Knot Group Epimorphisms

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Abstract: Let G be a finitely generated group, and let $\lambda \in G$. If there exists a knot k such that $\pi k = \pi_1(S^3 \setminus k)$ can be mapped onto G sending the longitude to λ, then there exists infinitely many distinct prime knots with the property. Consequently, if πk is the group of any knot (possibly composite), then there exists an infinite number of prime knots $k_1, k_2, \cdots$ and epimorphisms $\cdots \to \pi k_2 \to \pi k_1 \to \pi k$ each perserving peripheral structures. Properties of a related partial order on knots are discussed.

1. Introduction. Suppose that $\phi : G_1 \to G_2$ is an epimorphism of knot groups preserving peripheral structure (see §2). We are motivated by the following questions.

Question 1.1. If G_1 is the group of a prime knot, can G_2 be other than G_1 or $\mathbf{Z}$?

Question 1.2. If G_2 can be something else, can it be the group of a composite knot?

Since the group of a composite knot is an amalgamated product of the groups of the factor knots, one might expect the answer to Question 1.1 to be no. Surprisingly, the answer to both questions is yes, as we will see in §2.

These considerations suggest a natural partial ordering on knots: $k_1 \geq k_2$ if the group of k_1 maps onto the group of k_2 preserving peripheral structure. We study the relation in §3.

2. Main result. As usual a knot is the image of a smooth embedding of a circle in S^3. Two knots are equivalent if they have the same *knot type*, that is, there exists an autohomeomorphism of S^3 taking one knot to the other.

Let k be a knot in S^3. We denote its group $\pi_1(S^3 \setminus \mathrm{int}V, *)$ by πk. Here $V \cong k \times D^2$ is a tubular neighborhood of k, and the basepoint $*$ is chosen on the boundary $\partial V \cong k \times S^1$. The element m represented by an essential simple closed curve in ∂V that is contractible in V is called a *meridian*; the element l represented by an essential simple closed curve in ∂V that is nullhomologous in $S^3 \setminus \mathrm{int}V$ is called a *longitude.* A well-known algorithm enables one to express l in terms of Wirtinger generators corresponding to a diagram for k. Details can be found on page 37 of [BZ85].

The inclusion map $\partial V \hookrightarrow S^3 \setminus \mathrm{int}V$ induces an injection of fundamental groups. Its image is the subgroup $\langle m, l\rangle$ generated by m and l.

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Let $k_i, i = 1, 2$, be knots with meridian-longitude pairs m_i, l_i. A homomorphism $\pi k_1 \to \pi k_2$ *preserves peripheral structure* if the image of $\langle m_1, l_1 \rangle$ is conjugate to a subgroup of $\langle m_2, l_2 \rangle$.

Definition 2.1. (i) k_1 *covers* k_2 (or k_2 *supports* k_1) if there is an epimorphism $\pi k_1 \to \pi k_2$;

(ii) If G is a finitely generated group normally generated by an element μ, and if $\lambda \in G$, then a knot k *covers* (G, μ, λ) (or briefly k *covers* G) if there exists an epimorphism $\phi : (\pi k, m, l) \to (G, \mu, \lambda)$, where (m, l) is a meridian-longitude pair.

If k covers G for a given $\lambda \in G$, then we say (after Johnson and Livingston [JL89]) that k *realizes* λ. For a given group G as above and $\lambda \in G$, [JL89] provides necessary and sufficient conditions for the existence of a knot k that covers (G, μ, λ). We will show that k can always be chosen to be a prime knot.

Theorem 2.2. *Let G be a finitely generated group that is normally generated by a single element μ, and let $\lambda \in G$. If there exists a knot k that realizes λ, then there exists an infinite number of distinct prime knots that realize λ.*

Proof. As the theorem easily follows when k is trivial, we assume that k is knotted. By Proposition 2.5 of [EKT03], we can regard k as the numerator closure T^N of a prime or rational tangle T. (See [EKT03], where the authors provide general terminology and prove an even stronger condition.) Set $C = (1/2)^N$, using Conway notation (Figure 1).

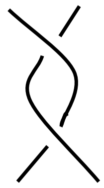

Figure 1: The tangle 1/2

We use a construction of [EKT03] to form a 2-component link $L = T^N \cup (1/2)^N = k \cup C$ (Figure 2). The components k and C are contained in disjoint solid tori V_1 and V_2, respectively, the cores of which form a Hopf link. Note that C is an untwisted double of the core of V_2. By Propositions 2.3 and 2.4 of [EKT03], $L = k \cup C$ is a prime link.

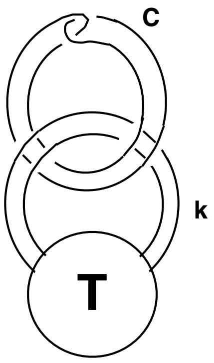

Figure 2: The link L

Let K_q denote the knot obtained from k by $1/q$-surgery on the trivial knot C. We take $q > 0$ for simplicity. It is helpful to view K_q as a satellite of a $(-2q)$-twist knot γ_{2q}. (The notation γ_{2q} here means that the usual regular projection of γ_{2q} has $-2q$ negative half-twists.) From this perspective, it follows immediately that each K_q is nontrivial.

If k is composite, then it follows from Theorem 4.1 of [G-S97] that K_q is prime for $q \geq 3$, since $k \cup C$ is prime. On the other hand, if k is prime and $K_{q'}$ is composite for some q', then K_q will be prime when $q \geq q' + 3$. Thus there exists $d \geq 3$ such that K_q is prime whenever $q \geq d$; cf. Remarks 3.5 (ii).

For a fixed meridian-longitude pair (m, l) of k, we have an epimorphism $\phi : \pi k \to G$ for which $\phi(m) = \mu$ and $\phi(l) = \lambda$. The elements m and l are represented by oriented simple closed curves m_1 and l_1 on the boundary of a tubular neighborhood of k in V_1. Assume that $m_1 \cap l_1$ is a single point $*$, which we take as the basepoint for $\pi(k \cup C), \pi K_q$ and πk, as previously mentioned. The curve m_1 certainly represents a meridian of K_q, and since the linking number of k and C is zero, the curve l_1 represents a longitude (see pages 266–267 of [R76]). Furthermore, if (w, c) is a meridian-longitude pair for C (based at $*$), we have $\pi k = \pi(k \cup C)/\langle\langle w \rangle\rangle$ and $\pi K_q = \pi(k \cup C)/\langle\langle wc^q \rangle\rangle$. (Here and throughout $\langle\langle\ \rangle\rangle$ denotes normal closure.) Thus

$$\pi K_q/\langle\langle w \rangle\rangle = \pi(k \cup C)/\langle\langle w, wc^q \rangle\rangle = \pi k/\langle\langle c^q \rangle\rangle.$$

But $c = 1$ in πk, since C represents the identity element of πk, and so $\pi K_q/\langle\langle w \rangle\rangle = \pi k$. The induced homomorphism $\eta : \pi K_q \to \pi k$ is an epimorphism taking meridian to meridian and longitude to longitude, preserving orientations; cf. Remarks 3.5 (ii). Therefore, $\phi_q = \phi \circ \eta : \pi K_q \to G$ is an epimorphism that maps the classes of m_1 and l_1 to μ and λ, respectively.

Recall that K_q ($q \geq d$ from now on) is a satellite of a $(-2q)$-twist knot γ_{2q}. Since γ_{2q} is the double of the unknot with twisting number $-2q$, it follows from the form of the Alexander polynomial for doubled knots (see p. 136 of [BZ85], for example) that the exteriors $\mathrm{Ext}(\gamma_{2q})$ and $\mathrm{Ext}(\gamma_{2\bar{q}})$ are not homeomorphic if $q \neq \bar{q}$.

Finally, consider the canonical splitting of the exterior E_q of K_q into a union Σ_q of Seifert pieces (the characteristic submanifold of E_q) and a union Λ_q of atoroidal pieces ($\Lambda_q = \mathrm{cl}(E_q \setminus \Sigma_q)$) [JS79], [J76]. Each of Σ_q and Λ_q has a finite number of components, and

$$\mathrm{cl}[(\Sigma_q \cup \Lambda_q) \setminus \mathrm{Ext}(\gamma_{2q})] \cong \mathrm{cl}[(\Sigma_r \cup \Lambda_r) \setminus \mathrm{Ext}(\gamma_{2r})],$$

for $q, r \geq d$. Now choose $p \in \{d, d+1, \ldots\}$ so large that a copy of $\mathrm{Ext}(\gamma_{2q})$ is not a component of $\Lambda_q \setminus \mathrm{Ext}(\gamma_{2q})$, for any $q \geq p$. Thus if $q \neq r$ and $q, r \geq p$, then E_q is not homeomorphic to E_r, since $\mathrm{Ext}(\gamma_{2q})$ is not homeomorphic to $\mathrm{Ext}(\gamma_{2r})$. Therefore, there exist infinitely many distinct prime knots realizing λ. ■

Remarks 2.3. (i) Notice the seemingly large number of choices we have in the above construction of prime knots that realize λ. For example, we might double and redouble C itself.

(ii) To see that the answer to each of Questions 1.1 and 1.2 is yes, see Example 2.6. Less specifically, let G be the group of a composite knot k with meridian-longitude pair (μ, λ), and let k_1 and k_2 be ambient isotopic copies of k. Assume that k is oriented and that each of k_1 and k_2 inherits this orientation. Let (m_i, l_i) be a meridian-longitude pair for k_i ($i = 1, 2$), and let $\phi_i : \pi k_i \to \pi k$ be an isomorphism such that $\phi(m_i) = \mu$ and $\phi_i(l_i) = \lambda$. Then ϕ_1 and ϕ_2 induce an epimorphism $\phi : \pi(k_1 \sharp k_2) \to G$ such that $\phi(m) = \mu$ and $\phi(l) = \lambda^2$, for some choice of meridian-longitude pair (m, l) for $k_1 \sharp k_2$. Thus $k_1 \sharp k_2$ covers (G, μ, λ^2), and so there exist prime knots K_q covering (G, μ, λ^2) by Theorem 2.2.

(iii) In the proof of Theorem 2.2, we have the epimorphism $\eta : \pi K_q \to \pi k$, which preserves both meridian and longitude. By construction [EKT03], k is geometrically essential in the solid torus V_1, and meets a meridianal disk of V_1 transversely in two points. Hence the wrapping number of k in V_1 must be 2 and the winding number of k (after it is given an orientation) in V_1 is either 0 or ± 2. If it is 0, then k and K_q have the same Alexander polynomial; in fact, they have the same Alexander invariant (see [BZ85], for example). If the winding number is ± 2, however, we can replace the tangle T by the vertical sum $T * 1$ (see Figure 3 of [EKT03]). This merely provides a different embedding of k in V_1, one for which the winding number is 0. The new link $k \cup C$ is still prime and the proof of Theorem 2.2 is unchanged. The knot K_q obtained with the tangle $T * 1$ might differ from the original.

Corollary 2.4. Let G be a knot group normally generated by $\mu \in G$, and let $\lambda \in G$. Then there exists an infinite number of prime knots realizing λ if and only if $\lambda \in G'' \cap Z(\mu)$, where G'' is the second commutator subgroup of G and $Z(\mu)$ is the centralizer of μ in G.

Proof. According to the main result, Proposition 1, of [JL89], λ is realizable (by some knot) if and only if $\lambda \in G'' \cap Z(\mu)$, since G is a knot group. But Theorem 2.2 implies that λ is realizable if and only if it is realizable by an infinite number of prime knots. ■

Remarks 2.5. (i) If we take λ to be the longitude of a knot group G_2 with meridian μ, then Corollary 2.4 ensures the existence of a knot group G_1 not isomorphic to G_2 and an epimorphism $\phi : G_1 \to G_2$ that preserves peripheral structure. This provides another answer to Questions 1.1 and 1.2.

(ii) One can avoid Corollary 2.4 by noting that the group of any knot K covers itself by the identity automorphism $(\pi K, m, l) \to (\pi K, m, l)$, and then taking $k = K$ in the construction given in the proof of Theorem 2.2.

Example 2.6. Consider the 2-component link $L = k \cup C$ in Figure 3. Regard the knotted component k as the connected sum $(k_W \sharp k_N)\sharp(k_E \sharp k_S)$, of two Granny knots. (Here W, N, E, S abbreviate west, north, east and south.) As in Remarks 2.3 (ii), there exist isomorphisms $\pi(k_W \sharp k_N) \to \pi(k_W \sharp k_E)$ and $\pi(k_E \sharp k_S) \to \pi(k_W \sharp k_E)$, taking the longitude of each of $\pi(k_W \sharp k_N)$ and $\pi(k_E \sharp k_S)$ to that of $\pi(k_W \sharp k_E)$. Hence there is an epimorphism $\phi : \pi k \to \pi(k_W \sharp k_E)$ that takes a longitude of πk to the square of that of $\pi(k_W \sharp k_E)$. Moreover, ϕ maps the class of C trivially.

Let K_q be the knot resulting from k after $1/q$ surgery on C. As in the proof of Theorem 2.2, the group πK_q admits an epimorphism onto the Granny knot group, sending meridian to meridan, longitude to the square of a longitude

We obtain the conclusion of Theorem 2.2 by another method, one that enables us to obtain hyperbolic knots K_q. Since the link L is prime and alternating, a theorem of W. Menasco [Me84] implies that L is hyperbolic. Results of W. Thurston [Th77/83] and W. Neumann and D. Zagier [NZ85] imply that for sufficiently large q the knots K_q are hyperbolic, with strictly increasing volumes (that approach 36.4732...); in particular, the knots are distinct.

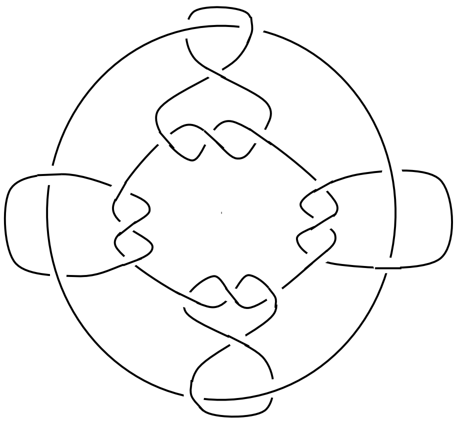

Figure 3: $k \cup C$

Remark 2.7. The choice of a curve C as in Example 2.6 and in Theorem 2.2 satisfies four basic requirements:

1. C is unknotted in S^3;
2. The linking number of C and $k_1 \sharp k_2$ is zero;
3. The link $C \cup (k_1 \sharp k_2)$ is prime; and
4. The epimorphism $\phi : \pi(k_1 \sharp k_2) \to \pi k$ maps the class of C trivially.

3. Partial order on knots. Motivated by the results of §2, we introduce a relation on the set of all knots.

Definition 3.1. $k_1 \succeq k_2$ if there is an epimorphism $\pi k_1 \to \pi k_2$ preserving peripheral structure.

Proposition 3.2. The relation $\succeq$ is a partial order.

Proof. Clearly $\succeq$ is reflexive and transitive. It remains to show that $\succeq$ is antisymmetric.

If $k_1 \succeq k_2$ and $k_2 \succeq k_1$, then there exist epimorphisms $\phi : \pi k_1 \to \pi k_2$ and $\eta : \pi k_2 \to \pi k_1$, preserving peripheral structure. The compositions $\phi \circ \eta$ and $\eta \circ \phi$ are epimorphisms from a knot group to itself. Since any knot group is residually finite [H87] and finitely generated, it has the Hopfian property: any epimorphism from the group to itself is an isomorphism [M40]. Hence both $\phi \circ \eta$ and $\eta \circ \phi$ are isomorphisms. In particular, it follows that ϕ is an isomorphism preserving peripheral structure. By [W68] k_1 and k_2 have homeomorphic complements. Finally [GL89] implies that k_1 and k_2 are of the same knot type (not necessarily of the same ambient isotopy type, however, since chiral knots exist).■

Remarks 3.3. (i) The condition that the epimorphism preserve peripheral structure is needed for the conclusion of Proposition 3.2. To see that this is so, let k_1 be the granny knot and k_2 the square knot. Since their groups are isomorphic, we certainly have epimorphisms from one to the other. However, no such epimorphism preserves peripheral structure, and indeed k_1 is not equal to k_2.

(ii) The relation $\succeq$ is compatible with some well-known invariants. For example, if $\Delta_k^{(i)}(t)$ $(i \geq 1)$ denotes the ith Alexander polynomial of k, then $k_1 \succeq k_2$ implies the existence of an epimorphism $\pi k_1 \to \pi k_2$, which in turn implies that, for each i, $\Delta_{k_1}^{(i)}(t)$ contains $\Delta_{k_2}^{(i)}(t)$ as a factor. Following the usual practice, we will abbreviate $\Delta_k^{(1)}(t)$ by $\Delta_k(t)$ and refer to it as *the* Alexander polynomial of k. Necessary algebraic background information can be found in [MKS76]).

(iii) If $k = k_1 \sharp k_2$, then $k \succeq k_1$ and $k \succeq k_2$. Hence $\succeq$ refines the crude partial order induced by knot factorization.

There are several natural methods to produce knots k_1, k_2 with $k_1 \succeq k_2$. For example, suppose that a diagram for a knot k_1 displays a rotational symmetry f. Let k_2 denote the quotient knot in the 3-sphere S^3/f. By identifying all pairs of Wirtinger generators corresponding to arcs in f-orbits we obtain a projection from πk_1 to πk_2 preserving peripheral structures. Hence $k \succeq \bar{k}$.

Another method is contained in the following.

Proposition 3.4. Assume that k is a satellite knot with pattern knot $\tilde{k}$. Then $k \succeq \tilde{k}$.

Proof. The satellite knot k is the image of a diffeomorphism $g : \tilde{V} \to \hat{V} \subset S^3$, where $\tilde{V}$ is a standard solid torus containing $\tilde{k}$, and $\hat{V}$ is a tubular neighborhood of a knot $\hat{k}$, the *companion knot*. As usual, we require that $\tilde{k}$ not be contained in any 3-ball of $\tilde{V}$, and also that g send the longitude l of $\tilde{V}$ to the longitude $\hat{l}$ of the solid torus $\hat{V}$. Denote the meridian of $\hat{V}$ by $\hat{m}$.

The group πk of the satellite is isomorphic to the free product of $\pi\hat{k}$ and $\pi_1(\tilde{V} \setminus \tilde{k})$ with amalgamation: subgroup $\langle \hat{m}, \hat{l} \rangle$ of $\pi\hat{k}$ is identified with the subgroup $\langle m, l \rangle$ of $\pi_1(\tilde{V} \setminus \tilde{k})$, matching $\hat{m}$ and $\hat{l}$ with m and l (see [BZ85], for example). By a well-known property of free products with amalgamation, both $\pi\hat{k}$ and $\pi_1(\tilde{V} \setminus \tilde{k})$ are subgroups of πk embedded in the obvious way. (see [LS77]).

Consider the natural projection $\phi : \pi k \to \pi k/N$ where N is the normal closure of the commutator subgroup of $\pi\hat{k}$. Regard $\pi k/N$ as the quotient group of πk obtained by allowing the elements of $\pi\hat{k}$ to commute. In the quotient, $\pi\hat{k}$ collapses to the infinite cyclic group generated by $\hat{m}$. The effect on $\pi_1(\tilde{V} \setminus \tilde{k})$ is to kill l, resulting in the group $\pi\tilde{k}$. Hence ϕ is an epimorphism from πk to $\pi\tilde{k}$. It is obvious that ϕ preserves peripheral structure. ∎

Remarks 3.5. (i) Proposition 3.4 states that a satellite knot k covers its pattern knot $\tilde{k}$. It is not generally true that k covers its companion $\hat{k}$. To see this, consider the untwisted double k of the trefoil knot. The Alexander polynomial $\Delta_k(t)$ is trivial (see [BZ85]). However, the companion knot is the trefoil, which has nontrivial Alexander polynomial. In view of Remark 3.3 (ii), k does not cover $\hat{k}$. We mention that for a given satellite knot, the pattern knot might well cover the companion or vice versa.

(ii) We can avoid the use of Theorem 4.1 of [G-S97] and apply Proposition 3.4 to give an alternative proof of Theorem 2.2 as follows. Think of K_q as the satellite knot with pattern k in the interior of the (standardly embedded) solid torus V_1 and with companion the twist-knot γ_{2q}; here V_1 is mapped by a longitude-preserving homeomorphism onto a tubular neighborhood of γ_{2q}. Since $\mathrm{Int}(V_1)$ contains no 2-sphere that decomposes K_q as a nontrivial connected sum, it follows easily that K_q is prime for $q \geq 2$, say. Proposition 3.4 immediately yields the epimorphism $\eta : \pi K_q \to \pi k$. We omit details.

Recall that the *genus* $g(k)$ of a knot k is the smallest genus of any Seifert surface of k.

Conjecture 3.6. If $k_1 \succeq k_2$ then $g(k_1) \succeq g(k_2)$.

As evidence for Conjecture 3.6 we offer the following.

Proposition 3.7. Assume that $k_1 \succeq k_2$. If either (i) k_1 is fibered or (ii) k_2 is alternating then $g(k_1) \succeq g(k_2)$.

Proof. (i) A knot k is fibered if and only if the commutator subgroup $[\pi k, \pi k]$ is finitely generated, in which case it is a free group of rank equal to $2g(k)$ (see [R76], for example). Assume that $k_1 \succeq k_2$. The epimorphism $\pi k_1 \to \pi k_2$ restricts to an epimorphism of commutator subgroups. Consequently, $[\pi k_2, \pi k_2]$ is finitely generated, and hence free of rank less than or equal that of $[\pi k_1, \pi k_1]$. Since the rank of $[\pi k_2, \pi k_2]$ is equal to $2g(k_2)$, the proof is complete.

(ii) By a theorem of Seifert [S34], for any knot k we have $2g(k) \geq \deg\Delta_k(t)$, while a theorem of Murasugi [M60] states that equality holds when k is alternating. Assume that $k_1 \succeq k_2$. Then $\Delta_{k_2}(t)$ divides $\Delta_{k_1}(t)$. Hence $2g(k_1) \geq \deg\Delta_{k_1}(t) \geq \deg\Delta_{k_2}(t) = 2g(k_2)$. ∎

There is further evidence for Conjecture 3.6. If $k_1 \succeq k_2$, then the epimorphism $\pi(k_1) \to \pi(k_2)$ is induced by a boundary-preserving map $f : \mathrm{Ext}(k_1) \to \mathrm{Ext}(k_2)$. If S is a minimal genus Seifert surface spanning k_1, and if $f_*([S]) \in H_2(\mathrm{Ext}(k_2), \partial\mathrm{Ext}(k_2))(\cong \mathbb{Z})$ is not zero, then it follows from Corollary 6.22 of [G83] that $g(k_1) \geq g(k_2)$. We are grateful to Ian Agol for pointing out the connection between Conjecture 3.6 and Gabai's result.

Proposition 3.8. If k_i is a sequence of hyperbolic knots such that $k_0 \succeq k_1 \succeq \cdots \succeq k_i \succeq \cdots$, then $k_i = k_{i+1}$ for sufficiently large i.

Proof. Assume that $k_0 \succeq k_1 \succeq k_2 \succeq \cdots$, for hyperbolic knots k_i. Consequently, there is a sequence of epimorphisms

$$\pi k_0 \xrightarrow{\phi_0} \pi k_1 \xrightarrow{\phi_1} \pi k_2 \xrightarrow{\phi_2} \cdots,$$

each preserving peripheral structure. By Theorem 1 of [S02], ϕ_i is an isomorphism for sufficiently large i. (The result of [S02] requires only knot group epimorphisms without any constraint on peripheral subgroups.) As in the proof of Proposition 3.2, [W68] and [GL89] together imply that $k_i = k_{i+1}$ for sufficiently large i.■

Conjecture 3.9. (Cf. J. Simon: Problem 1.12 (D) in [K95])] Any knot covers only finitely many knots. In other words, if k_1 is a knot, then the collection of all knots k_2 such that $k_1 \succeq k_2$ is finite.

Conjecture 3.9 is true if k_1 is fibered. To see this, note that k_2 must also be fibered. In [S95] an entropy invariant h_k was defined for any fibered knot k. If $k_1 \succeq k_2$, then $h_{k_1} \ge h_{k_2}$ and also $g(k_1) \ge g(k_2)$. By Theorem 3.4 of [S95], there exist only finitely many fibered knots with genus and entropy no greater than given bounds.

Definition 3.10. A knot k is *minimal* (with respect to the partial order) if $k \succeq k'$ implies that $k = k'$ or else k' is trivial.

Proposition 3.11. If k is a fibered knot with irreducible Alexander polynomial, then k is minimal.

Proof. Assume that $k \succeq k'$. Since $\Delta_{k'}(t)$ divides $\Delta_k(t)$ and $\Delta_k(t)$ is irreducible, either $\Delta_{k'}(t) = \Delta_k(t)$ or else $\Delta_{k'}(t) = 1$. In the first case, the epimorphism $\phi : \pi k \to \pi k'$ restricts to an isomorphism of commutator subgroups, as these groups are both free of rank equal to $\deg\Delta_k(t)$; it follows that ϕ is itself an isomorphism, and as in the proof of Proposition 3.2, $k = k'$. In the second case, $\Delta_{k'}(t) = 1$, and this together with the fact that k' is fibered imply that k' is trivial. ■

The figure eight knot 4_1 is minimal by Proposition 3.11. However, Example 3.12 below shows that 4_1 does not remain minimal in the larger category of virtual knots. It shows also that Conjecture 3.9 is not true in the category.

A knot is often studied as an equivalence class of planar knot diagrams, two diagrams being equivalent if one can be obtained from the other by a sequence of Reidemeister moves. In 1997 Kauffman introduced virtual knot diagrams, allowing a new type of crossing, called a *virtual crossing* and indicated by a small circle surrounding the site. After suitably extending the usual Reidemeister moves to allow certain deformations involving virtual crossings, Kauffman defined a *virtual knot* to be an equivalence class of virtual diagrams. The reader is referred to [K97], [K99], [K00] for details. It is a remarkable feature of the theory that two classical knots are equivalent under generalized Reidemeister moves if and only if they are equivalent under the classical ones [GPV00]. In this sense, virtual knot theory is an extension of the classical theory.

Many classical invariants of knot theory extend naturally in the larger virtual category. In particular, one can associate a knot group to an equivalence class of diagrams. Virtual knot groups were classified in [SW00] both in terms of combinatorial presentations and topologically (see also [Ki00]). The peripheral structure of a virtual knot is defined just as for classical knots (see [Ki00] for details).

Example 3.12. A diagram for the figure eight knot appears in Figure 4 with labeled Wirtinger generators. The group of k has a presentation $\langle x, y, z, w \mid zx = yz, yw = zy, wx = zw\rangle$. (The fourth Wirtinger relation $xw = yx$ is a consequence of the other three, and so we have omitted it.) Using the second and third relations to express w and z in terms of x and y, we see that

$$\pi k \cong \langle x, y \mid x^{-1}y^{-1}xy^{-1}x^{-1}yxy^{-1}xy\rangle.$$

Consider the diagram for the virtual knot k_q ($q \geq 2$) in Figure 4 with labeled Wirtinger generators. The corresponding group generated by $x, y, z, w, w_1, \ldots, w_{q-1}$ with relations $zx = yz, yw = zy, wx = zw, xw_1 = wx, xw_2 = w_1x, \ldots, xw_{q-2} = w_{q-3}x, wy = w_{q-2}x$.

The fourth relation implies that $w_1 = x^{-1}wx$. The fifth implies that $w_2 = x^{-2}wx^2$, and so forth. The next to last relation implies that $w_{q-2} = x^{-(q-2)}ww^{q-2}$. We use these to eliminate $w_1, \ldots, w_{q-2}$. The last relation then becomes $x^{-q}yx^q = y$. Consequently,

$$\pi k_q \cong \langle x, y, z, w \mid zx = yz, yw = zy, wx = zw, y = x^{-(q-1)}yx^{q-1} = w\rangle.$$

Recall that the three relations $zx = yz, yw = zy, wx = zw$ together imply that $xw = yx$, which can be rewritten as $w = x^{-1}yx$. Substitution in the relation $y = x^{-(q-1)}yx^{q-1} = w$ yields

$$\begin{aligned}\pi k_q &\cong \langle x, y, z, w \mid zx = yz, yw = zy, wx = zw, x^{-q}yx^q = y\rangle\\ &\cong \pi k/\langle\langle x^{-q}yx^q = y\rangle\rangle\\ &\cong \langle x, y \mid x^{-1}y^{-1}xy^{-1}x^{-1}yxy^{-1}xy, x^{-q}yx^q = y\rangle\end{aligned}$$

.

The Reidemeister-Schreier method (see [LS77], for example) can be used to find a presentation for the commutator subgroup of πk_q:

$$\langle a_j \mid a_{j+2} = a_{j+1}^2 a_j^{-1} a_{j+1},\ a_{j+q} = a_j \quad (j \in \mathbf{Z})\rangle.$$

(This group is in fact the fundamental group of the q-fold cyclic cover of S^3 branched over k. However, we do not require this fact.) Its abelianization is finite, and it has order equal to the absolute value of the cyclic resultant $\mathrm{Res}(t^2 - 3t + 1, t^q - 1)$. These values grow exponentially with q (see [SW02]). Consequently, the groups πk_q are pairwise nonisomorphic for sufficiently large q. (In fact, they are all nonisomorphic, but this is again a fact that we do not require.)

The canonical projection $\phi_q : \pi k \to \pi k_q$ is a surjection that maps the elements x and $z^{-1}xy^{-1}w$, forming a peripheral pair for k, to their cosets in πk_q. It is easy to see that $(x, z^{-1}x^{-(q-1)}y^{-1}wx^{n-2})$ is a peripheral pair for k_q. Using the relations above, one checks that $z^{-1}x^{-(q-1)}y^{-1}wx^{n-2}$ is equal to $z^{-1}xy^{-1}w$. Hence the projection ϕ_q preserves peripheral structure, and we have produced infinitely many virtual knots k_q such that $k \succeq k_q$.

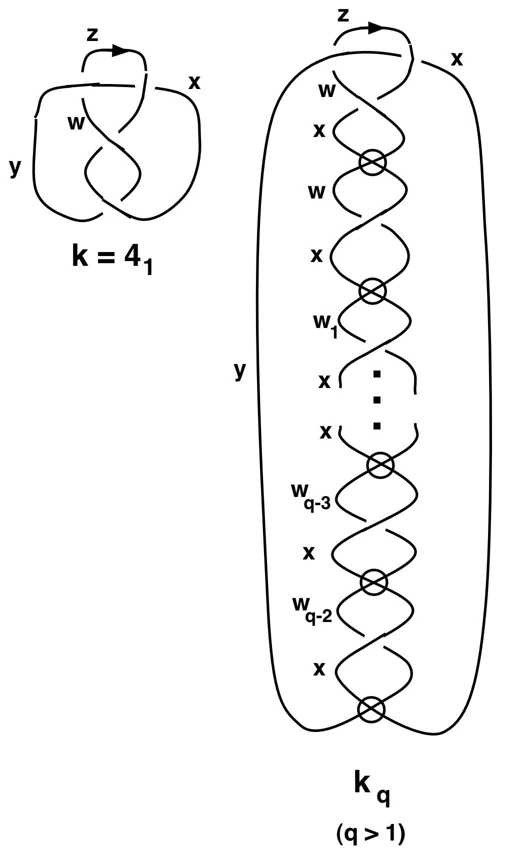

Figure 4: Figure eight knot and K_q

4. Degree one maps and a related partial order. The partial order in Definition 3.1 is related to another, a partial order on compact 3-manifolds, one that has been studied by Y. Rong [R92], S. Wang [W02], and others: If M and N are compact, oriented 3-manifolds, then $M \succeq_1 N$ if there exists a degree one proper map from M to N. (A manifold map is *proper* if it maps boundary to boundary.) When applied to exteriors of knots k_1, k_2, the relation becomes: $k_1 \succeq_1 k_2$ if there is an epimorphism $\pi k_1 \to \pi k_2$ mapping meridian m_1 to $m_2 l_2^p$ and longitude l_1 to $m_2^q l_2^{\pm 1}$, for some integers p, q. Since the normal subgroup generated by $m_2 l_2^p$ must be all of πk_2, Corollary 2 of [CGLS87] implies that $p \in \{0, 1, -1\}$. (The recent proof that every nontrivial knot satisfies Property P [KM04] implies that p must in fact be 0. However, we do not require that fact.) Also, since $m_2^q l_2^{\pm 1}$ must be in $(\pi k_2)'' \cap Z(m_2)$ [JL89], we have $q = 0$.

Clearly, $k_1 \succeq_1 k_2$ implies $k_1 \succeq k_2$.

Theorem 4.1. In general, the relation $k_1 \succeq k_2$ does not imply $k_1 \succeq_1 k_2$.

Proof. Let k_1, k_2 be the torus knots $9_1, 3_1$, respectively. Since k_1 has a rotational symmetry with k_2 as quotient, it follows from the remark before Proposition 3.4 that $k_1 \succeq k_2$.

Note that the epimorphism $\pi k_1 \to \pi k_2$ induced by the symmetry maps meridian m_1 to meridian m_2 and longitude l_1 to the third power l_2^3. It suffices to show there exists no epimorphism mapping (m_1, l_1) to $(m_2 l_2^p, l_2^{\pm 1})$, for $p \in \{0, 1, -1\}$. Since torus knots satisfy property P [S33] (see also [H64] or [BZ85], p.274), the integer p must be zero. The desired conclusion follows from the next proposition. (Details are given in Example 4.3.) ∎

The A-polynomial was introduced in [CCGLS94]. We briefly review its definition, following [CL96]. For any knot k with meridian-longitude pair (m, l), consider the affine algebraic set $R = \mathrm{Hom}(\pi k, SL_2(\mathbf{C}))$. Let $R_\triangle$ be the algebraic subset consisting of all $\rho \in R$ such that $\rho(l)$ and $\rho(m)$ are upper triangular. Let $\xi : R_\triangle \to \mathbf{C}^2$ be the eigenvalue map $\rho \mapsto (M, L)$, where M and L are the top-left entries (eigenvalues) of $\rho(m)$ and $\rho(l)$, respectively. The closure of any component of $\xi(R_\triangle)$ has complex dimension 0 or 1. Each 1-dimensional component is the zero set of a polynomial that is unique up to multiplication by a constant; the product of all such polynomials, divided by $L - 1$ (the polynomial corresponding to abelian representations) is A_k. It is possible to normalize, choosing a suitable multiplicative constant, so that the coefficients of A_k are integers with no common divisor, and we do so.

Proposition 4.2. Let k_1, k_2 be knots. If $\phi : \pi k_1 \to \pi k_2$ is a homomorphism mapping (m_1, l_1) to $(m_2^a l_2^b, m_2^c l_2^d)$, for integers a, b, c, d, then each irreducible factor of $A_{k_2}(M, L)$ divides $(M^c L^d - 1) \cdot A_{k_1}(M^a L^b, M^c L^d)$.

Proof. Denote the eigenvalue maps for k_1 and k_2 by ξ_1 and ξ_2, respectively. If (M, L) is in the image of ξ_2, then $(M^a L^b, M^c L^d)$ is in the image of ξ_1. Hence $A_{k_2}(M, L) = 0$ implies that $(M^c L^d - 1) \cdot A_{k_1}(M^a L^b, M^c L^d) = 0$. The result follows from Hilbert's Nullstellensatz (see [H77], for example). ∎

Example 4.3. The A-polynomial of $k_1 = 9_1$ is $1 + M^{18}L$ while that of the trefoil $k_2 = 3_1$ is $1 + M^6L$ (see [CCGLS94] or [N02]). The polynomial $A_{3_1}(M, L)$ divides $A_{9_1}(M, L^3)$, reflecting the fact that there is a homomorphism $\pi k_1 \to \pi k_2$ mapping (m_1, l_1) to (m_2, l_2^3). However, the polynomial $A_{3_1}(M, L)$ is irreducible and divides neither $A_{9_1}(M, L)$ nor $A_{9_1}(M, L^{-1})$. Hence by Proposition 4.2, there is no homomorphism $\pi k_1 \to \pi k_2$ mapping (m_1, l_1) to (m_2, l_2) or (m_2, l_2^{-1}).

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