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ALEXANDER-LIN TWISTED POLYNOMIALS

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ABSTRACT

X.S. Lin's definition of twisted Alexander knot polynomial is extended for finitely presented groups. J. Cha's fibering obstruction theorem is generalized. The group of the 2-twist-spun trefoil knot is seen to have a faithful representation that yields a trivial twisted Alexander polynomial.

Keywords: Knot, twisted Alexander polynomial, Wada invariant, fibering obstruction.

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1. Introduction.

Twisted Alexander polynomials were introduced by X.S. Lin in a 1990 Columbia University preprint [15]. Lin's invariant, defined in terms of Seifert surfaces, incorporated information from linear representations of the knot group π . While the classical Alexander polynomial is an invariant of the $\mathbb{Z}[t^{\pm 1}]$ -module π'/π'' , its twisted counterpart can incorporate information from any term of the derived series of π' . As a result, it is often a more sensitive invariant.

In [22], M. Wada modified Lin's original polynomial. He developed a general invariant for any finitely presented group provided with an epimorphism to a free abelian group and also a linear representation. The definition is given below.

When applied to knot groups, Wada's invariant is often called the "twisted Alexander polynomial." (Twisted Reidemeister torsion is a more appropriate label. See [10].) A few years later, P. Kirk and C. Livingston, working from an algebraic topological perspective, offered another candidate for the title (see [9]).

We revisit Lin's original invariant, putting it in Wada's general context. It is a "pointed invariant," depending on a group and distinguished conjugacy class of elements. Thus it gives obstructions to knot group epimorphisms preserving meridians. We generalize a fibering obstruction theorem of J. Cha. We also give an example of a 2-knot group with trivial twisted invariants.

For background on twisted Alexander invariants, see the comprehensive survey paper [5].

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2. Alexander-Lin polynomials.

For any positive integer d , we regard $\mathbb{Z}^d$ as a multiplicative group $\langle t_1, \dots, t_d \mid t_i t_j = t_j t_i \ (1 \leq i, j \leq d) \rangle$. The ring $\mathbb{Z}[\mathbb{Z}^d]$ of Laurent polynomials in variables $t_1, \dots, t_d$ will be denoted by Λ . When $d = 1$, we write t instead of t_1 .

Definition 2.1. An *augmented group system* is a triple $\mathcal{A} = (G, \epsilon, x)$ consisting of a finitely presented group G , epimorphism $\epsilon : G \rightarrow \mathbb{Z}^d$, and conjugacy class of an element $x \in G$ such that $\epsilon(x)$ is nontrivial.

Augmented group systems are objects of a category. A morphism from $\mathcal{A} = (G, \epsilon, x)$ to $\bar{\mathcal{A}} = (\bar{G}, \bar{\epsilon}, \bar{x})$ is defined to be a group homomorphism $f : G \rightarrow \bar{G}$ such that $f(x) = \bar{x}$ and $\epsilon = \bar{\epsilon}f$. When f can be made an isomorphism, we regard $\mathcal{A}$ and $\bar{\mathcal{A}}$ as the same. If f can be made an epimorphism, we write $\mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$.

Remark 2.2. If we restrict ourselves to groups that are hopfian (that is, every epimorphism from the group to itself is an automorphism), then the relation $\twoheadrightarrow$ is a partial order on augmented group systems (see [19]). Knot and link groups are hopfian, as are any finitely generated, residually finite groups.

Augmented group systems were introduced in [18]. Definition 2.1 is a mild generalization.

Let $\mathcal{A} = (G, \epsilon, x)$ be an augmented group system, and let

$$\langle x_0, x_1, \dots, x_n \mid r_1, \dots, r_m \rangle \quad (2.1)$$

be a presentation of G . Without loss of generality we can assume that $x = x_0$ and also $m \geq n$.

Throughout, R is assumed to be a Noetherian unique factorization domain.

Given a linear representation $\gamma : G \rightarrow \mathrm{GL}_N R$, we define a representation $\gamma \otimes \epsilon : \mathbb{Z}[G] \rightarrow M_N \Lambda$ by sending $x_i \mapsto \epsilon(x_i) \gamma(x_i)$, and extending. Here $M_N \Lambda$ denotes the ring of $N \times N$ matrices with entries in Λ . Pre-compose with the natural projection $F = \langle x_0, x_1, \dots, x_n \mid \rangle \rightarrow G$ to obtain a homomorphism

$$\Phi : \mathbb{Z}[F] \rightarrow M_N \Lambda.$$

Let $\mathcal{M}_\gamma$ be the $m \times n$ matrix with i, j th entry equal to the image under Φ of the Fox partial derivative $\partial r_i / \partial x_j$, denoted by

$$\left(\frac{\partial r_i}{\partial x_j} \right)^\Phi \in M_N \Lambda,$$

where $i = 1, \dots, m$ and $j = 1, \dots, n$. Regard $\mathcal{M}_\gamma$ as an $mN \times nN$ matrix over Λ by ignoring inner parentheses.

Definition 2.3. Let $\mathcal{A} = (G, \epsilon, x)$ be an augmented group system. The *Alexander-Lin polynomial* $D_\gamma(\mathcal{A})$ is the greatest common divisor of all $nN \times nN$ minors of $\mathcal{M}_\gamma$.

A priori, $D_\gamma(\mathcal{A})$ depends on the presentation that we used. However, it is a consequence of [22] that up to multiplication by a unit $\pm t_1^{n_1} \cdots t_d^{n_d} \in \Lambda$, the polynomial $D_\gamma(\mathcal{A})$ depends only on $\mathcal{A}$. (Furthermore, the sign $\pm$ is $+$ if N is even.) For such invariance holds for the quotient

$$W_\gamma(\mathcal{A}) = \frac{D_\gamma(\mathcal{A})}{\det(x-1)^\Phi},$$

commonly known as *Wada's invariant*. Since the denominator depends only on the conjugacy class of x , invariance holds for $D_\gamma(\mathcal{A})$ as well.

Remark 2.4. The quotient $D_\gamma(\mathcal{A})/\det(x-1)^\Phi$ is independent of x provided that $\det(x-1)^\Phi \neq 0$ [22]. Hence $W_\gamma(\mathcal{A})$ is an invariant of the pair (G, ϵ) and representation γ . We will denote it simply by W_γ .

In a similar way, the divisibility results of [12] proved for Wada's invariant give the following obstruction to the existence of epimorphisms preserving distinguished elements.

Proposition 2.5. *Let $\mathcal{A} = (G, \epsilon, x)$ and $\bar{\mathcal{A}} = (\bar{G}, \bar{\epsilon}, \bar{x})$ be augmented group systems such that $\mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ via $f : G \rightarrow \bar{G}$. Then for any representation $\gamma : \bar{G} \rightarrow \mathrm{GL}_N R$, $D_{\gamma f}(\mathcal{A})$ is divisible by $D_\gamma(\bar{\mathcal{A}})$.*

Example 2.6. If k is an oriented knot with group π_k , then there is a well-defined epimorphism $\epsilon : \pi_k \rightarrow \mathbb{Z} = \langle t \mid \rangle$ taking each oriented meridian to t . We associate to k an augmented group system $\mathcal{A}_k = (\pi_k, \epsilon, x)$, where x represents the class of an oriented meridian. If $\gamma : \pi_k \rightarrow \mathrm{GL}_N \mathbb{C}$, then $D_\gamma(\mathcal{A}_k)$ agrees with the invariant originally defined by Lin [15]. Lin used a Seifert surface S that is *free* in the sense that $\mathcal{S}^3$ split along S is a handlebody H with fundamental group $\langle a_1, \dots, a_{2g} \mid \rangle$. He then described the knot group π_k as

$$\langle x, a_1, \dots, a_{2g} \mid xu_i x^{-1} = v_i \ (i = 1, \dots, 2g) \rangle,$$

where u_i, v_i are words in $a_1^{\pm 1}, \dots, a_{2g}^{\pm 1}$ describing positive and negative push-offs of simple closed curves representing a basis for $\pi_1 S$. Lin's invariant is the determinant of the matrix M_γ computed as above.

The following is a group-theoretic version of 3-manifold fibering results proved in different degrees of generality by Cha [2], Kitano and Morifuji [11], Goda, Kitano and Morifuji [6], Friedl and Kim [4], Pajitnov [17], and Kitayama [13].

Theorem 2.7. *Let $\mathcal{A} = (G, \epsilon, x)$ be an augmented group system with $d = 1$ and $\epsilon(x) = t$. Assume that the kernel of $\epsilon : G \rightarrow \mathbb{Z}$ is a finitely generated free group of rank r . Then for any representation $\gamma : G \rightarrow \mathrm{G}_N R$,*

1. $\deg D_\gamma(\mathcal{A}) = rN$;
2. the leading and trailing coefficients of $D_\gamma(\mathcal{A})$ are units.

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Proof. Let $a_1, \dots, a_r$ be generators for the kernel of ϵ . Since G is a semi-direct product of $\mathbb{Z} \cong \langle x \mid \rangle$ and the free group $F = \langle a_1, \dots, a_r \mid \rangle$, it has a presentation of the form

$$G = \langle x, a_1, \dots, a_r \mid xa_1x^{-1} = \mu(a_1), \dots, xa_rx^{-1} = \mu(a_r) \rangle,$$

for some automorphism μ of F . The matrix $\mathcal{M}_\gamma$ in the definition of $D_\gamma(\mathcal{A})$ is equal to

$$\mathcal{M}_\gamma = t \operatorname{diag}(\gamma(x), \dots, \gamma(x)) - \left(\frac{\partial \mu(a_i)}{\partial a_j} \right)^\Phi.$$

It follows that the leading and constant terms of $D_\gamma(\mathcal{A}) = \det \mathcal{M}_\gamma$ are, respectively,

$$t^{rN} (\det \gamma(x))^r, \quad \pm \det \left(\frac{\partial \mu(a_i)}{\partial a_j} \right)^\Phi.$$

It suffices to show that the constant term is a unit. By a theorem of Birman [1], elements $y_1, \dots, y_r$ in F generate if and only if the $r \times r$ Jacobian matrix with ij th entry $\partial y_i / \partial x_j$ has a right inverse in $M_r(\mathbb{Z}[F])$. Hence

$$J = \left(\frac{\partial \mu(a_i)}{\partial a_j} \right)$$

has a right inverse $B \in M_r(\mathbb{Z}[F])$. Replacing each occurrence of $a_i^{\pm 1}$ in B by the matrix $\gamma(a_i^{\pm 1})$, for $i = 1, \dots, r$, produces a right inverse for J^Φ . Hence its determinant is a unit. $\square$

Example 2.8. The group π_k of the trefoil has presentation

$$\langle x, a \mid x^2ax^{-2} \cdot xa^{-1}x^{-1} \cdot a \rangle.$$

A presentation for the commutator subgroup π'_k can easily be found using the Reidemeister-Schreier method [16]. Denoting $x^j a x^{-j}$ by a_j ,

$$\pi'_k = \langle a_j \mid a_{j+2} a_{j+1}^{-1} a_j, (j \in \mathbb{Z}) \rangle.$$

From this one sees that $\pi'_k \cong \langle a_0, a_1 \mid \rangle$ is a finitely generated free group.

Consider the related group G with presentation

$$G = \langle x, a \mid x^2ax^{-2} \cdot a^2 \cdot xa^{-1}x^{-1} \cdot a^{-1} \rangle$$

and commutator subgroup

$$G' = \langle a_j \mid a_{j+2} a_j^2 a_{j+1}^{-1}, (j \in \mathbb{Z}) \rangle.$$

The $\mathbb{Z}[t^{\pm 1}]$ -modules π'_k / π''_k and G' / G'' are isomorphic. In particular, the Alexander polynomial is $t^2 - t + 1$ in both cases. However, using Magnus's Freiheitsatz as in [20], one sees that G' is not a finitely generated free group. Theorem 2.7 provides another way to see this.

Consider the augmented group system $\mathcal{A} = (G, \epsilon, x)$, where $\epsilon : G \rightarrow \mathbb{Z}$ is the abelianization map sending x to 1. One checks that there exists a representation $\gamma : G \rightarrow \mathrm{GL}_3(\mathbb{Z})$ such that

$$\gamma(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \gamma(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

A simple calculation shows that $D_\gamma(\mathcal{A}) = (t^2 - 4)(t^2 - 1)(t^2 - t + 1)$. Since the constant term is not a unit, G' is not a finitely generated free group.

3. Twisted invariants and triviality.

Given any group G with finite presentation (2.1), epimorphism $\epsilon : G \rightarrow \mathbb{Z}^d$ and representation $\gamma : G \rightarrow \mathrm{GL}_N R$, there is another invariant $\Delta_{\gamma,1}$ that competes with W_γ for the title “twisted Alexander polynomial” of G . Homological definitions of the polynomials $\Delta_{\gamma,0}$ and $\Delta_{\gamma,1}$, well defined up to multiplication by units in $R[t^{\pm 1}]$, are given in [9]. We can compute $\Delta_{\gamma,0}$ as the greatest common divisor of maximal minors of the matrix

$$\begin{pmatrix} (x_0 - 1)^\Phi \\ (x_1 - 1)^\Phi \\ \vdots \\ (x_n - 1)^\Phi \end{pmatrix},$$

which we regard as an $nN \times N$ matrix with entries in $R[t^{\pm 1}]$.

The relationship between $\Delta_{\gamma,1}$ and W_γ is given by the following proposition. A version of Proposition 3.1 is proven in [9]. See [5] for a proof of the general result.

Proposition 3.1. $W_\gamma = \Delta_{\gamma,1}/\Delta_{\gamma,0}$.

Since for the infinite cyclic group and any representation γ , the polynomial $\Delta_{\gamma,1}$ is equal to 1, we ask:

Question. Does there exist a nontrivial knot group π_k and faithful representation $\gamma : \pi_k \rightarrow \mathrm{GL}_N R$ for which $\Delta_{\gamma,1} = 1$?

Remark 3.2. By Proposition 3.1, the condition $\Delta_{\gamma,1} = 1$ is equivalent to $D_\gamma = \det(x_0 - 1)^\Phi / \Delta_{\gamma,0}$.

If we consider 2-knot groups, then the question above has an affirmative answer.

Example 3.3. Consider the semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$:

$$\pi_k = \langle x, a \mid xa = a^2x, a^3 = 1 \rangle.$$

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This group arises as the group of the 2-twist-spun trefoil $k \subset \mathbb{S}^4$ [23]. The reader can easily check that there is a representation $\gamma : \pi_k \rightarrow \mathrm{GL}_2\mathbb{C}$ sending x and a to

$$\gamma(x) = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad \gamma(a) = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix},$$

where α is a nonzero complex number while ω is a primitive 3rd root of unity. If α is not a root of unity, then the matrix $\gamma(x)$ has infinite order, and the representation γ is faithful. A straightforward calculation shows that

$$\mathcal{M}_\gamma = \begin{pmatrix} t\gamma(x) - I - \gamma(a) \\ I + \gamma(a) + \gamma(a)^2 \end{pmatrix} = \begin{pmatrix} -\omega - 1 & \alpha t \\ \alpha t & -\bar{\omega} - 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Alexander-Lin polynomial is $D_\gamma = 1 - \alpha^2 t^2$. Since this is the same as $\det(x - 1)^\Phi = \det(t\gamma(x) - I)$, the Wada invariant W_γ is equal to 1. Since $\det(a - 1)^\Phi = \det(\gamma(a) - I) = 1$, we have $\Delta_{\gamma,0} = 1$. Hence $\Delta_{\gamma,1} = 1$.

The group π_k in Example 3.3 also arises as the group of a noteworthy virtual knot.

Virtual knots were introduced by L. Kauffman in 1997. They are described by diagrams that can have “virtual crossings” (indicated by circles) as well as classical over/under crossings. A virtual knot is an equivalence class of diagrams under generalized Reidemeister moves. Details about virtual knots and their groups, a generalization of the classical notions, can be found in [7]. Many invariants of classical knots, such as the knot group and Jones polynomial, are defined in a natural way for virtual knots. Details can be found in [8].

Consider the virtual knot k in Figure 1 with generators for a Wirtinger presentation indicated. In [8] Kauffman gave it as an example of a virtual knot with trivial Jones polynomial.

Fig. 1. Virtual knot and Wirtinger generators

The group of k , which has presentation

$$\pi_k = \langle x, y \mid xyx = yxy, xy^{-1}x = yxy^{-1} \rangle,$$

is known to be isomorphic to that of the 2-twist-spun trefoil. One may see this directly by substituting $y = ax$, thereby obtaining

$$\pi_k = \langle x, a \mid x^2ax^{-2} = a^{-1} \cdot xax^{-1}, xax^{-1} = a^2 \rangle,$$

and then simplifying the presentation.

Remark 3.4. In [21], M. Suzuki showed that the braid group B_4 with the faithful Lawrence-Krammer representation has $W_\gamma = 1$. One checks that $\Delta_{\gamma,1} = 1$ as well.

Neither B_4 nor the group of the 2-twist spun trefoil has a presentation with deficiency 1 (that is, one more generator than the number of relators).

Following [20], we will say that a group is *knot-like* if it has infinite cyclic abelianization and a presentation with deficiency 1.

Question: Does there exist a knot-like group with a faithful representation $\gamma : G \rightarrow \mathrm{GL}_N \mathbb{C}$ such that $\Delta_{\gamma,1}$ or W_γ is equal to 1?

Remark 3.5. S. Friedl has shown [3] that $\Delta_{\gamma,0} = 1$ whenever $\gamma : G \rightarrow \mathrm{GL}_N \mathbb{C}$ is an irreducible faithful representation. In this case, $\Delta_{\gamma,1} = W_\gamma$.

Any knot-like group providing an affirmative action must have a non-finitely generated commutator subgroup, as the following corollary of Theorem 2.7 shows.

Proposition 3.6. *If G is a knot-like group with finitely generated commutator subgroup, $G \neq \mathbb{Z}$, then $\Delta_{\gamma,1}$ and W_γ are nontrivial, for any representation $\gamma : G \rightarrow \mathrm{GL}_N \mathbb{C}$.*

Proof. We can assume without loss of generality that G has a presentation of the form (2.1) with $m = n$ and such that the image of $x = x_0$ under the abelianization homomorphism ϵ is equal to t . By [14], the commutator subgroup of G is free. Theorem 2.7 implies that $\deg D_\gamma = rN$, where r is the rank of the commutator subgroup. Since $G \neq \mathbb{Z}$, the rank r must be at least 2. The degree of $\det(x - 1)^\Phi$ cannot exceed N , and hence both $W_\gamma = D_\gamma / \det(x - 1)^\Phi$ and $\Delta_{\gamma,1} = D_\gamma \cdot \Delta_{\gamma,0} / \det(x - 1)^\Phi$ are nontrivial. $\square$

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