Extensions for finite groups of Lie type: twisted groups

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Abstract. In [BNP1] [BNP2] [BNP3] the authors relate the extensions for modules for finite Chevalley groups to certain extensions for the corresponding reductive groups and Frobenius kernels. In this paper we will show that these results can be extended to the twisted finite groups of Lie type. The applications are most pertinent to the twisted groups of types A, D, and E_6.

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1. Introduction

1.1 Let G be a connected semisimple algebraic group defined and split over the field $\mathbb{F}_p$ with p elements, and k be the algebraic closure of $\mathbb{F}_p$. Assume further that G is almost simple and simply connected. Let $G(\mathbb{F}_q)$ be the finite Chevalley group consisting of $\mathbb{F}_q$-rational points of G where $q = p^r$ for a non-negative integer r and let G_r denote the rth Frobenius kernel of G. In a series of papers [BNP1] [BNP2] [BNP3] [BNP4] the authors investigated the connections between the extension theory of modules for the semisimple algebraic group G, the finite Chevalley group $G(\mathbb{F}_q)$ and the Frobenius kernel G_r via the use of spectral sequences involving truncated (bounded) subcategories of G-modules. In particular these methods enabled us to provide explicit formulas for the extensions between simple modules of finite Chevalley groups via algebraic groups and Frobenius kernels. As a direct application, several open problems involving the existence of self-extensions for finite Chevalley groups posed by Humphreys [Hum] were resolved by proving suitable generalizations of Andersen's work on line bundle cohomology for the flag variety [And1] [And2].

Recent work by Tiep and Zalesskiĭ [TZ, Prop. 1.4] has shown that questions involving extensions between simple modules for finite groups arise naturally when

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one studies the modular representation theory of finite groups. They prove that the ability to lift certain simple modular representations to characteristic zero can happen only when the simple module admits a self-extension. In the mid 1980s, Smith [Smi] showed that the existence of self-extensions is also relevant in the study of sheaf homology.

1.2 The purpose of this paper is to demonstrate that the ideas from the authors earlier work can be used to study the extension theory for the twisted finite groups of Lie type. Since our results hold mostly for large primes, we will exclude the Suzuki and Ree groups from our discussion unless otherwise stated. Let σ denote an automorphism of the Dynkin diagram of G. Then σ induces a group automorphism of G that commutes with the Frobenius morphism, which we will also denote by σ. Let $G_\sigma(\mathbb{F}_q)$ be the finite group consisting of the fixed points of σ composed with the rth Frobenius morphism on G. Our results will compare the extension theory of modules for $G_\sigma(\mathbb{F}_q)$ with that of G and G_r. In our discussion, we will leave out several of the details of earlier proofs which have straight forward generalizations to the twisted case. Instead we will focus on places where there is care needed in accounting for the twist σ. The new formulas that we obtain involving extensions of simple modules for $G_\sigma(\mathbb{F}_q)$ do indeed involve using the automorphism σ in the calculations. We also indicate several instances in the text where improvements have been made to our earlier work.

1.3 Notation: The conventions in the paper will follow the ones used in [Jan1]. Throughout this paper G is a connected semisimple algebraic group defined and split over the finite field $\mathbb{F}_p$ with p elements. Assume further that G is almost simple and simply connected. The field k is the algebraic closure of $\mathbb{F}_p$ and G will be considered as an algebraic group scheme over k. Let Φ be a root system associated to G with respect to a maximal split torus T. Moreover, let Φ^+ (resp. Φ^-) be positive (resp. negative) roots and Δ be a base consisting of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots.

Let $X(T)$ be the integral weight lattice obtained from Φ contained in the Euclidean space $\mathbb{E}$ with the the inner product denoted by $\langle\ ,\ \rangle$. The set $X(T)$ has a partial ordering given by $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha\in\Delta}\mathbb{N}\alpha$ for $\lambda, \mu \in X(T)$. Let $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ be the coroot corresponding to $\alpha \in \Phi$. The set of dominant integral weights is defined by

$$X(T)_+ = \{\lambda \in X(T) :\ 0 \leq \langle\lambda, \alpha^\vee\rangle \text{ for all } \alpha \in \Delta\}.$$

Furthermore, the set of p^r-restricted weights is

$$X_r(T) = \{\lambda \in X(T) :\ 0 \leq \langle\lambda, \alpha^\vee\rangle < p^r \text{ for all } \alpha \in \Delta\}.$$

The Weyl group W is the group generated by the reflections $s_\alpha : \mathbb{E} \to \mathbb{E}$ given by $s_\alpha(\lambda) = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$. The group W acts on $X(T)$ via the "dot action" given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ where ρ is the half sum of the positive roots. The long word in W will be denoted by w_0 and the Coxeter number for Φ is $h = \langle\rho, \alpha_0^\vee\rangle + 1$, where α_0 is the maximal short root.

1.4 Unless otherwise stated we assume here that the finite group is a split or quasi-split form of G [Car, 1.19] and adopt the set-up described in [Jan3, 1.3]. Let σ denote an automorphism of the Dynkin diagram of G. The automorphism σ can be extended to a linear bijection on $X(T)$ that permutes the fundamental weights and preserves the inner product $\langle\ ,\ \rangle$ as well as the partial order on $X(T)$. Moreover, $\sigma(\alpha_0) = \alpha_0$. The graph automorphism σ induces an automorphism on G which will also be denoted by σ. The group automorphism σ commutes with the standard Frobenius morphism F and is compatible with the action of σ on $X(T)$. Let $G_\sigma(\mathbb{F}_q)$ be the group of fixed points of $F^r \circ \sigma = \sigma \circ F^r$, where $q = p^r$.

For each $\lambda \in X(T)_+$, let $H^0(\lambda) = \text{ind}_B^G \lambda$. The simple modules for G are labeled by the set $X(T)_+$ and are given by the correspondence $\lambda \to L(\lambda) = \text{soc}_G H^0(\lambda)$. Every G-module V becomes a $G_\sigma(\mathbb{F}_q)$-module by restriction. A complete set of non-isomorphic simple $G_\sigma(\mathbb{F}_q)$-modules and simple G_r-modules are obtained by restricting $\{L(\lambda) :\ \lambda \in X_r(T)\}$. For any G-module V and simple $G_\sigma(\mathbb{F}_q)$-module $L(\lambda)$ we denote by $[V : L(\lambda)]_{G_\sigma(\mathbb{F}_q)}$ the multiplicity of $L(\lambda)$ as a $G_\sigma(\mathbb{F}_q)$-composition factor of V. One obtains

$$[V : L(\lambda)]_{G_\sigma(\mathbb{F}_q)} = \sum_{\gamma \in X(T)_+} [V : L(\gamma)]_G \ [L(\gamma) : L(\lambda)]_{G_\sigma(\mathbb{F}_q)}.$$

Moreover, twisting a G-module V with the rth Frobenius morphism yields the same result as twisting by σ^{-1} upon restriction to $G_\sigma(\mathbb{F}_q)$. Therefore we have

$$[V^{(r)} : L(\lambda)]_{G_\sigma(\mathbb{F}_q)} = [V : L(\sigma(\lambda))]_{G_\sigma(\mathbb{F}_q)}. \tag{1.4.1}$$

2. General Ext-formulas

2.1 We briefly review the techniques developed in [BNP1]. For a fixed positive integer r let $\mathcal{C}_s$ be the full subcategory of $\text{Mod}(G)$ with objects having composition factors whose highest weights lies in π_s where

$$\pi_s = \{\lambda \in X(T)_+ :\ \langle \lambda + \rho, \alpha_0^\vee \rangle \ < \ 2p^r s(h-1)\}.$$

Notice that the category $\mathcal{C}_1$ differs slightly from Jantzen's p^r-bounded category where it is assumed that $\langle \lambda, \alpha_0^\vee \rangle < 2p^r(h-1)$ [Jan1, p. 360]. Let $\mathcal{F}_{\mathcal{C}_s}$ be the truncation functor from $\text{Mod}(G)$ to $\text{Mod}(\mathcal{C}_s)$ which takes $M \in \text{Mod}(G)$ to the largest submodule of M in $\mathcal{C}_s$. Let $\mathcal{G}_s = \mathcal{F}_{\mathcal{C}_s} \circ \text{ind}_{G_\sigma(\mathbb{F}_q)}^G$ and $R^j\mathcal{G}_s$ be the higher right derived functors of $\mathcal{G}$. Since σ preserves the inner product and the order on $X(T)_+$ the arguments in [BNP1, 4.1- 4.4] yield an analog of [BNP1, Thm. 4.4a] as follows.

Theorem . *For $M \in \mathcal{C}_s$ and $N \in \text{Mod}(G_\sigma(\mathbb{F}_q))$ there exists a first quadrant spectral sequence*

$$E_2^{i,j} = \text{Ext}^i_G(M, R^j\mathcal{G}_s(N)) \Rightarrow \text{Ext}^{i+j}_{G_\sigma(\mathbb{F}_q)}(M, N).$$

2.2 If $p \geq 2(h-1)$ the rth Steinberg module St_r is projective and injective in all three categories $\mathrm{Mod}(G_\sigma(\mathbb{F}_q))$, $\mathrm{Mod}(G_r)$, and $\mathcal{C}_1$. The injective hulls of all p^r-restricted simple modules in all three categories appear as respective direct summands of tensor products of St_r with appropriate p^r-restricted simple modules. Moreover, the injective hull $Q_r(\lambda)$ of a simple G_r-module $L(\lambda)$ lifts to a G-structure and is also injective in $\mathcal{C}_1$ [Jan2]. A theorem due to Chastkofsky [Cha] and Jantzen [Jan3] says that the injective hull of $L(\lambda)$ in $\mathrm{Mod}(G_\sigma(\mathbb{F}_q))$ appears exactly once as a summand of $Q_r(\lambda)$, both for the split forms and for the quasi-split forms. Therefore, the arguments in [BNP1, 4.6-5.2], which were addressing only the split forms extend to the twisted groups. The main theorem is the following.

Theorem . *Let $s \geq 1$, $p \geq 2s(h-1)$, $M \in \mathrm{Mod}(G_\sigma(\mathbb{F}_q))$, and let $L \in \mathcal{C}_s$ such that L has only p^r-restricted composition factors in its head (as a G-module), then*

$$\mathrm{Ext}^i_{G_\sigma(\mathbb{F}_q)}(L, M) \cong \mathrm{Ext}^i_G(L, \mathcal{G}_s(M))$$

for all $0 \leq i \leq s$.

2.3 For the remainder of the paper we will look at the Ext^1 groups. We will simply denote $\mathcal{G}_1$ by $\mathcal{G}$, π_1 by π, and $\mathcal{C}_1$ by $\mathcal{C}$. It turns out that the module $\mathcal{G}(k)$ is of particular interest. The theorem below justifies this statement. It is an analog to [BNP2, Thm. 2.2]. The proof in [BNP2] can easily be adapted to the twisted case. Notice that the result holds for any prime.

Theorem . *Let $\lambda, \mu \in X_r(T)$. Then*

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu) \otimes \mathcal{G}(k)).$$

2.4 Suzuki and Ree groups: For the Suzuki and Ree groups a similar theorem can be obtained. Let G be of type C_2 or F_4 and $p = 2$ or G of type G_2 and $p = 3$. Then there exists a special isogeny σ on G with $\sigma^2 = F$, the Frobenius morphism, which leads to a stronger Steinberg Tensor Product Theorem [Ste, Section 11], [Sin, 1.4(2)]. Define $X_\sigma(T) \subset X_1(T)$ to be the subset of those weights that are orthogonal to all long simple roots. Then every $\lambda \in X(T)_+$ has a σ-adic expansion $\lambda = \sum_{i=0}^m \sigma^i \lambda_i, \quad \lambda_i \in X_\sigma(T)$.

For an odd positive integer m we denote by $G(m)$ the subgroup of G fixed by σ^m. These are the Suzuki (for type C_2) and Ree groups (for types G_2, F_4). The set of σ^m-restricted weights $X_{\sigma^m}(T) = \{\sum_{i=0}^{m-1} \sigma^i \lambda_i \mid \lambda_i \in X_\sigma(T)\}$ parametrizes the simple modules of $G(m)$. The Steinberg module $\mathrm{St}(m)$ with highest weight $(\sigma^m - 1)\rho$ is injective and projective.

For fixed m we denote by $\mathcal{C}$ the subcategory of $\mathrm{Mod}(G)$ whose highest weights γ satisfy

$$\langle \gamma, \alpha_0^\vee \rangle \leq 2\langle (\sigma^m - 1)\rho, \alpha_0^\vee \rangle.$$

As in Section 2.1 we let $\mathcal{F}_\mathcal{C}$ be the truncation functor from $\mathrm{Mod}(G)$ to $\mathrm{Mod}(\mathcal{C})$ which takes $M \in \mathrm{Mod}(G)$ to the largest submodule of M in $\mathcal{C}$ and set $\mathcal{G} =$

$\mathcal{F}_{\mathcal{C}} \circ \operatorname{ind}_{G(m)}^{G}$. As before we obtain for any $M \in \mathcal{C}$ and any $N \in \operatorname{Mod}(G(m))$ a spectral sequence

$$E_2^{i,j} = \operatorname{Ext}_G^i(M, R^j\mathcal{G}(N)) \Rightarrow \operatorname{Ext}_{G(m)}^{i+j}(M, N). \tag{2.4.1}$$

In the proof of the following theorem we make use of the fact that the $E_2^{1,0}$-term of the spectral sequence embeds in the E^1-term.

Theorem . *Let $\lambda, \mu \in X_{\sigma^m}(T)$. Then*

$$\operatorname{Ext}_{G(m)}^1(L(\lambda), L(\mu)) \cong \operatorname{Ext}_G^1(L(\lambda), L(\mu) \otimes \mathcal{G}(k)).$$

Proof. All simple modules are self-dual and

$$\operatorname{Ext}_{G(m)}^1(L(\lambda), L(\mu)) \cong \operatorname{Ext}_{G(m)}^1(L(\mu), L(\lambda)).$$

Therefore, we may assume that $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$ and furthermore that $\mu \not> \lambda$. Set $M = \operatorname{St}(m) \otimes L((\sigma^m - 1)\rho - \lambda)$. Then M is injective and projective as a $G(m)$-module and $M \otimes L(\mu)$ is in $\mathcal{C}$.

In order to proceed as in the proof of [BNP2, Thm. 2.2] we need the following

$$\operatorname{Hom}_{G(m)}(M \otimes L(\mu), k) \cong \operatorname{Hom}_{G(m)}(L(\lambda) \otimes L(\mu), k) \text{ and} \tag{2.4.2}$$

$$\operatorname{Hom}_G(M, L(\lambda)) \cong k. \tag{2.4.3}$$

Clearly,

$$\operatorname{Hom}_{G(m)}(L(\lambda) \otimes L(\mu), k) \cong \operatorname{Hom}_G(L(\lambda) \otimes L(\mu), k) \cong \begin{cases} k & \text{for } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Both (2.4.2) and (2.4.3) follow immediately from the Lemma below. Equation (2.4.2) now replaces [BNP2, (2.2.1)] and (2.4.3) allows us to define the G-module R via

$$0 \to R \to M \otimes L(\mu) \to L(\lambda) \otimes L(\mu) \to 0.$$

This last equation replaces [BNP2, (2.2.2)]. For the remainder of the proof one can proceed as in the proof of [BNP2, Thm. 2.2] by replacing the spectral sequence [BNP2, (2.1.1)] with (2.4.1). □

Lemma . *Let $\lambda, \mu \in X_{\sigma^m}(T)$ with $\mu \not> \lambda$. Then*

(a) $\operatorname{Hom}_G(\operatorname{St}(m) \otimes L((\sigma^m - 1)\rho - \lambda), L(\lambda)) \cong k$.

(b) $\operatorname{Hom}_{G(m)}(\operatorname{St}(m) \otimes L((\sigma^m - 1)\rho - \lambda), L(\mu)) \cong \begin{cases} k & \textit{for } \lambda = \mu \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (a) Set $r = (m-1)/2$ and let $\tilde{\rho}$ be the sum of all fundamental weights in $X_{\sigma^m}(T)$. Then $\operatorname{St}(m) \cong \operatorname{St}_r \otimes L((p-1)\tilde{\rho})^{(r)}$. There are unique $\lambda^0 \in X_r(T)$ and $\lambda^1 \in X_{\sigma^m}(T)$ with $\lambda = \lambda^0 + p^r\lambda^1$. Let $G_\sigma = \operatorname{Ker} \sigma$. Then G_σ is an infinitesimal group scheme [Sin, 1.2]. One can argue as follows:

$$\begin{aligned}
&\mathrm{Hom}_G(\mathrm{St}(m)\otimes L((\sigma^m-1)\rho-\lambda), L(\lambda))\\
&\quad\cong \mathrm{Hom}_G(\mathrm{St}(m), L((\sigma^m-1)\rho-\lambda)\otimes L(\lambda))\\
&\quad\cong (\mathrm{Hom}_{G_r}(\mathrm{St}_r, L((p^r-1)\rho-\lambda^0)\otimes L(\lambda^0))^G\\
&\qquad \otimes \mathrm{Hom}_G(L((p-1)\tilde\rho), L((p-1)\tilde\rho-\lambda^1)\otimes L(\lambda^1))\\
&\quad\cong \mathrm{Hom}_G(L((p-1)\tilde\rho), L((p-1)\tilde\rho-\lambda^1)\otimes L(\lambda^1)) \text{ (by [Jan1, II.11.9(2)])}\\
&\quad\cong (\mathrm{Hom}_{G_\sigma}(L((p-1)\tilde\rho), L((p-1)\tilde\rho-\lambda^1)\otimes L(\lambda^1)))^G \text{ (by [Sin, 1.8(1)]).}
\end{aligned}$$

The assertion now follows from the fact that $L(\tilde\rho)$ is projective as a G_σ-module [Sin, 1.7(2)] and that the multiplicity of $L(\tilde\rho)$ in $L(\tilde\rho-\lambda^1)\otimes L(\lambda^1)$ is one.

(b) $\mathrm{St}(m)$ is projective as a $G(m)$-module. Therefore,

$$\begin{aligned}
&\dim \mathrm{Hom}_{G(m)}(\mathrm{St}(m)\otimes L((\sigma^m-1)\rho-\lambda)\otimes L(\mu), k)\\
&= \dim \mathrm{Hom}_{G(m)}(\mathrm{St}(m), L((\sigma^m-1)\rho-\lambda)\otimes L(\mu))\\
&= [L((\sigma^m-1)\rho-\lambda)\otimes L(\mu):\mathrm{St}(m)]_{G(m)}\\
&= \sum_{\nu\in X(T)_+} [L((\sigma^m-1)\rho-\lambda)\otimes L(\mu):L(\nu)]_G\cdot[L(\nu_0)\otimes L(\sigma^m\nu_1):\mathrm{St}(m)]_{G(m)}\\
&= \sum_{\nu\in X(T)_+} [L((\sigma^m-1)\rho-\lambda)\otimes L(\mu):L(\nu)]_G\cdot[L(\nu_0)\otimes L(\nu_1):\mathrm{St}(m)]_{G(m)}.
\end{aligned}$$

Notice that $\nu_1=0$ forces $\nu_0=(\sigma^m-1)\rho$ and

$$[L((\sigma^m-1)\rho-\lambda)\otimes L(\mu):\mathrm{St}(m)]_{G(m)} = [L((\sigma^m-1)\rho-\lambda)\otimes L(\mu):\mathrm{St}(m)]_G.$$

The latter is zero unless $\mu\geq\lambda$, which implies $\mu=\lambda$. In this case the multiplicity is one, as desired.

Assume that there exists a ν with $\nu_1\neq 0$ that contributes to the above sum. Observe that $\sigma^m\nu_1>\nu_1$. It follows that

$$(\sigma^m-1)\rho-\lambda+\mu\geq\nu=\nu_0+\sigma^m\nu_1>\nu_0+\nu_1.$$

By using
$[L(\nu_0)\otimes L(\nu_1):\mathrm{St}(m)]_{G(m)}$

$$\begin{aligned}
&= \sum_{\gamma\in X(T)_+} [L(\nu_0)\otimes L(\nu_1):L(\gamma)]_G\cdot[L(\gamma_0)\otimes L(\sigma^m\gamma_1):\mathrm{St}(m)]_{G(m)}\\
&= \sum_{\gamma\in X(T)_+} [L(\nu_0)\otimes L(\nu_1):L(\nu)]_G\cdot[L(\gamma_0)\otimes L(\gamma_1):\mathrm{St}(m)]_{G(m)},
\end{aligned}$$

we can argue inductively on the size of the weight ν that $[L(\nu):\mathrm{St}(m)]_{G(m)}=0$ unless $\nu\geq(\sigma^m-1)\rho$. But in that case

$$(\sigma^m-1)\rho-\lambda+\mu\geq\nu=\nu_0+\sigma^m\nu_1>\nu_0+\nu_1\geq(\sigma^m-1)\rho$$

gives the desired contradiction to $\mu\not>\lambda$. □

2.5 The remainder of this section is devoted to collecting information on $\mathcal{G}(k)$ for split and quasi-split forms of G. Observe that if $\nu \in \pi$ is expressed as $\nu = \nu_0 + p^r\nu_1$ for weights $\nu_0 \in X_r(T)$ and $\nu_1 \in X(T)_+$, then the weight ν_1 satisfies $\langle \nu_1, \alpha_0^\vee\rangle < 2(h-1)$. We will show that the highest weight of any composition factor of $\mathcal{G}(k)$ is an element of a finite bounded set of weights. The bound is independent of p and r.

For a positive integer m, we define the set

$$\Gamma_m = \{\nu \in X(T)_+ : \langle \nu, \alpha_0^\vee\rangle < m\}.$$

Proposition . *If $L(\nu_0 + p^r\nu_1)$ is a composition factor of $\mathcal{G}(k)$ then $\nu_0, \nu_1 \in \Gamma_{2(h-1)}$.*

Proof. We know already that $\nu_1 \in \Gamma_{2(h-1)}$. Assume that

$$[\mathcal{G}(k) : L(\nu_0 + p^r\nu_1)]_G \neq 0.$$

Then $[\mathcal{G}(k) : L(\nu_0)]_{G_r} \neq 0$. This implies that $\mathrm{Hom}_{G_r}(Q_r(\nu_0), \mathcal{G}(k)) \neq 0$. Set $\widetilde{\nu} = (p^r-1)\rho + w_0\nu_0$. Then $Q_r(\nu_0)$ is a G_r-summand of $\mathrm{St}_r \otimes L(\widetilde{\nu})$ [Jan1, II.11.9]. One obtains

$$\mathrm{Hom}_{G_r}(\mathrm{St}_r \otimes L(\widetilde{\nu}), \mathcal{G}(k)) = \mathrm{Hom}_{G_r}(\mathrm{St}_r, \mathcal{G}(k) \otimes L(\widetilde{\nu})^*) \neq 0.$$

The St_r-isotypical component of the G_r-socle of $\mathcal{G}(k) \otimes L(\widetilde{\nu})^*$ is not zero. Therefore there exists a weight γ such that

$$\mathrm{Hom}_G(\mathrm{St}_r \otimes L(\gamma)^{(r)}, \mathcal{G}(k) \otimes L(\widetilde{\nu})^*) = \mathrm{Hom}_G(\mathrm{St}_r \otimes L(\gamma)^{(r)} \otimes L(\widetilde{\nu}), \mathcal{G}(k)) \neq 0.$$

The G-module $\mathrm{St}_r \otimes L(\gamma)^{(r)}$ is simple with highest weight $(p^r-1)\rho + p^r\gamma$. This weight has to be less than or equal to any highest weight μ of $\mathcal{G}(k) \otimes L(\widetilde{\nu})^*$. But any such μ satisfies $\langle \mu + \rho, \alpha_0^\vee\rangle < 3p^r(h-1)$. Hence, $\langle \gamma, \alpha_0^\vee\rangle < 2(h-1)$.

Since $\mathcal{G}(k)$ is the truncation of $\mathrm{ind}_{G_\sigma(\mathbb{F}_q)}^G(k)$ it follows that

$$\mathrm{Hom}_G(\mathrm{St}_r \otimes L(\gamma)^{(r)} \otimes L(\widetilde{\nu}), \mathrm{Ind}_{G_\sigma(\mathbb{F}_q)}^G(k)) \neq 0.$$

and by adjointness

$$\mathrm{Hom}_{G_\sigma(\mathbb{F}_q)}(\mathrm{St}_r \otimes L(\gamma)^{(r)} \otimes L(\widetilde{\nu}), k) \neq 0.$$

As a $G_\sigma(\mathbb{F}_q)$-module $L(\gamma)^{(r)} \cong L(\sigma(\gamma))$ and one obtains

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(L(\sigma(\gamma)) \otimes L(\widetilde{\nu}), St_r) \neq 0.$$

It follows from [Jan2, Satz 1.5] that

$$\dim \mathrm{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\sigma(\gamma)) \otimes L(\widetilde{\nu}), \mathrm{St}_r) = [L(\sigma(\gamma)) \otimes L(\widetilde{\nu}) : \mathrm{St}_r]_{G_\sigma(\mathbb{F}_q)}$$

$$= \sum_{\mu \in X(T)_+} [L(\sigma(\gamma)) \otimes L(\widetilde{\nu}) \otimes L(\sigma(\mu)) : \mathrm{St}_r \otimes L(\mu)^{(r)}]_G.$$

Therefore $\sigma(\gamma) \geq (p^r-1)\rho + p^r\mu - \sigma(\mu) - \widetilde{\nu} \geq (p^r-1)\rho - \widetilde{\nu} = -w_0\nu_0$. Since σ preserves the inner product one obtains $2(h-1) > \langle \gamma, \alpha_0^\vee\rangle \geq \langle \nu_0, \alpha_0^\vee\rangle$. □

2.6 An immediate consequence of the previous theorem and proposition is that any extension between simple modules of the finite group has to come from some extension between a finite set of modules over the algebraic group. For large primes this will be made more explicit later.

Corollary . *Let* $\lambda, \mu \in X_r(T)$. *If* $\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \neq 0$ *then there exist weights* ν_0, $\nu_1 \in \Gamma_{2(h-1)}$ *such that* $\mathrm{Ext}^1_G(L(\lambda) \otimes L(\nu_0), L(\mu) \otimes L(\nu_1)^{(r)}) \neq 0$.

Proof. By Theorem 2.3 we have $\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu) \otimes \mathcal{G}(k))$. In the previous proposition it was shown that $\mathcal{G}(k)$ has a filtration with factors of the form $L(\gamma + p^r\nu_1)$ where $\gamma, \nu_1 \in \Gamma_{2(h-1)}$. If $\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \neq 0$ then there exist γ, $\nu_1 \in \Gamma_{2(h-1)}$ such that $\mathrm{Ext}^1_G(L(\lambda), L(\mu) \otimes L(\gamma) \otimes L(\nu_1)^{(r)}) \cong \mathrm{Ext}^1_G(L(\lambda) \otimes L(-w_0\gamma), L(\mu) \otimes L(\nu_1)^{(r)}) \neq 0$. Set $\nu_0 = -w_0\gamma$ and the assertion follows. ◻

2.7 Proposition 2.5 gives useful information only for $q > 2(h-1)$. In that case we are also able to describe the G-socle of $\mathcal{G}(k)$.

Lemma . *Let* $p^r \geq 2(h-1)$ *then*

$$\mathrm{soc}_G\, \mathcal{G}(k) = \bigoplus_{\nu \in \Gamma_{2(h-1)}} L(\sigma(\nu) + p^r(-w_0\nu)).$$

Proof. The condition $p^r \geq 2(h-1)$ forces any weight in $\Gamma_{2(h-1)}$ to be p^r-restricted. Using Proposition 2.5 and adjointness we can argue as follows

$$\begin{aligned}
&\mathrm{soc}_G\, \mathcal{G}(k)\\
&= \bigoplus_{\nu_0,\nu_1 \in \Gamma_{2(h-1)}} \mathrm{Hom}_G(L(\nu_0 + p^r\nu_1), \mathcal{G}(k)) \otimes L(\nu_0 + p^r\nu_1)\\
&= \bigoplus_{\nu_0,\nu_1 \in \Gamma_{2(h-1)}} \mathrm{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\nu_0) \otimes L(\nu_1)^{(r)}, k) \otimes L(\nu_0 + p^r\nu_1) \text{ (by adjointness)}\\
&= \bigoplus_{\nu_0,\nu_1 \in \Gamma_{2(h-1)}} \mathrm{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\nu_0) \otimes L(\sigma(\nu_1)), k) \otimes L(\nu_0 + p^r\nu_1) \text{ (by (1.4.1))}\\
&= \bigoplus_{\nu_0,\nu_1 \in \Gamma_{2(h-1)}} \mathrm{Hom}_{G_\sigma(\mathbb{F}_q)}(L(\nu_0), L(-w_0\sigma(\nu_1))) \otimes L(\nu_0 + p^r\nu_1)\\
&= \bigoplus_{\nu \in \Gamma_{2(h-1)}} L(-w_0\sigma(\nu) + p^r\nu)\\
&= \bigoplus_{\nu \in \Gamma_{2(h-1)}} L(\sigma(\nu) + p^r(-w_0\nu)).
\end{aligned}$$

◻

2.8 The nicest and most explicit extension results can be obtained when $\mathcal{G}(k)$ is semisimple. The semisimplicity of $\mathcal{G}(k)$ was shown for the Chevalley groups in [BNP1, 7.4]. We present a slightly different proof that involves only the algebraic group and is independent of the finite group, twisted or untwisted. This proof has the advantage that it does not rely on the existence of a G-structure on projective indecomposable G_r-modules (for $p \geq 2(h-1)$). For technical reasons we have to exclude type A_1.

Proposition . *Let $p \geq 3(h-1)$, then $\mathcal{G}(k)$ is semisimple. Moreover,*

$$\mathcal{G}(k) = \bigoplus_{\nu \in \Gamma_{2(h-1)}} L(\sigma(\nu) + p^r(-w_0\nu)).$$

Proof. For the untwisted groups this was shown in [BNP1, Thm 7.4]. We may therefore assume that G is not of type A_1. Assume further that $L(\lambda_0 + p^r\lambda_1)$ and $L(\nu_0 + p^r\nu_1)$ are composition factors of $\mathcal{G}(k)$. We look at the first three terms of the five-term exact sequence corresponding to the Lyndon-Hochschild-Serre spectral sequence [Jan1, I.6.6] .

$$\begin{aligned}
0 \quad &\to \quad \mathrm{Ext}^1_{G/G_1}(L(\lambda_1)^{(1)}, \mathrm{Hom}_{G_1}(L(\lambda_0), L(\nu_0)) \otimes L(\nu_1)^{(1)}) \\
&\to \quad \mathrm{Ext}^1_G(L(\lambda_0 + p^r\lambda_1), L(\nu_0 + p^r\nu_1)) \\
&\to \quad \mathrm{Hom}_{G/G_1}(L(\lambda_1)^{(1)}, \mathrm{Ext}^1_{G_1}(L(\lambda_0), L(\nu_0)) \otimes L(\nu_1)^{(1)}) \\
&\to \quad \ldots .
\end{aligned}$$

By Proposition 2.5 we have $\lambda_0, \lambda_1, \nu_0, \nu_1 \in \Gamma_{2(h-1)}$. The size of p forces all these weights to be inside the lowest alcove. Clearly we have $\mathrm{Hom}_{G_1}(L(\lambda_0), L(\nu_0)) \cong \mathrm{Hom}_G(L(\lambda_0), L(\nu_0))$ and by the Linkage Principle $\mathrm{Ext}^1_G(L(\lambda_1), L(\nu_1)) = 0$. Therefore, the first term of the sequence vanishes. Since λ_0 and ν_0 are inside the lowest alcove it follows from [BNP3, 5.4 Cor B] that $\mathrm{Ext}^1_{G_1}(L(\lambda_0), L(\nu_0)) = 0$. Thus the third term in the sequence also vanishes. We conclude that there are no extensions between simple composition factors of $\mathcal{G}(k)$. The module is semisimple and the assertion follows from Lemma 2.7. □

2.9 The semisimplicity of $\mathcal{G}(k)$ for $p \geq 3(h-1)$ gives us an extremely useful Ext^1-formula for simple $G_\sigma(\mathbb{F}_q)$-modules. For the Chevalley groups this was shown in [BNP2, 2.5].

Theorem . *For $p \geq 3(h-1)$ and $\lambda, \mu \in X_r(T)$,*

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \Gamma_h} \mathrm{Ext}^1_G(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\sigma(\nu))).$$

Proof. For $\lambda, \mu \in X_r(T)$, we have

$$
\begin{aligned}
&\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu))\\
&\quad\cong \mathrm{Ext}^1_G(L(\lambda), L(\mu)\otimes \mathcal{G}(k)) \text{ (by Thm. 2.3)}\\
&\quad\cong \bigoplus_{\nu\in\Gamma_{2(h-1)}} \mathrm{Ext}^1_G(L(\lambda), L(\mu)\otimes L(\sigma(\nu)+p^r(-w_0\nu))) \text{ (by Prop. 2.8)}\\
&\quad\cong \bigoplus_{\nu\in\Gamma_{2(h-1)}} \mathrm{Ext}^1_G(L(\lambda)\otimes L(\nu)^{(r)}, L(\mu)\otimes L(\sigma(\nu)))\\
&\quad\cong \bigoplus_{\nu\in\Gamma_{h}} \mathrm{Ext}^1_G(L(\lambda)\otimes L(\nu)^{(r)}, L(\mu)\otimes L(\sigma(\nu))) \text{ (by [BNP2, Prop. 2.4c]).}
\end{aligned}
$$

□

3. Extensions for $r \geq 2$

The formula given in the previous theorem involves extensions between a simple p^r-bounded G-module and the tensor product of two simple p^r-restricted G-modules, one of them having a small highest weight. It turns out that for $r \geq 2$ the extension and the tensor product can be regarded separately. This allows for even more explicit formulas.

3.1 For $\gamma, \delta \in X(T)_+$, let $S(\gamma,\delta) = \mathrm{Hom}_G(L(\gamma), L(\delta))$. Also, for $r \geq 3$, if $\lambda \in X_r(T)$ and $\lambda = \sum_{i=0}^{r-1} p^i\lambda_i$ where $\lambda_i \in X_1(T)$, set $\ddot{\lambda} = \sum_{i=1}^{r-2} p^{i-1}\lambda_i$ and for $r = 2$ set $\ddot{\lambda} = 0$. The following theorem is the equivalent of [BNP3, Thm. 2.4b]. It follows from Theorem 2.9 by reidentifying the G-extensions as done in [BNP3] using [BNP2, Thm. 3.2] (equivalently [BNP3, Thm. 2.1]) and [BNP3, Prop. 2.3].

Theorem . *Let $p \geq 3(h-1)$, $r \geq 2$, and $\lambda, \mu \in X_r(T)$. Then*

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \oplus R$$

where

$$
\begin{aligned}
R = \bigoplus_{\nu\in\Gamma_h-\{0\}} \quad & \mathrm{Ext}^1_G(L(\lambda_{r-1})\otimes L(\nu)^{(1)}, L(\mu_{r-1}))\\
\otimes\quad & \mathrm{Hom}_G(L(\lambda_0), L(\mu_0)\otimes L(\sigma(\nu)))\otimes S(\ddot{\lambda},\ddot{\mu}).
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
R = \bigoplus_{\nu\in\Gamma_h-\{0\}} \quad & \mathrm{Hom}_G(L(\nu)^{(1)}, \mathrm{Ext}^1_{G_r}(L(\lambda_{r-1}), L(\mu_{r-1}))\\
\otimes\quad & \mathrm{Hom}_G(L(\lambda_0), L(\mu_0)\otimes L(\sigma(\nu)))\otimes S(\ddot{\lambda},\ddot{\mu}).
\end{aligned}
$$

3.2 The following theorem, which is an analog of [BNP3, Thm. 3.1], identifies the extensions between a pair of simple $G(\mathbb{F}_q)$-modules with the G-extensions of a single pair of p^r-restricted simple G-modules. Note that if $n = r-1$, then the weights $\widetilde{\lambda}$ and $\widetilde{\mu}$ defined below simply equal λ and μ respectively.

Theorem . *Let $p \geq 3(h-1)$ and $\lambda, \mu \in X_r(T)$. Set $n = \min(\{i \mid \lambda_i = \mu_i\} \cup \{r-1\})$ and define p^r-restricted weights*

$$\widetilde{\lambda} = \sum_{i=0}^{r-n-2} p^i \sigma(\lambda_{n+1+i}) + \sum_{i=r-n-1}^{r-1} p^i \lambda_{i-r+n+1}, \quad \text{and}$$
$$\widetilde{\mu} = \sum_{i=0}^{r-n-2} p^i \sigma(\mu_{n+1+i}) + \sum_{i=r-n-1}^{r-1} p^i \mu_{i-r+n+1}.$$

(a) If $r \geq 3$, then

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\widetilde{\lambda}), L(\widetilde{\mu})).$$

(b) If $r = 2$ and $\lambda_0 = \mu_0$ or $\lambda_1 = \mu_1$, then

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q))}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\widetilde{\lambda}), L(\widetilde{\mu})).$$

(c) If $r = 2$ and $\lambda_i \neq \mu_i$ for $i = 0, 1$, set $\hat{\lambda} = \sigma(\lambda_1) + p\lambda_0$ and $\hat{\mu} = \sigma(\mu_1) + p\mu_0$. Then

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q))}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \oplus \mathrm{Ext}^1_G(L(\hat{\lambda}), L(\hat{\mu})).$$

Proof. The Frobenius morphism is an automorphism on $G_\sigma(\mathbb{F}_q)$. Hence,

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda)^{(r-1-n)}, L(\mu)^{(r-1-n)}),$$

where $n = \min(\{i \mid \lambda_i = \mu_i\} \cup \{r-1\})$. Moreover, $L(\widetilde{\lambda}) \cong L(\lambda)^{(r-1-n)}$ and $L(\widetilde{\mu}) \cong L(\mu)^{(r-1-n)}$ as $G_\sigma(\mathbb{F}_q)$-modules. Therefore, we may identify

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\widetilde{\lambda}), L(\widetilde{\mu})).$$

(a) For $r \geq 3$, Theorem 3.1 and [BNP3, Thm. 2.4a] imply that $\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) = \mathrm{Ext}^1_G(L(\widetilde{\lambda}), L(\widetilde{\mu})) = 0$ unless there exists an i with $0 \leq i \leq r-1$ and $\lambda_i = \mu_i$. We assume such an i exists and apply Theorem 3.1 to the pair of weights $\widetilde{\lambda}$ and $\widetilde{\mu}$. Observe that $\widetilde{\lambda}_{r-1} = \widetilde{\mu}_{r-1}$ which forces $\mathrm{Ext}^1_{G_1}(L(\widetilde{\lambda}_{r-1}), L(\widetilde{\mu}_{r-1})) = 0$ by [And1]. The remainder term R in Theorem 3.1 vanishes and

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\widetilde{\lambda}), L(\widetilde{\mu})) \cong \mathrm{Ext}^1_G(L(\widetilde{\lambda}), L(\widetilde{\mu})).$$

This proves part (a).

(b) Assume $r = 2$. If either $\lambda_0 = \mu_0$ or $\lambda_1 = \mu_1$, then $\widetilde{\lambda}_{r-1} = \widetilde{\mu}_{r-1}$ and one can argue as above.

(c) Assume $r = 2$ and $\lambda_i \neq \mu_i$ for $i = 0, 1$. By Theorem 3.1, we have

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \oplus R$$

where

$$R=\bigoplus_{\nu\in\Gamma_h-\{0\}}\operatorname{Ext}^1_G(L(\lambda_1)\otimes L(\nu)^{(1)},L(\mu_1))\otimes\operatorname{Hom}_G(L(\lambda_0),L(\mu_0)\otimes L(\sigma(\nu))).$$

Since $\lambda_0\neq\mu_0$, we have

$$R=\bigoplus_{\nu\in\Gamma_h}\operatorname{Ext}^1_G(L(\lambda_1)\otimes L(\nu)^{(1)},L(\mu_1))\otimes\operatorname{Hom}_G(L(\lambda_0),L(\mu_0)\otimes L(\sigma(\nu))).$$

Since σ is an automorphism on G, we have

$$\operatorname{Ext}^1_G(L(\lambda_1)\otimes L(\nu)^{(1)},L(\mu_1))\cong\operatorname{Ext}^1_G(L(\sigma(\lambda_1))\otimes L(\sigma(\nu))^{(1)},L(\sigma(\mu_1))).$$

We can re-write R as

$$R=\bigoplus_{\nu\in\Gamma_h}\operatorname{Ext}^1_G(L(\sigma(\lambda_1))\otimes L(\nu)^{(1)},L(\sigma(\mu_1)))\otimes\operatorname{Hom}_G(L(\lambda_0),L(\mu_0)\otimes L(\nu)).$$

By [BNP3, Thm. 2.4a], the remainder term R is exactly $\operatorname{Ext}^1_G(L(\hat{\lambda}),L(\hat{\mu}))$. □

3.3 Various conjectures have been made about the dimensions of Ext^1-groups. The following shows that in most cases the dimensions of the Ext^1 groups between simple modules of the finite groups are bounded by the dimensions of Ext^1 groups for the reductive algebraic groups. The corollary below is a generalization of [BNP3, Thm. 3.3]. The proof follows along the same lines and details are left to the reader.

Corollary . *Let $p\geq 3(h-1)$, $r\geq 2$, and $\lambda,\mu\in X_r(T)$. Then*

$$\max\{\dim_k\operatorname{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda),L(\mu))\mid\lambda,\mu\in X_r(T)\}=\max\{\dim_k\operatorname{Ext}^1_G(L(\lambda),L(\mu))\mid\lambda,\mu\in X_r(T)\},$$

unless, $r=2$, $\sigma=1$, and the underlying root system is of type $C_n, n\geq 1$. In that case

$$\max\{\dim_k\operatorname{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda),L(\mu))\mid\lambda,\mu\in X_r(T)\}\leq 2\max\{\dim_k\operatorname{Ext}^1_G(L(\lambda),L(\mu))\mid\lambda,\mu\in X_r(T)\}.$$

4. Self-Extensions

Let $p\geq 3(h-1)$. In [BNP2] it was shown that self-extensions, i.e. extensions of a simple $G(\mathbb{F}_q)$-module by itself, do not exist unless $q=p$ and the root system of the underlying algebraic group is of type C_n. Since there are no twisted groups of type C_n it is not surprising that the result generalizes to the twisted case.

4.1 Self-extensions, $r \geq 2$: In Theorem 3.2 it was shown that there is a nice one-to-one correspondence between extensions of simple modules for the finite group and extensions of p^r-restricted simple modules for the algebraic group. Since it is well-known that self-extensions do not exist for algebraic groups it is an immediate consequence of Theorem 3.2 that self-extensions also vanish for simple $G_\sigma(\mathbb{F}_q)$-modules as long as $q > p$ and $p \geq 3(h-1)$.

Theorem . *For $p \geq 3(h-1)$, $r \geq 2$, and $\lambda \in X_r(T)$, $\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_q)}(L(\lambda), L(\lambda)) = 0$.*

4.2 Self-extensions, $r = 1$: In the case $p = q$ the analysis in [BNP2] was considerably harder since self-extensions were known to exist for groups of type C_n [Hum], [TZ, Rem. 3.18], [Pil]. It turns out, however, that the twisted groups do not add to this list. One obtains the following result.

Theorem . *Let $p \geq 3(h-1)$ and $\lambda \in X_1(T)$. If either*

(a) G does not have underlying root system of type C_n $(n \geq 1)$ or

(b) $\langle \lambda, \alpha_n^\vee \rangle \neq \frac{p-2-c}{2}$, where α_n is the unique long simple root and c is odd with $0 < |c| \leq h-1$,

then $\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_p)}(L(\lambda), L(\lambda)) = 0$.

Proof. By Theorem 2.9 one has

$$\mathrm{Ext}^1_{G_\sigma(\mathbb{F}_p)}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \Gamma_h} \mathrm{Ext}^1_G(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\sigma(\nu))),$$

where $\Gamma_h = \{\nu \in X(T)_+ | \langle \nu, \alpha_0^\vee \rangle < h\}$. The only complication in the twisted case versus the split case is that the weights ν and $\sigma(\nu)$ may be distinct. However, Corollary 4.9 in [BNP2] says that $\mathrm{Ext}^1_G(L(\lambda) \otimes L(\nu_1)^{(r)}, L(\mu) \otimes L(\sigma(\nu_2)))$ vanishes for any $\nu_1, \nu_2 \in \Gamma_h$ as long (a) or (b) holds. □

5. First Cohomology for Simple Modules

In this section we consolidate our results on the first cohomology for simple groups that can be found in [BNP1] [BNP3] and extend them to the twisted groups. Notice that Theorem 5.1 offers some improvements on the condition of the prime (compare to [BNP1, Thm. 7.4]) and the size of the weights (Γ_{h-1} versus Γ_h). For technical reasons we exclude the well-known results for groups of type A_1 [AJL]. The most striking result is a one-to-one correspondence between p^r-bounded simple G-modules with non-vanishing cohomology and simple $G_\sigma(\mathbb{F}_q)$-modules with non-zero cohomology. The correspondence is simply the identity for p^r-restricted G-modules, while non-restricted G-modules correspond to simple $G_\sigma(\mathbb{F}_q)$-modules whose p^r-restricted highest weights are not linked to 0. Naturally, this gives also a one-to-one correspondence between the cohomology of simple modules for the quasi-split and the split forms over a given G.

5.1 By $\widetilde{W}_p$ we denote the affine Weyl group generated by the ordinary Weyl group and the translation by elements in $pX(T)$. Two weights λ and ν are said to be G_1-*linked* if $\lambda = w \cdot \nu + p\gamma$ for some $w \in W$ and $\gamma \in X(T)$ or, equivalently, if there exists an element $u \in \widetilde{W}_p$ such that $\lambda = u \cdot \nu$. In this section we make use of the strong linkage principle [Jan1, II.6.13] and replace the order relation "$\leq$" on $X(T)$ by the stronger relation "$\uparrow$" as defined in [Jan1, II.6.4].

Theorem . *Let $p \geq 3(h-1)$, $\lambda \in X_r(T)$, and $\Gamma_{h-1} = \{\nu \in X(T)_+ \mid \langle \nu, \alpha_0^\vee \rangle < h-1\}$. Assume that G is not of type A_1.*

(a) If λ is not G_1-linked to any weight in Γ_{h-1}, then $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) = 0$.

(b) If λ is G_1-linked to the zero weight, then $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{H}^1(G, L(\lambda))$.

(c) If $r = 1$ and λ is G_1-linked to some element in Γ_{h-1}, then there exist unique $\nu \in \Gamma_{h-1}$ and $u \in \widetilde{W}_p$ such that $\lambda = u \cdot \nu$ and $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{H}^1(G, L(u \cdot 0 + p\sigma^{-1}(\nu)))$.

In addition, if $p\sigma^{-1}(\nu) \not\uparrow u \cdot 0$ then

$$\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) = \begin{cases} k & \text{if } \lambda = p\omega_\alpha + s_\alpha \cdot (-w_0\sigma(\omega_\alpha)) \text{ where } \alpha \in \Delta \\ & \text{and } \omega_\alpha \text{ is the corresponding fundamental weight,} \\ 0 & \text{else.} \end{cases}$$

(d) If $r > 1$, then $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) = 0$ unless there exist unique $\nu \in \Gamma_{h-1}$ and $u \in \widetilde{W}_p$ such that $\lambda = p^{r-1}u \cdot 0 + \nu$. In that case $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{H}^1(G, L(u \cdot 0 + p\sigma^{-1}(\nu)))$.

In addition, if $p\sigma^{-1}(\nu) \not\uparrow u \cdot 0$ then

$$\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) = \begin{cases} k & \text{if } \lambda = p^r\omega_\alpha - p^{r-1}\alpha - w_0\sigma(\omega_\alpha) \text{ where } \alpha \in \Delta \\ & \text{and } \omega_\alpha \text{ is the corresponding fundamental weight,} \\ 0 & \text{else.} \end{cases}$$

Proof. Since $p \geq 3(h-1)$ it follows that any $\nu \in \Gamma_h$ is in the interior of the lowest alcove. Consequently, from Jantzen's translation principle [Jan1, II. 7.9],

$L(\nu) \cong T_0^{\nu}(k)$. One can now argue as follows:

$$\begin{aligned}
\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) &\cong \mathrm{Ext}^1_{G(\mathbb{F}_q)}(k, L(\lambda)) \\
&\cong \bigoplus_{\nu\in\Gamma_h} \mathrm{Ext}^1_G(L(\nu)^{(r)}, L(\lambda)\otimes L(\sigma(\nu))) \quad \text{(by Theorem 2.9)} \\
&\cong \bigoplus_{\nu\in\Gamma_h} \mathrm{Ext}^1_G(L(-w_0\sigma(\nu)), L(\lambda)\otimes L(-w_0\nu)^{(r)}) \\
&\cong \bigoplus_{\nu\in\Gamma_h} \mathrm{Ext}^1_G(L(\nu), L(\lambda)\otimes L(\sigma^{-1}(\nu))^{(r)}) \\
&\cong \bigoplus_{\nu\in\Gamma_h} \mathrm{Ext}^1_G(T_0^{\nu}(k), L(\lambda + p^r\sigma^{-1}(\nu))) \\
&\cong \bigoplus_{\nu\in\Gamma_h} \mathrm{Ext}^1_G(k, T_{\nu}^0 L(\lambda + p^r\sigma^{-1}(\nu))) \quad \text{([Jan1, II.7.6])}
\end{aligned}$$

Assume that $T_\nu^0 L(\lambda + p^r\sigma^{-1}(\nu))$ is not zero and denote by λ_ν the unique p^r-restricted weight such that $L(\lambda_\nu + p^r\sigma^{-1}(\nu)) \cong T_\nu^0 L(\lambda + p^r\sigma^{-1}(\nu))$. We have

$$\begin{aligned}
\mathrm{Ext}^1_G(k, T_\nu^0 L(\lambda + p^r\sigma^{-1}(\nu))) &\cong \mathrm{Ext}^1_G(k, L(\lambda_\nu + p^r\sigma^{-1}(\nu))) \\
&\cong \mathrm{Ext}^1_G(L(-w_0\sigma^{-1}(\nu))^{(r)}, L(\lambda_\nu)).
\end{aligned}$$

Define Q via the exact sequence of G-modules

$$0 \to L(\lambda_\nu) \to H^0(\lambda_\nu) \to Q \to 0$$

and obtain the exact sequence

$$\begin{aligned}
\mathrm{Hom}_G(L(-w_0\sigma^{-1}(\nu))^{(r)}, Q) &\to \mathrm{Ext}^1_G(L(-w_0\sigma^{-1}(\nu))^{(r)}, L(\lambda_\nu)) \\
&\to \mathrm{Ext}^1_G(L(-w_0\sigma^{-1}(\nu))^{(r)}, H^0(\lambda_\nu)).
\end{aligned}$$

We conclude that $\mathrm{Ext}^1_G(k, T_\nu^0 L(\lambda + p^r\sigma^{-1}(\nu))) = 0$ unless either $-w_0 p^r\sigma^{-1}(\nu) \uparrow \lambda_\nu$ by [Jan1, II.6.6] or $\lambda_\nu = p^r\omega_\alpha - p^i\alpha$ and $-w_0\sigma^{-1}(\nu) = \omega_\alpha$, where α is a simple root and ω_α the corresponding fundamental weight. The last assertion follows from [BNP3, Prop. 4.3b]. In the first case we conclude that $p^r\langle -w_0\sigma^{-1}(\nu), \alpha_0^\vee\rangle = p^r\langle \nu, \alpha_0^\vee\rangle \le (p^r-1)(h-1)$. In either case $\langle \nu, \alpha_0^\vee\rangle < h-1$, unless G is of type A_1. Obviously the cohomology vanishes unless λ is G_1-linked to some $\nu \in \Gamma_{h-1}$. This implies part (a) of the Theorem.

Next assume that λ is G_1 linked to two elements ν_1 and ν_2 in Γ_{h-1}. Then there exist $w \in W$ and $\gamma \in X(T)$ such that $w\cdot\nu_1 + p\gamma = \nu_2$. This implies for any simple root α that $\langle \nu_2 + \rho, \alpha^\vee\rangle - \langle w(\nu_1+\rho), \alpha^\vee\rangle = p\langle\gamma, \alpha^\vee\rangle$. Since $\langle \nu_i, \alpha_0^\vee\rangle \le h-2$, the absolute value of the left hand side is at most $h-2+1+2h-3 = 3(h-1)-1 < p$. Thus $\gamma = 0$, which forces $w = 1$ and $\nu_1 = \nu_2$. Consequently, if λ is G_1-linked to some element in Γ_{h-1}, then there exist unique $u \in \widetilde{W}_p$ and $\nu \in \Gamma_{h-1}$ such that $\lambda = u\cdot\nu$. This implies that $\lambda_\nu = u\cdot 0$ and $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{Ext}^1_G(k, L(u\cdot 0 + p^r\sigma^{-1}(\nu))) \cong \mathrm{H}^1(G, L(u\cdot 0 + p^r\sigma^{-1}(\nu)))$. Now part (b) and the first part of (c) follow immediately.

Assume further that $r = 1$ and $p\sigma^{-1}(\nu) \not\uparrow u\cdot 0$. Then it follows from our earlier observations that the cohomology vanishes unless $u\cdot 0 = p\omega_\alpha - \alpha = p\omega_\alpha + s_\alpha\cdot 0$ for some simple root α and $\nu = -w_0\sigma(\omega_\alpha)$. This forces $\lambda = p\omega_\alpha + s_\alpha \cdot (-w_0\sigma(\omega_\alpha))$ and $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{Ext}^1_G(k, L(p\omega_\alpha - \alpha + p(-w_0\omega_\alpha)) \cong \mathrm{Ext}^1_G(L(\omega_\alpha)^{(1)}, L(p\omega_\alpha - \alpha))$. The remainder of part (c) now follows from [BNP3, Prop. 4.5].

Finally, we assume that $r > 1$ and that λ is G_1-linked to some non-zero $\nu \in \Gamma_{h-1}$. Then ν is unique, $\lambda = w \cdot \nu$ with unique $w \in \widetilde{W}_p$, and $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{H}^1(G, L(w\cdot 0 + p^r\sigma^{-1}(\nu)))$. It follows from [BNP3, Thm. 2.4a] that $\mathrm{H}^1(G, L(w\cdot 0 + p^r\sigma^{-1}(\nu))) = 0$, unless $w \cdot 0 = p^{r-1}u \cdot 0$ for a unique $u \in \widetilde{W}_p$. In this case [BNP3, Thm. 2.4a] implies further that $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \cong \mathrm{H}^1(G, L(w \cdot 0 + p^r\sigma^{-1}(\nu))) \cong \mathrm{H}^1(G, L(u \cdot 0 + p\sigma^{-1}(\nu)))$. Moreover, $\lambda = \nu + p^{r-1}u \cdot 0$.

Now assume that $p\sigma^{-1}(\nu) \not\uparrow u \cdot 0$. Again it follows from [BNP3, Prop. 4.5] that $\mathrm{H}^1(G, L(u \cdot 0 + p\sigma^{-1}(\nu))) = k$, if $u \cdot 0 = p\omega_\alpha - \alpha$ for some simple root α and $\nu = -w_0\sigma(\omega_\alpha)$. This implies that $\lambda = p^r\omega_\alpha - p^{r-1}\alpha - w_0\sigma(\omega_\alpha)$. In all other cases the cohomology vanishes. □

The modules $L(p\omega_\alpha + s_\alpha \cdot (-w_0\sigma(\omega_\alpha)))$ and $L(p^r\omega_\alpha - p^{r-1}\alpha - w_0\sigma(\omega_\alpha))$ are of interest because for appropriate choices of α they are the smallest simple modules with $\mathrm{H}^1(G_\sigma(\mathbb{F}_p), L(\lambda)) \neq 0$ and $\mathrm{H}^1(G_\sigma(\mathbb{F}_q), L(\lambda)) \not\cong \mathrm{H}^1(G, L(\lambda))$, respectively (see [Jan4, Prop. 2.2], [BNP3, 4.8]).

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