha_i} \cdot (\gamma - p\epsilon_i)$ fails to be dominant and the expression vanishes. This implies part (b).

From Lemma 2.1 one obtains

$$\text{Ext}_B^2(V, \gamma - p\epsilon_i) \cong \begin{cases} 0 & \text{if } \langle \gamma, \alpha_i^\vee \rangle = p-1 \\ \text{Ext}_B^1(V, s_{\alpha_i} \cdot (\gamma - p\epsilon_i)) & \text{else.} \end{cases}$$

We have seen that $s_{\alpha_i} \cdot (\gamma - p\epsilon_i)$ is dominant if and only if $\langle \gamma, \alpha_i^\vee \rangle + \langle \gamma, \alpha_{i+1}^\vee \rangle \geq p-1$. In this case one obtains $\text{Ext}_B^1(V, s_{\alpha_i} \cdot (\gamma - p\epsilon_i)) \cong \text{Ext}_B^1(V, H^0(s_{\alpha_i} \cdot (\gamma - p\epsilon_i)))$. If $V = V(s_{\alpha_n} \cdot (\gamma - p\epsilon_n))$ the expression vanishes due to [Jan1, II.4.13].

Finally assume $\langle \gamma, \alpha_i^\vee \rangle + \langle \gamma, \alpha_{i+1}^\vee \rangle < p-1$ and $\mu = s_{\alpha_n} \cdot (\gamma - p\epsilon_n)$. Then Lemma 2.1 yields

$$\begin{aligned}
\text{Ext}_B^1(V, s_{\alpha_i} \cdot (\gamma - p\epsilon_i)) & \cong \text{Hom}_B(V, s_{\alpha_{i+1}} s_{\alpha_i} \cdot (\gamma - p\epsilon_i)) \\
& \cong \begin{cases} k & \text{if } s_{\alpha_n} \cdot (\gamma - p\epsilon_n) = s_{\alpha_{i+1}} s_{\alpha_i} \cdot (\gamma - p\epsilon_i) \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Recall that $s_{\alpha_n} \cdot (\gamma - p\epsilon_n) = \gamma \pm \epsilon_n$. If $s_{\alpha_n} \cdot (\gamma - p\epsilon_n) = s_{\alpha_{i+1}} s_{\alpha_i} \cdot (\gamma - p\epsilon_i)$ then $s_{\alpha_{i+1}} \cdot (\gamma \pm \epsilon_n) = s_{\alpha_i} \cdot (\gamma - p\epsilon_i)$. This implies that $\gamma \pm \epsilon_n - ((\langle \gamma \pm \epsilon_n, \alpha_{i+1}^\vee \rangle + 1) \alpha_{i+1} + \gamma - p\epsilon_i + (p-1 - \langle \gamma, \alpha_i^\vee \rangle) \alpha_i)$, which forces

$$(p-1 - \langle \gamma, \alpha_i^\vee \rangle) \alpha_i + ((\langle \gamma \pm \epsilon_n, \alpha_{i+1}^\vee \rangle + 1) \alpha_{i+1} + p\epsilon_i \pm \epsilon_n) = p \left(\sum_{k=i}^{n-1} \alpha_k \right) + \frac{p \pm 1}{2} \alpha_n$$

Comparing the coefficients for α_i shows that this is impossible. Parts (c) and (d) follow. \square

2.3. Here it is shown that exceptions to [BNP2, Prop. 5.2] indeed exist for groups of type C_n .

PROPOSITION . *Let G be of type C_n , p odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$, then*

$$\text{(a) } \text{Ext}_G^1(V(\lambda - \frac{1}{2} \alpha_n), H^0(\lambda) \otimes L(\omega_1)^{(1)}) \cong k,$$

- (b) If $\text{Ext}_B^2(L(\lambda - \frac{1}{2}\alpha_n), \lambda - p\epsilon_i) = 0$ for all $i < n$, then
 $\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)^{(1)}) \cong k$.

PROOF. Let $V \in \{V(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n)\}$. The G -module $L(\omega_1)$ is multiplicity free with weight spaces $\pm\epsilon_i$, $i = 1, \dots, n$. Let S denote the B -submodule consisting of the weight spaces $-\epsilon_1, \dots, -\epsilon_n$, R be the B -submodule consisting of the weight spaces $-\epsilon_1, \dots, -\epsilon_{n-1}$, and Q the B -quotient $L(\omega_1)/S$. The weight spaces of Q are $\epsilon_1, \dots, \epsilon_n$.

The short exact sequence $0 \rightarrow S \rightarrow L(\omega_1) \rightarrow Q \rightarrow 0$ yields the exact sequence

$$\begin{aligned} \text{Hom}_B(V, \lambda \otimes Q^{(1)}) &\rightarrow \text{Ext}_B^1(V, \lambda \otimes S^{(1)}) \\ &\rightarrow \text{Ext}_B^1(V, \lambda \otimes L(\omega_1)^{(1)}) \rightarrow \text{Ext}_B^1(V, \lambda \otimes Q^{(1)}). \end{aligned}$$

The first term in this sequence is zero because $\lambda - \frac{1}{2}\alpha_n$ is not a weight of $\lambda \otimes Q^{(1)}$. The last term vanishes because the height of a weight of $\lambda \otimes Q^{(1)}$ is greater than the height of a weight of V [Jan1, II.4.10(b)]. Therefore

$$\text{Ext}_B^1(V, \lambda \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_B^1(V, \lambda \otimes S^{(1)}).$$

Next we use the short exact sequence $0 \rightarrow R \rightarrow S \rightarrow -\epsilon_n \rightarrow 0$ to obtain

$$\begin{aligned} \text{Ext}_B^1(V, \lambda \otimes R^{(1)}) &\rightarrow \text{Ext}_B^1(V, \lambda \otimes S^{(1)}) \\ &\rightarrow \text{Ext}_B^1(V, \lambda - p\epsilon_n) \rightarrow \text{Ext}_B^2(V, \lambda \otimes R^{(1)}) \end{aligned}$$

Recall that $s_{\alpha_n} \cdot (\lambda - p\epsilon_n) = \lambda - \epsilon_n = \lambda - \frac{1}{2}\alpha_n$. Lemma 2.2(a) implies that the first term in the above sequence is zero and the last term vanishes by Lemma 2.2(c) and the assumption in part (b). One concludes from Lemma 2.2(a) and [Jan1, II.4.7(1)] that

$$\text{Ext}_G^1(V, H^0(\lambda) \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_B^1(V, \lambda \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_B^1(V, \lambda - p\epsilon_n) \cong k. \quad \square$$

COROLLARY . Let G be of type C_n , p odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$, then $\text{Hom}_G(\text{Ext}_{G_1}^1(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda)), L(\omega_1)^{(1)}) \cong k$.

PROOF. Consider the Lyndon-Hochschild-Serre spectral sequence

$$\begin{aligned} E_2^{i,j} &= \text{Ext}_{G/G_1}^i(\text{Ext}_{G_1}^j(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda)), L(\omega_1)^{(1)}) \\ &\Rightarrow \text{Ext}_G^{i+j}(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)^{(1)}). \end{aligned}$$

Since $\text{Hom}_{G_1}(L(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda)) = 0$, we have $E_2^{1,0} = E_2^{2,0} = 0$ and from the corresponding five-term sequence $E^1 \cong E_2^{0,1}$. \square

2.4. We establish the existence of certain non-trivial G -extensions between simple G -modules, one with restricted highest weight and the other non-restricted. These will later yield the self-extensions for the finite symplectic groups.

PROPOSITION . Let G be of type C_n , p odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$. In addition, assume that

- (i) $\langle \lambda, \alpha_i^\vee \rangle + \langle \lambda, \alpha_{i+1}^\vee \rangle < p-1$, for $1 \leq i \leq n-1$,
- (ii) $H^0(\lambda)$ and $H^0(\lambda - \frac{1}{2}\alpha_n)$ have only p -restricted composition factors,

then $\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda) \otimes L(\omega_1)^{(1)}) \cong k$.

PROOF. It follows from Lemma 2.2(d) and Proposition 2.3(b) that $\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)^{(1)}) \cong k$. Therefore there exists a composition factor $L(\mu)$ of $H^0(\lambda)$ such that $\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\mu) \otimes L(\omega_1)^{(1)}) \neq 0$. We make use of the short exact sequence $0 \rightarrow R \rightarrow V(\lambda - \frac{1}{2}\alpha_n) \rightarrow L(\lambda - \frac{1}{2}\alpha_n) \rightarrow 0$ to obtain the exact sequence

$$\begin{aligned} \text{Hom}_G(R, L(\mu) \otimes L(\omega_1)^{(1)}) &\rightarrow \text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\mu) \otimes L(\omega_1)^{(1)}) \\ &\rightarrow \text{Ext}_G^1(V(\lambda - \frac{1}{2}\alpha_n), L(\mu) \otimes L(\omega_1)^{(1)}). \end{aligned}$$

The first term vanishes because μ and the composition factors of R are restricted. The last term is isomorphic to $\text{Ext}_G^1(L(\mu), H^0(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_B^1(L(\mu), \lambda - \frac{1}{2}\alpha_n \otimes L(\omega_1)^{(1)})$. This implies that $\text{Ext}_B^1(L(\mu), \lambda - \frac{1}{2}\alpha_n \pm p\epsilon_i) \neq 0$ for some i . Hence $\text{Ext}_B^1(L(\mu), \lambda - \frac{1}{2}\alpha_n + p\epsilon_i) \neq 0$ is not possible by height comparison. If $\text{Ext}_B^1(L(\mu), \lambda - \frac{1}{2}\alpha_n - p\epsilon_i) \neq 0$, Lemma 2.2(b) yields $i = n$ and 2.2(a) forces $\mu = \lambda$ (recall that $s_{\alpha_n} \cdot (\lambda - \frac{1}{2}\alpha_n) = \lambda$). \square

3. Constructing self-extensions via the algebraic group

Here we use methods due to Andersen to generalize Humphreys' construction of self-extensions.

3.1. In this section we allow for algebraic groups G other than type C_n . The associated finite groups of Lie type obtained as fixed points of the r -th Frobenius morphism twisted by an automorphism σ coming from an automorphism of the Dynkin diagram are denoted by $G_\sigma(\mathbb{F}_q)$, where $q = p^r$ (see [Jan2, 1.3]).

The following is a generalization of work by H.H. Andersen [And2, Prop 2.7]. If $p \geq 2(h-1)$ the set π_λ can be replaced by the set of p^r -bounded weights.

PROPOSITION . *Let $\lambda, \mu \in X_r(T)$ with $\lambda \not\leq_{\mathbb{Q}} \mu$. Set $\pi_\lambda = \{\nu = \nu_0 + p^r\nu_1 \mid \nu_0 \in X_r(T) \text{ and } \nu_1 \in X(T)_+ \text{ such that } \nu_0 - \lambda \not\geq p^r\eta \text{ for any } \eta \in X(T)_+\}$. Then*

$$\bigoplus_{\nu \in \pi_\lambda} \text{Hom}_G(L(\mu), L(\nu_0) \otimes L(\sigma\nu_1)) \otimes \text{Ext}_G^1(L(\nu), L(\lambda)) \hookrightarrow \text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\mu), L(\lambda)).$$

PROOF. Set $\tilde{\lambda} = (p^r - 1)\rho + w_0\lambda$. If $p^r\gamma$ is a dominant weight of

$$\text{Hom}_{G_r}(L(\nu_0), \text{St}_r \otimes L(\tilde{\lambda})) \cong \text{Hom}_{G_r}(L(\nu_0) \otimes L((p^r - 1)\rho - \lambda), \text{St}_r),$$

then one of the weights appearing in $L(\nu_0) \otimes L((p^r - 1)\rho - \lambda)$ is $(p^r - 1)\rho + p^r\gamma$. This forces $\nu_0 - \lambda \geq p^r\gamma$, which is not allowed. Hence

$$(3.1.1) \quad \text{Hom}_{G_r}(L(\nu_0), \text{St}_r \otimes L(\tilde{\lambda})) = 0.$$

Moreover

$$(3.1.2) \quad \text{Hom}_G(L(\nu), \text{St}_r \otimes L(\tilde{\lambda})) \cong \text{Hom}_G(L(\nu_1)^{(r)}, \text{Hom}_{G_r}(L(\nu_0), \text{St}_r \otimes L(\tilde{\lambda}))) = 0.$$

The Steinberg module is injective as a G_r -module. It follows from the five-term exact sequence of the Lyndon-Hochschild-Serre spectral sequence and (3.1.1) that

$$(3.1.3) \quad \text{Ext}_G^1(L(\nu), \text{St}_r \otimes L(\tilde{\lambda})) \cong \text{Ext}_{G/G_r}^1(L(\nu_1)^{(r)}, \text{Hom}_{G_r}(L(\nu_0), \text{St}_r \otimes L(\tilde{\lambda}))) = 0.$$

Define Q via the exact sequence of G -modules

$$(3.1.4) \quad 0 \rightarrow L(\lambda) \rightarrow \text{St}_r \otimes L(\tilde{\lambda}) \rightarrow Q \rightarrow 0.$$

From the corresponding long exact sequence as well as (3.1.2) and (3.1.3) one obtains that

$$(3.1.5) \quad \text{Ext}_G^1(L(\nu), L(\lambda)) \cong \text{Hom}_G(L(\nu), Q).$$

Next, we use the injectivity of St_r as a $G_\sigma(\mathbb{F}_q)$ -module and weight comparison to argue, by [Jan2, Satz 1.5]),

$$\begin{aligned} \dim \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), \text{St}_r \otimes L(\tilde{\lambda})) &= [L(\mu) \otimes L((p^r - 1)\rho - \lambda) : \text{St}_r]_{G_\sigma(\mathbb{F}_q)} \\ &= \sum_{\gamma \in X(T)_+} [L(\mu) \otimes L((p^r - 1)\rho - \lambda) \otimes L(\sigma\gamma) : \text{St}_r \otimes L(\gamma)^{(r)}]_G. \end{aligned}$$

The last expression vanishes unless $\mu \geq \lambda + (p^r - \sigma)\gamma \geq \lambda$. Now $\lambda \not\leq_{\mathbb{Q}} \mu$ forces $\gamma = 0$ and $\mu = \lambda$. One concludes that

$$(3.1.6) \quad \dim \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), \text{St}_r \otimes L(\tilde{\lambda})) \cong \begin{cases} k & \text{if } \mu = \lambda \\ 0 & \text{else.} \end{cases}$$

The long exact sequence arising from (3.1.4) together with (3.1.6) and the injectivity of St_r as a $G_\sigma(\mathbb{F}_q)$ -module now imply

$$(3.1.7) \quad \text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\mu), L(\lambda)) \cong \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), Q).$$

Restriction from G to $G_\sigma(\mathbb{F}_q)$ induces an embedding of

$$(3.1.8) \quad \begin{aligned} \text{Hom}_G(L(\mu), L(\nu_0) \otimes L(\sigma\nu_1)) &\hookrightarrow \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), L(\nu_0) \otimes L(\sigma\nu_1)) \\ &\cong \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), L(\nu_0) \otimes L(\nu_1)^{(r)}) \cong \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), L(\nu)). \end{aligned}$$

The module $\bigoplus_{\nu \in \pi_\lambda} L(\nu) \otimes \text{Hom}_G(L(\nu), Q)$ is a G -submodule of the G -socle of Q . It is also a $G_\sigma(\mathbb{F}_q)$ -submodule of Q . Therefore

$$\begin{aligned} &\bigoplus_{\nu \in \pi_\lambda} \text{Hom}_G(L(\mu), L(\nu_0) \otimes L(\sigma\nu_1)) \otimes \text{Ext}_G^1(L(\nu), L(\lambda)) \\ &\cong \bigoplus_{\nu \in \pi_\lambda} \text{Hom}_G(L(\mu), L(\nu_0) \otimes L(\sigma\nu_1)) \otimes \text{Hom}_G(L(\nu), Q) \quad (\text{by (3.1.5)}) \\ &\hookrightarrow \bigoplus_{\nu \in \pi_\lambda} \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), L(\nu)) \otimes \text{Hom}_G(L(\nu), Q) \quad (\text{by (3.1.8)}) \\ &\hookrightarrow \text{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\mu), Q) \cong \text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\mu), L(\lambda)) \quad (\text{by (3.1.7)}). \end{aligned}$$

□

Setting $\gamma = \lambda = \mu$ and applying Proposition 3.1 yields:

COROLLARY . *Let $\gamma, \nu_0 \in X_r(T)$ and $\nu_1 \in X(T)_+$ such that $\nu_0 \not\leq \gamma$. If*

- (i) $\text{Hom}_G(L(\gamma), L(\nu_0) \otimes L(\sigma\nu_1)) \neq 0$, and
- (ii) $\text{Ext}_G^1(L(\gamma), L(\nu_0) \otimes L(\nu_1)^{(r)}) \neq 0$,

then $\text{Ext}_{G_\sigma(\mathbb{F}_q)}^1(L(\gamma), L(\gamma)) \neq 0$.

4. Self-extensions for $Sp_{2n}(\mathbb{F}_p)$ and odd primes.

Here we use the algebraic group to construct families of self-extensions for $Sp_{2n}(\mathbb{F}_p)$ including ones discovered by Tiep and Zalesskii (see [TZ, 3.18]). The examples exist for all odd primes.

4.1. In order to make use of Proposition 3.1 one needs to show that $L(\lambda)$ appears in the G -socle of $L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)$.

LEMMA . *Let p be odd, G be of type C_n , and α_n be the unique long simple root. Assume that $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$. For all $1 \leq i \leq n-1$ with $\langle \lambda, \alpha_i^\vee \rangle > 0$ we assume in addition that $\langle \lambda + \rho, (\alpha_i + \dots + \alpha_{n-1})^\vee \rangle \not\equiv 0 \pmod{p}$. Then $\text{Hom}_G(L(\lambda), L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)) \neq 0$.*

PROOF. The tensor product $H^0(\lambda - \frac{1}{2}\alpha_n) \otimes H^0(\omega_1) = H^0(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)$ has a filtration with factors $H^0(\gamma)$ where $\gamma \in S = \{\lambda \pm \epsilon_i - \epsilon_n \mid 1 \leq i \leq n\} \cap X(T)_+$. Notice that for $1 \leq i \leq n-2$, $\lambda - (\epsilon_i + \epsilon_n) = \lambda - (\alpha_i + \dots + \alpha_n) \in S$ if and only if $\langle \lambda, \alpha_i^\vee \rangle > 0$. Assume that $\lambda - (\alpha_i + \dots + \alpha_n)$ is dominant and that $\lambda - (\alpha_i + \dots + \alpha_n) \uparrow \lambda$. This implies that a hyperplane of the form $H_{\alpha_i + \dots + \alpha_n, mp} = \{x + \rho \in \mathbb{R}^n \mid \langle x, (\alpha_i + \dots + \alpha_n)^\vee \rangle = mp\}$ lies between the two weights. One concludes that $\langle \lambda + \rho, (\alpha_i + \dots + \alpha_n)^\vee \rangle \equiv 1 \pmod{p}$, which yields $\langle \lambda + \rho, (\alpha_i + \dots + \alpha_{n-1})^\vee \rangle \equiv 0 \pmod{p}$. Such weights λ have been excluded. If $\langle \lambda, \alpha_{n-1}^\vee \rangle = 0$ then $\langle \lambda + \rho, (\alpha_{n-1} + \alpha_n)^\vee \rangle = p+2$. Moreover, $\langle \lambda + \rho, \alpha_n^\vee \rangle = \frac{p+1}{2} < p$. Therefore neither $\lambda - (\alpha_{n-1} + \alpha_n)$ nor $\lambda - \alpha_n$ is strongly linked to λ . One concludes that λ is minimal in the “ \uparrow ”-order among the weights in S . As a consequence of [Jan1, II.4.18 and II.6.13] the module $L(\lambda) \subset H^0(\lambda)$ is a submodule of $H^0(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)$.

At this point of the proof we know that $L(\lambda)$ is a submodule of $H^0(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)$ and we want to prove that it is actually in $L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)$. It is clear that $\text{Hom}_G(L(\lambda), L(\gamma) \otimes L(\omega_1)) \neq 0$ for some composition factor $L(\gamma)$ of $H^0(\lambda - \frac{1}{2}\alpha_n)$, so we have to show that $\gamma = \lambda - \frac{1}{2}\alpha_n = \lambda - \epsilon_n$. For such γ we have $\dim \text{Hom}_G(V(\gamma), H^0(\lambda) \otimes H^0(\omega_1)) = \dim \text{Hom}_G(V(\lambda), H^0(\gamma) \otimes H^0(\omega_1)) \geq \dim \text{Hom}_G(L(\lambda), L(\gamma) \otimes L(\omega_1)) \neq 0$. This together with the fact that $\gamma \uparrow \lambda - \epsilon_n$ forces γ to be either equal to $\lambda - \epsilon_n$ or dominant of the form $\lambda - \epsilon_i = \lambda - \frac{1}{2}\alpha_n - (\alpha_i + \dots + \alpha_{n-1})$ for some $1 \leq i \leq n-1$. The latter implies that $\langle \lambda - \frac{1}{2}\alpha_n + \rho, (\alpha_i + \dots + \alpha_{n-1})^\vee \rangle \equiv 1 \pmod{p}$, which forces $\langle \lambda + \rho, (\alpha_i + \dots + \alpha_{n-1})^\vee \rangle \equiv 0 \pmod{p}$. Now the premises of the lemma says that $\langle \lambda, \alpha_i^\vee \rangle = 0$. Hence $\gamma = \lambda - \epsilon_i$ is not dominant. This leaves $\gamma = \lambda - \epsilon_n$, as claimed. \square

4.2. Here we give sufficient conditions for the existence of self-extensions. Their usefulness will become apparent in the following sections.

PROPOSITION . *Let p be odd, G be of type C_n , and α_n be the unique long simple root. Assume that $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$. In addition, assume that*

- (i) $\langle \lambda, \alpha_i^\vee \rangle + \langle \lambda, \alpha_{i+1}^\vee \rangle < p-1$, for $1 \leq i \leq n-1$,
- (ii) $H^0(\lambda)$ and $H^0(\lambda - \frac{1}{2}\alpha_n)$ have only p -restricted composition factors,
- (iii) For $1 \leq i \leq n-1$, $\langle \lambda, \alpha_i^\vee \rangle > 0$ implies that $\langle \lambda + \rho, (\alpha_i + \dots + \alpha_{n-1})^\vee \rangle \not\equiv 0 \pmod{p}$,

then $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) \neq 0$ and $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n)) \neq 0$.

PROOF. Proposition 2.4 implies that

$$\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda) \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_G^1(L(\lambda), L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}) \cong k.$$

Lemma 4.1 yields

$$\text{Hom}_G(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda) \otimes L(\omega_1)) \cong \text{Hom}_G(L(\lambda), L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)) \cong k.$$

The assertion follows from Proposition 3.1. \square

4.3. The weight $\frac{p-1}{2}\omega_n$ satisfies the conditions (i) through (iii) of Proposition 4.2 because $\langle \frac{p-1}{2}\omega_n, \alpha_0^\vee \rangle = p-1$. This yields a family of self-extensions discovered by Tiep and Zalesskiĭ (see [TZ, 3.18]). Their method of proof is quite different. They show that in order for certain irreducible p -modular representations to be lifted to characteristic zero the representation has to admit self-extensions. Corollary A follows now from a result due to Zalesskiĭ and Suprunenko [ZS] that says that $L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n)$ and $L(\frac{p-1}{2}\omega_n)$ are reduction modulo p of irreducible complex Weil representations for $Sp_{2n}(\mathbb{F}_p)$. Using the algebraic group to construct these extensions, has the advantage that additional families of self-extensions arise.

COROLLARY (A). *Let p be odd, G be of type C_n , α_n be the unique long simple root, and ω_n the corresponding fundamental weight. Then*

- (i) $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\frac{p-1}{2}\omega_n), L(\frac{p-1}{2}\omega_n)) \neq 0$ and
- (ii) $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n), L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n)) \neq 0$.

Assume that $p \geq 2n$ and that λ is p -regular, i. e. $\langle \lambda + \rho, \alpha^\vee \rangle \not\equiv 0 \pmod{p}$ for any root α . Clearly any p -regular weight satisfies condition (iii) of Proposition 4.1. Moreover, the weight $\lambda - \frac{1}{2}\alpha_n$ is a reflection of λ across the hyperplane $\langle x + \rho, \alpha_n^\vee \rangle = \frac{p}{2}$ that bisects the alcove containing λ . It follows that $\lambda - \frac{1}{2}\alpha_n$ is a p -regular weight inside the same alcove and the translation principle allows for the following simplification of condition (ii) in Proposition 4.2. Later we will see that conditions (i) and (ii) can be dropped if one assumes the Lusztig conjecture.

COROLLARY (B). *Let $p \geq 2n$, G of type C_n . Assume that $\lambda \in X_1(T)$ is a p -regular weight with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$, where α_n denotes the unique long simple root in the root system. Assume further that*

- (i) $\langle \lambda, \alpha_i^\vee \rangle + \langle \lambda, \alpha_{i+1}^\vee \rangle < p-1$ for all $1 \leq i \leq n-1$, and
- (ii) $H^0(\lambda)$ has only p -restricted composition factors,

then $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) \neq 0$ and $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n)) \neq 0$.

5. Self-extensions for $Sp_{2n}(\mathbb{F}_p)$ via the Lusztig conjecture

5.1. Lusztig conjecture and equivalences. Throughout this section we assume that G is of type C_n and $p \geq h = 2n$. Set $\Gamma = \{\gamma \in X(T)_+ \mid \langle \gamma + \rho, \alpha_0^\vee \rangle \leq p(p-h+2)\}$. If $\lambda \in X_1(T)$ with $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p-h+1)$ then the non-restricted weights $\lambda + p\omega_1$ and $\lambda - \frac{1}{2}\alpha_n + p\omega_1$ are also contained in Γ . Furthermore, we assume throughout this section that the Lusztig conjecture, as stated in [Jan1, II.8.22], holds for G .

For any p -regular weight $\gamma \in X(T)$ and any $\alpha \in \Phi^+$ there exists a unique integer n_α with $n_\alpha p < \langle \gamma + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p$. As in [Jan1, II.6.6] we set $d(\gamma) = \sum_{\alpha \in \Phi^+} n_\alpha$. A result due to Cline, Parshall, and Scott says that the Lusztig conjecture is equivalent to each of the following statements [CPS, 5.4].

$$(5.1.1) \quad \text{For } \gamma, \eta \in \Gamma, \text{Ext}_G^i(L(\gamma), H^0(\eta)) \neq 0 \text{ implies } d(\gamma) - d(\eta) \equiv i \pmod{2}.$$

$$(5.1.2) \quad \text{For } \gamma, \eta \in \Gamma, \text{ the natural map } \text{Ext}_G^1(L(\gamma), L(\eta)) \rightarrow \text{Ext}_G^1(L(\gamma), H^0(\eta)) \text{ is surjective.}$$

This will allow us to eliminate conditions (i) and (ii) of Proposition 2.4.

LEMMA . Let G be of type C_n and $p \geq 2n$. Assume that $\lambda \in X_1(T)$ is a p -regular weight with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$, where α_n denotes the unique long simple root. Assume further that $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p-h+1)$ and that the Lusztig Conjecture holds for G .

Then $\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda) \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_G^1(L(\lambda), L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}) \neq 0$.

PROOF. Proposition 2.3 and (5.1.2) reduce the assertion to

$$\text{Ext}_B^2(L(\lambda - \frac{1}{2}\alpha_n), \lambda - p\epsilon_i) = 0 \text{ for all } i < n.$$

By Lemma 2.2(d) it is sufficient to show that $\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), H^0(s_{\alpha_i} \cdot (\lambda - p\epsilon_i))) = 0$. We will show that $d(\lambda - \frac{1}{2}\alpha_n) - d(s_{\alpha_i} \cdot (\lambda - p\epsilon_i))$ is even. The assertion then follows from the Lusztig Conjecture via (5.1.1).

Observe that $s_{\alpha_i} \cdot (\lambda - p\epsilon_i) = s_{\alpha_i} \cdot \lambda - p\epsilon_{i+1}$. The reflection s_{α_i} permutes all the positive roots other than α_i . Since λ is p -restricted, one concludes that $d(s_{\alpha_i} \cdot \lambda) = d(\lambda) - 1$. Next we compare $d(s_{\alpha_i} \cdot \lambda)$ to $d(s_{\alpha_i} \cdot \lambda - p\epsilon_{i+1})$. For $1 \leq l \leq i$ one has $\langle \epsilon_{i+1}, (\epsilon_l - \epsilon_{i+1})^\vee \rangle = -1$ and $\langle \epsilon_{i+1}, (\epsilon_l + \epsilon_{i+1})^\vee \rangle = 1$. For $i+1 \leq l \leq n$ one has $\langle \epsilon_{i+1}, (\epsilon_i - \epsilon_l)^\vee \rangle = 1$ and $\langle \epsilon_{i+1}, (\epsilon_i + \epsilon_l)^\vee \rangle = 1$. All other short roots are perpendicular to ϵ_{i+1} . For the long roots one obtains $\langle \epsilon_{i+1}, (2\epsilon_l)^\vee \rangle = \delta_{i+1,l}$. It follows that $d(s_{\alpha_i} \cdot \lambda) - d(s_{\alpha_i} \cdot \lambda - p\epsilon_{i+1}) = 2(n - (i+1)) - 1$. Recall that λ and $\lambda - \frac{1}{2}\alpha_n$ are contained in the same alcove. One concludes that $d(\lambda - \frac{1}{2}\alpha_n) - d(s_{\alpha_i} \cdot (\lambda - p\epsilon_i)) = d(\lambda) - d(s_{\alpha_i} \cdot \lambda) + d(s_{\alpha_i} \cdot \lambda) - d(s_{\alpha_i} \cdot \lambda - p\epsilon_{i-1}) = 2(n - (i+1))$ is even. \square

Using the same argument as in the proof of Corollary 2.3 one can show that exceptions to [BNP2, Thm 5.3(A) part (a)] indeed exist.

COROLLARY . Let G be of type C_n and $p \geq 2n$. Assume that $\lambda \in X_1(T)$ is a p -regular weight with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$, where α_n denotes the unique long simple root. Assume further that $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p-h+1)$ and that the Lusztig Conjecture holds for G . Then $\text{Hom}_G(\text{Ext}_{G_1}^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda)), L(\omega_1)^{(1)}) \cong k$.

5.2. With the help of Corollary 3.1, Lemma 4.1, and Lemma 5.1 we can now generalize Humphreys' $Sp_4(\mathbb{F}_p)$ example to higher ranks, at least for p -regular weights.

PROPOSITION (A). Let G be of type C_n and $p \geq 2n$. Assume that $\lambda \in X_1(T)$ is a p -regular weight with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$, where α_n denotes the unique long simple root. Assume further that $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p-h+1)$ and that the Lusztig Conjecture holds for G .

Then $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) \neq 0$ and $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n)) \neq 0$.

Not all p -singular weights adjacent to the hyperplane $H_{\alpha_n, p/2}$ will admit self-extensions. This can already be observed in the case of $Sp_4(\mathbb{F}_p)$, where self-extensions do not occur when λ is contained in a α_1 -wall (see [And2, Note on p. 402]). Our methods show that self-extensions exist for all p -singular weights that are adjacent to $H_{\alpha_n, p/2}$, as long as they are not contained in any $(\alpha_i + \dots + \alpha_{n-1})$ -wall, with $1 \leq i \leq n-1$.

PROPOSITION (B). Let G be of type C_n and $p \geq 2n$. Assume that $\lambda \in X_1(T)$ is a p -singular weight with $\langle \lambda + \rho, (\alpha_i + \dots + \alpha_{n-1})^\vee \rangle \not\equiv 0 \pmod{p}$, for $1 \leq i \leq n-1$, and $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$. Assume further that $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p(p-h+1)$ and that the Lusztig Conjecture holds for G .

Then $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda), L(\lambda)) \neq 0$ and $\text{Ext}_{Sp_{2n}(\mathbb{F}_p)}^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n)) \neq 0$.

PROOF. For each $\alpha \in \Phi^+$ there exists a unique non-negative integer n_α such that $n_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle \leq (n_\alpha + 1)p$. Then $C_\lambda = \{\gamma \in X(T)^+ \mid n_\alpha p < \langle \gamma + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p \text{ for all } \alpha \in \Phi^+\}$ describes the alcove that contains λ in its upper closure.

Set $R_\lambda = \{\alpha \in \Phi^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle = (n_\alpha + 1)p\}$. Notice that the conditions on λ imply that no root of the form $\alpha_i + \dots + \alpha_{n-1}$ is contained in R_λ . At the same time one observes that the weight $\lambda - \frac{1}{2}\alpha_n$ is contained in the upper closure of C_λ if and only if R_λ contains no elements of the form $\alpha_i + \dots + \alpha_{n-1}$ with $1 \leq i \leq n-1$. Hence both λ and $\lambda - \frac{1}{2}\alpha_n$ lie in the upper closure of the same alcove.

The alcove C_λ is bisected by the hyperplane $H_{\alpha_n, p/2} = \{x \in \mathbb{R}^n \mid \langle x + \rho, \alpha_n^\vee \rangle = p/2\}$. Since $p \geq h$ there exists a p -regular weight μ inside C_λ . We denote its reflection across the hyperplane $H_{\alpha_n, p/2}$ by $\bar{\mu}$. We choose a pair $(\mu, \bar{\mu})$ with minimal distance $\mu - \bar{\mu}$. This implies that $\langle \mu + \rho, \alpha_n^\vee \rangle = (p+1)/2$. Lemma 5.1 implies that $\text{Ext}_G^1(L(\mu), L(\bar{\mu}) \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_G^1(L(\mu), L(\bar{\mu} + p\omega_1)) \neq 0$. We define E_μ via a non-split sequence $0 \rightarrow L(\bar{\mu} + p\omega_1) \rightarrow E_\mu \rightarrow L(\mu) \rightarrow 0$. Clearly E_μ embeds in $H^0(\bar{\mu} + p\omega_1)$. We set $E_\lambda = T_\mu^\lambda E_\mu$, where T_μ^λ denotes the translation functor as defined in [Jan1, II.7.6]. The exactness of the translation functor [Jan1, II.7.7.6] yields an embedding of E_λ in $T_\mu^\lambda H^0(\bar{\mu} + p\omega_1) \cong H^0(\lambda - \frac{1}{2}\alpha_n + p\omega_1)$. Moreover, since the weight $\lambda - \frac{1}{2}\alpha_n + p\omega_1$ is in the upper closure of the alcove containing $\bar{\mu} + p\omega_1$, one concludes that E_λ has exactly two composition factors, namely $L(\lambda)$ and $L(\lambda - \frac{1}{2}\alpha_n + p\omega_1)$. Therefore,

$$\text{Ext}_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda) \otimes L(\omega_1)^{(1)}) \cong \text{Ext}_G^1(L(\lambda), L(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}) \neq 0.$$

The assertions follows from Lemma 4.1 and Corollary 3.1. \square

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