Select problems from 2004, 2005, and 2006 Mathematics Contests

We will discuss these problems on Thursday, April 5, 2-3 p.m. in ILB 430.

1. a and b are two positive numbers such that $a + b = 1$. Show that $a^2 + b^2 \geq \frac{1}{2}$.

2. There 6 men and 6 women in a dance class. How many ways are there to divide them into 6 (man-woman) pairs?

3. Is it possible to put numbers 0, 1, and -1 into a 4 by 4 table in such a way that the sums of the numbers in the columns, the rows, and the two main diagonals are all different?

4. A car is speeding along a highway at 35 m/s. The moment it passes a police car standing at the side of the road, the police car starts chasing it. The police car accelerates for 10 seconds with constant acceleration of 4 m/s^2, and then continues at a constant speed. The speeding car keeps going at 35 m/s.

   How long will it take for the police car to catch up with the speeding car?

5. Find the sum of all angles of a convex n-sided polygon.

6. Find the limit $\lim_{x \to \pi} \tan^{-1}(\ln(\sin^2 x))$.

7. Consider the function
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   (a) Is this function continuous at $x = 0$?

   (b) Is this function differentiable at $x = 0$?

8. Prove that for any integer $n \geq 1$, $\quad \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} \leq \ln 2$.

9. Let f be a continuous non-negative function on $[0, \infty)$ such that $\int_0^\infty f(x)\,dx$ converges. Is it true that $f(x) \to 0$ as $x \to \infty$?

   If you think that this is true, give a proof. If you think that this is not true, give an example of a function that satisfies the assumptions but does not tend to 0 as $x \to \infty$.