Math 237   Final Exam   Review

Final Exam will be on **Friday, December 16, 8-10 a.m.**

The exam will be cumulative, it will cover Sections 1.1-1.5, 2.1-2.4, 2.7, 3.1-3.4, 4.1-4.3, 5.1-5.4, and 6.2. The material of Sections 5.1-5.4, and 6.2 will be emphasized.

No calculators, books, or notes, except for three 3″ by 5″ note cards, are allowed on the test.

The exam will contain 10 True/False questions and a number of problems where calculations and/or explanations are required. The problems on the test will be similar to the review problems below. If you are not comfortable with a review problem, go over similar Hw problems. Also, review the definitions and properties discussed in the course.

A complete list of Hw assignments is available at the course home page

`http://www.southalabama.edu/mathstat/personal_pages/sadovska/237/237.html`

Review problems:

1. Let $\mathbf{u} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$.

(a) Find the length of $\mathbf{u}$.

(b) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

(c) Find the projection of $\mathbf{u}$ on $\mathbf{v}$.

(d) Write a vector equation of the line through $P(1, 2, 3)$ in the direction of $\mathbf{u}$.

(e) Is it true that $\mathbf{u}$ and $\mathbf{v}$ span a plane in $\mathbb{R}^3$?

(f) Is it true that $\mathbf{u}$ is orthogonal to $\mathbf{w} = \begin{bmatrix} -1 \\ -7 \\ 5 \end{bmatrix}$?

2. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}$. Find AB^T and $A^{-1}(B^{-1})^T(AB)^T$ (simplify first).

3. For each of the following matrices, find its determinant.
If a matrix is invertible, find the inverse.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$,   (b) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,   (c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 0 & 3 & 3 \end{bmatrix}$.

4. $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) For each eigenvalue, find the corresponding eigenspace.

(d) Is A diagonalizable? If it is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If it is not, explain why not.

5. (a) Are the vectors $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ linearly independent?

(b) Find a basis and the dimension of their span. (c) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ in their span?

6. $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 1 \end{bmatrix}$.

(a) Find the rank of A.

(b) Find the null space of A.

(c) Find a basis of the column space of A.

(d) Left multiplication by A defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. What is n and what is m?

7. Is U a subspace of $\mathbb{R}^3$? Justify your answer.

(a) $U = \left\{ \begin{bmatrix} x \\ y \\ x - y - 2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$ (b) $U = \left\{ \begin{bmatrix} x \\ x \\ 3x \end{bmatrix} \mid x \in \mathbb{R} \right\}$

8. Determine whether each of the following functions is a linear transformation.
Justify your answers.

(a) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 0 \end{bmatrix}$, (b) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+z \\ xy \end{bmatrix}$.

9. Does there exist a linear transformation T such that

(a) $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, and $T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$?

(b) $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} -6 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$?

Justify your answers.

10. Let $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ 3x_1 + x_4 \end{bmatrix}$

(a) $T : \mathbb{R}^n \to R^m$. What is n and what is m?

(b) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(c) Find a basis for the kernel of T.

(d) Find a basis for the range of T.

(e) Verify that dim (Ker T) + dim (Range T)= n.

11. Let $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3y \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2z \\ x-y \end{bmatrix}$.

(a) Is ST defined? Is TS defined?

(b) Find $ST\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

(c) Find the matrix A of S, the matrix B of T, the matrix C of ST, and verify that $C = AB$.

12. Let $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ y \\ z \end{bmatrix}$. Is T one-to-one? Is T onto? Is T invertible?

13. Let $\mathcal{V} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{W} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

(a) Let $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Find $x_{\mathcal{V}}$ and $x_{\mathcal{W}}$.

(b) Find $P_{\mathcal{W} \leftarrow \mathcal{V}}$ and $P_{\mathcal{W} \leftarrow \mathcal{V}}$.

(c) If $\mathbf{x}_{\mathcal{W}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, use $P_{\mathcal{V} \leftarrow \mathcal{W}}$ to find $\mathbf{x}_{\mathcal{V}}$.

14. Let $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$,

let $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$,

and let $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \\ y + z \end{bmatrix}$.

Find $M_{\mathcal{E} \leftarrow \mathcal{E}}(T)$, $M_{\mathcal{E} \leftarrow \mathcal{V}}(T)$, $M_{\mathcal{V} \leftarrow \mathcal{E}}(T)$, $M_{\mathcal{V} \leftarrow \mathcal{V}}(T)$, and $M_{\mathcal{W} \leftarrow \mathcal{V}}(T)$.

15. Let $U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Apply the Gram-Schmidt algorithm to find an orthogonal basis of U.