On Reciprocality of Twisted Alexander Invariants

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Abstract

Given a knot and an $\mathrm{SL}_n\mathbb{C}$ representation of its group that is conjugate to its dual, the representation that replaces each matrix with its inverse-transpose, the associated twisted Reidemeister torsion is reciprocal. An example is given of a knot group and $\mathrm{SL}_3\mathbb{Z}$ representation that is not conjugate to its dual for which the twisted Reidemeister torsion is not reciprocal.

Keywords: Knot, twisted Reidemeister torsion, twisted Alexander polynomial[1]

1 Introduction

The Alexander polynomial $\Delta(t)$ of a knot k can be computed from a diagram of k or from a presentation of the knot group (see [5], for example). It is an integral Laurent polynomial, well defined up to multiplication by units $\pm t^i \in \mathbb{Z}[t^{\pm 1}]$, and it is usually normalized to be a polynomial with nonzero constant coefficient.

It is well known that $\Delta(t)$ is *reciprocal* in the sense that

$$\Delta(t^{-1}) \doteq \Delta(t), \tag{1.1}$$

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where $\doteq$ indicates equality up to multiplication by units. This is a consequence of Poincaré duality of the knot exterior (see [15] for an alternative approach based on duality in the knot group).

In 1990 X.S. Lin introduced a more sensitive invariant using information from nonabelian representations of the knot group [10]. Later, refinements were described by M. Wada [16] and others including P. Kirk and C. Livingston [6], J. Cha [1], and others. These twisted Alexander invariants have proven to be useful for a variety of questions about knots including questions about concordance [6], knot symmetry [4] and fibrations [3]. See [2] for a survey.

We briefly review the definition of perhaps the best-known twisted Alexander invariant. Let k be a knot with exterior X, endowed with the structure of a CW complex. We fix a Wirtinger presentation $\langle x_0, x_1, \ldots, x_k \mid r_1, \ldots r_k \rangle$ for the knot group $\pi = \pi_1(X)$. Let $\phi : F_k \to \pi$ be the associated projection of the free group $F_k = \langle x_0, x_1, \ldots, x_k \mid \ \rangle$ to π. It induces a ring homomorphism $\tilde{\phi} : \mathbb{Z}[F_k] \to \mathbb{Z}[\pi]$.

Let $\epsilon : \pi \to H_1(X;\mathbb{Z}) \cong \langle t \mid \ \rangle$ be the abelianization mapping each x_i to t. It induces a ring homomorphism $\tilde{\epsilon} : \mathbb{Z}[\pi] \to \mathbb{Z}[t^{\pm 1}]$.

Assume that $\gamma : \pi \to \mathrm{SL}_n\mathbb{C}$ is a linear representation. Let $\tilde{\gamma} : \mathbb{Z}[\pi] \to M_n(\mathbb{C})$ be the associated ring homomorphism to the algebra of $n \times n$ matrices over $\mathbb{C}$. We obtain a homomorphism

$$\tilde{\gamma} \otimes \tilde{\epsilon} : \mathbb{Z}[\pi] \to M_n(\mathbb{C}[t^{\pm 1}]), \tag{1.2}$$

mapping g to $t^{\epsilon(g)}\gamma(g)$, that we denote more simply by Φ.

Let $M_{\gamma\otimes\epsilon}$ denote the $k \times (k+1)$ matrix with (i,j)-component equal to the $n \times n$ matrix $\Phi(\frac{\partial r_i}{\partial x_j}) \in M_n(\mathbb{C}[t^{\pm 1}])$. Here $\frac{\partial r_i}{\partial x_j}$ denotes Fox partial derivative. Let $M^0_{\gamma\otimes\epsilon}$ denote the $k \times k$ matrix obtained by deleting the column corresponding to x_0. We regard $M^0_{\gamma\otimes\epsilon}$ as a $kn \times kn$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$.

Definition 1.1. The *Wada invariant* $W_\gamma(t)$ is

$$\frac{\det M^0_{\gamma\otimes\epsilon}}{\det \Phi(x_0 - 1).}$$

When γ is the trivial 1-dimensional representation, $M^0_{\gamma\otimes\epsilon}$ is a matrix $M(t)$ that we call the *Alexander matrix* of k. (This terminology is used, for example, in [13], but it is not standard.) The determinant of $M(t)$ is the (untwisted) Alexander polynomial $\Delta(t)$ of k.

Remark 1.2. Although the rational function $W_\gamma(t)$ is often a polynomial, it need not be. However, in general it is well defined up to multiplication by $(-t)^{ni}$. See [16].

The matrix $M_{\gamma\otimes\epsilon}$ represents a boundary homomorphism for a twisted chain complex

$$C_*(X; V[t^{\pm 1}]_\gamma) = (\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V) \otimes_\gamma C_*(\tilde{X}). \tag{1.3}$$

Here $V = \mathbb{C}^n$ is a vector space on which π acts via γ, while $C_*(\tilde{X})$ denotes the cellular chain complex of the universal cover $\tilde{X}$ with the structure of a CW complex that is lifted from X. The group ring $\mathbb{Z}[\pi]$ acts on the left via deck transformations. On the other hand, $\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V$ has the structure of of a right $\mathbb{Z}[\pi]$-module via

$$(p \otimes v) \cdot g = (pt^{\epsilon(g)}) \otimes (v\gamma(g)), \text{ for } \gamma \in \pi.$$

Remark 1.3. The homology group $H_1(X; V[t^{\pm 1}])$ of the chain complex (1.3) is a finitely generated $\mathbb{C}[t^{\pm 1}]$-module. Its 0th elementary divisor $\Delta_\gamma(t)$, which is well defined up to multiplication by units in $\mathbb{C}[t^{\pm 1}]$, lately competes with $W_\gamma(t)$ for the name "twisted Alexander polynomial." In many cases they are equal (up to multiplication by units); generally, $\Delta_\gamma(t)$ is $\det M^0_{\gamma\otimes\epsilon}$ divided by a factor of $\det \Phi(x_0 - 1)$. See [6] or [14] for details.

Let $\mathbb{C}(t)$ denote the field of rational functions. When $\det M^0_{\gamma\otimes\epsilon} \neq 0$, the chain complex

$$C_*(X; V(t)) = (\mathbb{C}(t) \otimes_{\mathbb{C}} V) \otimes_\gamma C_*(\tilde{X}) \tag{1.4}$$

is acyclic [7], and hence the (Reidemeister) torsion $\tau_\gamma(t)$ is defined. In [6] it is shown that $\tau_\gamma(t)$ coincides with the Wada invariant $W_\gamma(t)$.

Remark 1.4. (1) Conjugating the representation γ corresponds to a change of basis for V. It is well known that the invariants $W_\gamma(t), \Delta_\gamma(t)$ and $\tau_\gamma(t)$ are unchanged.

(2) The indeterminacy of sign in the definition of $\tau_\gamma(t)$ can be removed (see [8]).

T. Kitano used Poincaré duality to prove in [7] that for orthogonal representations $\gamma : \pi \to \mathrm{SO}_n(\mathbb{R})$, the torsion $\tau_\gamma(t)$ is reciprocal; that is, $\tau_\gamma(t^{-1}) \doteq \tau_\gamma(t)$. (In fact Kitano shows that $\tau_\gamma(t^{-1})$ and $\tau_\gamma(t)$ are equal up to multiplication by $\pm t^{ni}$.) He asked whether reciprocality holds for general representations $\gamma : \pi \to \mathrm{SL}_n(\mathbb{C})$. Several years later, Kirk and Livingston showed in [6] that reciprocality holds whenever γ is unitary.

It is not difficult to find representations $\gamma : \pi \to \mathrm{GL}_n\mathbb{C}$ such that $\tau_\gamma(t)$ is non-reciprocal. For example, consider the Wirtinger presentation $\langle x_0, x_1, x_2 \mid x_0x_1 = x_2x_0, x_1x_2 = x_0x_1\rangle$ of the trefoil knot group π. The assignment $x_i \mapsto X_i \in \mathrm{GL}_1\mathbb{C}$, such that $X_i = \begin{pmatrix} 2 \end{pmatrix}$, $i = 0, 1, 2$, yields the non-reciprocal invariant

$$\tau_\gamma(t) = \frac{4t^2 - 2t + 1}{2t - 1}.$$

(This simple example was suggested to us by S. Friedl.) The question of reciprocality for representations in $\mathrm{SL}_n\mathbb{C}$ is more subtle. The question was proposed by Kitano [7]; it appeared recently in [2].

In Section 2 we show that reciprocality need not hold for general representations in $\mathrm{SL}_n\mathbb{C}$. The representations γ that we consider have the property that the dual representation $\bar\gamma$, obtained by replacing each matrix $\gamma(g), g \in \pi$, by its inverse-transpose, is not conjugate to γ.

In Section 3 we prove that if a representation $\gamma : \pi \to \mathrm{SL}_n\mathbb{C}$ is conjugate to its dual, then the torsion $\tau_\gamma(t)$ is reciprocal.

2 Examples

Any reciprocal even-degree integral polynomial $\Delta(t)$ such that $\Delta(1) = \pm 1$ arises as the Alexander polynomial of a knot (see [5], for example). Let $f(t)$ be any monic integral polynomial with constant coefficient -1 and $f(1) = \pm 1$. Choose a knot k with Alexander polynomial $\Delta(t) = f(t)f(t^{-1})$.

Let C be the companion matrix of $(t-1)f(t)$. Then $C \in \mathrm{SL}_n\mathbb{Z}$, where $\deg f = n - 1$. Consider the cyclic representation $\gamma : \pi \to \mathrm{SL}_n\mathbb{Z}$ sending each generator $x_0, x_1, \ldots, x_k$ of a Wirtinger presentation of π to C. We have

$$\tau_\gamma(t) \doteq \frac{\det M^0_{\gamma\otimes\epsilon}}{\det \Phi(x_0 - 1)} \doteq \frac{\det M^0_{\gamma\otimes\epsilon}}{f(t^{-1})(t-1)}. \tag{2.1}$$

The matrix $M^0_{\gamma\otimes\epsilon}$ can be obtained from the $(k \times k)$ Alexander matrix $M(t)$ by replacing each polynomial entry $\sum a_i t^i$ with the $(n \times n)$ block matrix $\sum a_i (tC)^i$. Since the $n \times n$ blocks commute,

$$\det M^0_{\gamma\otimes\epsilon} = \prod_\lambda \det M(t\lambda),$$

where λ ranges over the eigenvalues of C, that is, the roots of $(t-1)f(t)$ (see [9] for details). Hence

$$\det M^0_{\gamma\otimes\epsilon} \doteq \prod_{\lambda} \Delta(t\lambda) \;=\; \Delta(t) \prod_{\lambda: f(\lambda)=0} f(t\lambda)f(t^{-1}\lambda^{-1}).$$

Since $\Delta(t)$ and $\det M^0_{\gamma\otimes\epsilon}(t)$ are integral polynomials, so is

$$g(t) = \prod_{\lambda: f(\lambda)=0} f(t\lambda)f(t^{-1}\lambda^{-1}).$$

Lemma 2.1. *If* $\deg f = 2$*, then* $g(t)$ *is reciprocal.*

Proof. Our assumptions about $f(t)$ imply that its roots have the form $\lambda, -\lambda^{-1}$, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $g(t) = f(t\lambda)f(t^{-1}\lambda^{-1})f(-t\lambda^{-1})f(-t^{-1}\lambda)$ while $g(t^{-1}) = f(t^{-1}\lambda)f(t\lambda^{-1})f(-t^{-1}\lambda^{-1})f(-t\lambda)$. Observe that $g(t)$ and $g(t^{-1})$ have the same roots:

- $f(t\lambda)$ and $f(-t^{-1}\lambda^{-1})$ have roots: $t = 1, -\lambda^{-2}$;
- $f(t^{-1}\lambda^{-1})$ and $f(-t\lambda)$ have roots: $t = -1, \lambda^{-2}$;
- $f(-t\lambda^{-1})$ and $f(t^{-1}\lambda)$ have roots: $t = 1, -\lambda^{2}$;
- $f(-t^{-1}\lambda)$ and $f(t\lambda^{-1})$ have roots: $t = -1, \lambda^{2}$.

It follows that $g(t^{-1}) = \alpha g(t)$, for some $\alpha \in \mathbb{C} \setminus \{0\}$. Letting $t = 1$, we see that $\alpha = 1$. Hence $g(t^{-1}) = g(t)$.

□

Remark 2.2. The numerator $\det M^0_{\gamma\otimes\epsilon}$ of (1.1) is a polynomial invariant $D_\gamma(t)$ of k, well defined up to multiplication by units in $\mathbb{C}[t^{\pm 1}]$ (see [14]). Since $\Delta(t)$ is reciprocal, Lemma 2.1 implies that $D_\gamma(t)$ is reciprocal whenever $\deg f = 2$. Example 2.5 below shows that this conclusion need not hold when $\deg f > 2$.

Proposition 2.3. *Let* $f(t)$ *be a polynomial as above with degree* 2*. If* $f(t)$ *is non-reciprocal, then* $\tau_\gamma(t)$ *is a non-reciprocal integral polynomial of the form* $(t-1)h(t)$*.*

Proof. From equation (2.1),

$$\tau_\gamma(t) \doteq \frac{f(t)f(t^{-1})g(t)}{f(t^{-1})(t-1)} \doteq \frac{f(t)g(t)}{t-1}. \tag{2.2}$$

Since $g(t)$ and $t-1$ are reciprocal but $f(t)$ is not, $\tau_\gamma(t)$ is non-reciprocal. To see that $\tau_\gamma(t)$ has the desired form, note that $(t-1)^2$ divides $g(t)$ since both factors $f(t\lambda), f(-t\lambda^{-1})$ of $g(t)$ vanish when $t=1$.

□

Example 2.4. Let $f(t) = t^2 - t - 1$. Then

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Computation shows that $g(t) = (t-1)^2(t+1)^2(t^2-3t+1)(t^2+3t+1)$. By equation (2.2),

$$\tau_\gamma(t) \doteq (t^2 - t + 1)(t-1)(t+1)^2(t^2-3t+1)(t^2+3t+1),$$

which is non-reciprocal.

Example 2.5. Let $f(t) = t^3 - t - 1$. Then

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Computation shows that $g(t) = (t-1)^3(t^3-t-1)^2(t^3-t^2+2t-1)(t^6+3t^5+5t^4+5t^3+5t^2+3t+1)$. The polynomial $f(t)f(t^{-1})g(t)$ is the numerator $D_\gamma(t)$ of Wada's invariant (1.1). It is non-reciprocal.

It is not difficult to see that for any cyclic representation, $D_\gamma(t) \doteq \Delta_\gamma(t)$ (see Section 3 of [14]) Hence this example shows that $\Delta_\gamma(t)$ can also be non-reciprocal.

3 Sufficient condition for reciprocality

If $\gamma : G \to \mathrm{GL}_n\mathbb{F}$ is a linear representation, then the *dual* (or *contragredient*) representation $\bar\gamma$ is defined by

$$\bar\gamma(g) = {}^t\gamma(g)^{-1},$$

where ^{t} denotes transpose.

The following elementary lemma is included for the reader's convenience.

Lemma 3.1. *A representation $\gamma : G \to \mathrm{GL}_n\mathbb{F}$ is conjugate to its dual if and only if there exists a nondegenerate bilinear form $(v, w) \mapsto \{v, w\} \in \mathbb{F}$ on V such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$ and $g \in G$.*

Proof. Assume that $\bar{\gamma}$ is conjugate to γ. Then there exists a matrix $A \in \mathrm{GL}_n\mathbb{F}$ such that $A^{-1}\gamma(g)A = {}^t\gamma(g)^{-1}$, for all $g \in G$. Define $\{v, w\} = vA\,{}^tw$. Since A is invertible, the bilinear form is nondegenerate. It is easy to check that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$.

Conversely, assume that γ preserves a nondegenerate bilinear form $(v, w) \mapsto \{v, w\}$. There exists an invertible matrix $A \in \mathrm{GL}_n\mathbb{F}$ such that $\{v, w\} = vA\,{}^tw$. Since γ preserves the form, we have $v\gamma(g)A\,{}^t\gamma(g)\,{}^tw = \{v\cdot g, w\cdot g\} = \{v, w\} = vA\,{}^tw$, for all $v, w \in V, g \in G$. It follows that $\gamma(g)A\,{}^t\gamma(g) = A$ for all $g \in G$. Hence $A^{-1}\gamma(g)A = {}^t\gamma(g)^{-1}$, and so $\bar{\gamma}$ is conjugate to γ.

□

As before, let k be a knot with group π. Assume that $\gamma : \pi \to \mathrm{SL}_n\mathbb{F}$ is a representation, where $\mathbb{F}$ is an arbitrary field. As above, $V = \mathbb{F}^n$ is a right $\mathbb{Z}[\pi]$-module via $v \cdot g = v\gamma(g)$, for all $v \in V$ and $\gamma \in \pi$. Let $W = \mathbb{F}^n$ with the dual $\mathbb{Z}[\pi]$-module structure given by $w \cdot g = w\,{}^t\gamma(t)^{-1}$.

Theorem 3.2. *Assume that $\det M^0_{\gamma\otimes\epsilon} \neq 0$. If γ is conjugate to its dual representation $\bar{\gamma}$, then the torsion $\tau_\gamma(t)$ is reciprocal.*

Proof. The following argument is similar to those of [7] and [6].

Recall that X is the exterior of k, endowed with a CW cell structure. Let X' be the same space but with the dual cell structure. Let $^- \colon \mathbb{F}(t) \to \mathbb{F}(t)$ be the involution induced by $t \mapsto t^{-1}$.

Assume that $\gamma : \pi \to \mathrm{SL}_n\mathbb{F}$ is a representation that is conjugate to its dual. By Lemma 3.1 there exists a nondegenerate bilinear form $(v, w) \mapsto \{v \cdot g, w \cdot g\}$ such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V, g \in \pi$. Consider the twisted chain complexes

$$C_* = (\mathbb{F}(t) \otimes V) \otimes C_*(\tilde{X}),\ D_* = (\mathbb{F}(t) \otimes W) \otimes C_*(\tilde{X}', \partial\tilde{X}'),$$

where $\tilde{X}$ and $\tilde{X}'$ denote universal covering spaces of X and X', respectively. We abbreviate these by $V_{\gamma\otimes\epsilon} \otimes C_*(\tilde{X})$ and $V_{\bar{\gamma}\otimes\epsilon} \otimes C_*(\tilde{X})$, respectively.

Define a bilinear pairing $C_q \times D_{3-q} \to F(t)$ by

$$\langle p \otimes v \otimes z_1, q \otimes w \otimes z_2\rangle = \sum_{g\in\pi}(z_1 \cdot gz_2)p\bar{q}\{v \cdot g, w\}, \tag{3.1}$$

where $z_1 \cdot gz_2$ is the algebraic intersection number in $\mathbb{Z}$ of cells z_1 and gz_2. We extend linearly.

The pairing induces a $\mathbb{F}(t)$-module isormorphism $D_{3-q} \to \overline{\mathrm{Hom}}(C_q, \mathbb{F}(t))$, where $\overline{\mathrm{Hom}}$ denotes the dual space with $(q \cdot h)(z) = \bar{q}(h(z))$, for all $q \in \mathbb{F}(t), z \in C_q$. Consequently, there exists a nondegenerate pairing $H_q(X; V(t)) \times H_{3-q}(S', \partial X'; W(t)) \to \mathbb{F}(t)$. Since the torsion of C_* is defined, by our hypothesis, the torsion of D_* is too.

Choose a basis $\{v_i\}$ over $\mathbb{F}$ for V and lifts to $\tilde{X}$ of simplices of X to get a preferred $\mathbb{F}(t)$-basis for C_*. Basis elements have the form $1 \otimes v_i \otimes z_j$. Then D_* has a natural dual basis over $\mathbb{F}(t)$ obtained by picking a basis for W that is dual to the basis for V with respect to $\{,\}$, and dual cells in $\tilde{X}'$ of the fixed lifts of simplices of X. As observed in [6], the bases for C_* and D_* that we build are dual with respect to bilinear form (3.1).

Let $\tau(X; V_{\gamma\otimes\epsilon})$ denote the torsion of C_*. Similarly, let $\tau(X', \partial X'; V_{\bar{\gamma}\otimes\epsilon})$ denote the torsion of D_*. Then $\tau(X; V_{\gamma\otimes\epsilon}) = \tau(X', \partial X'; V_{\bar{\gamma}\otimes\bar{\epsilon}})$ by Theorem $1'$ of [11]. Futhermore,

$$\begin{aligned}
\tau(X', \partial X'; V_{\bar{\gamma}\otimes\bar{\epsilon}}) &= \tau(X, \partial X; V_{\bar{\gamma}\otimes\bar{\epsilon}}) \quad \text{(by subdivision)} \\
&= \tau(X, \partial X; V_{\gamma\otimes\bar{\epsilon}}) \quad \text{(since } \gamma \text{ is conjugate to } \bar{\gamma}) \\
&= \bar{\tau}(X, \partial X; V_{\gamma\otimes\epsilon}) \\
&= \bar{\tau}(X; V_{\gamma\otimes\epsilon}),
\end{aligned}$$

using Lemma 2 of [12] and $\tau(\partial X; V_{\gamma\otimes\epsilon}) = 1$ (see [6]). Hence

$$\tau_\gamma(t) = \tau(X; V_{\gamma\otimes\epsilon}) = \bar{\tau}(X; V_{\gamma\otimes\epsilon}) = \bar{\tau}_\gamma(t).$$

□

Remark 3.3. If $\mathbb{F} = \mathbb{R}$, and the bilinear form in Lemma 3.1 is positive-definite, then by considering a basis for V that is orthonormal with respect to the form, we see that A is the identity matrix. In this case, $\gamma(g) = {}^t\gamma(g)^{-1}$ for all $g \in G$, and hence γ is conjugate to an orthogonal representation. Similarly, if $\mathbb{F} = \mathbb{C}$ and the bilinear form is hermitian and positive-definite, γ is conjugate to a unitary representation.

Corollary 3.4. *If $\gamma : \pi \to \mathrm{Sp}_{2n}\mathbb{C}$ is a symplectic representation, then $\tau_\gamma(t)$ is reciprocal.*

Proof. The representation preserves the bilinear form given by $A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$.
□

Since $\mathrm{Sp}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}$, the following is immediate.

Corollary 3.5. *If γ is any representation of π in $\mathrm{SL}_2\mathbb{C}$, then $\tau_\gamma(t)$ is reciprocal.*

Corollary 3.5 shows that Example 2.4 is, in a sense, the simplest possible.

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References

[1] J. Cha, *Fibred knots and twisted Alexander invariants*, Trans. Amer. Math. Soc. **355** (2003), 4187–4200.

[2] S. Friedl and S. Vidussi, *A survey of twisted Alexander polynomials*, preprint, 2009, arXiv:0905.0591.v2.

[3] S. Friedl and S. Vidussi, *Twisted Alexander polynomials detect fibered 3-manifolds*, preprint, 2008, arXiv:0805.1234v2.

[4] J.A. Hillman, C. Livingston and S. Naik, *Twisted Alexander polynomials of periodic knots*, Algebraic and Geom. Top. **6** (2006), 145–169.

[5] A. Kawauchi, A Survey of Knot Theory, Birkhäuser Verlag, Basel, 1996.

[6] P. Kirk and C. Livingston, *Twisted Alexander invariants, Reidemeister torsion and Casson-Gordon invariants*, Topology **38** (1999), 635–661.

[7] T. Kitano, *Twisted Alexander Polynomial and Reidemeister Torsion*, Pacific J. Math. **174** (1996), 431–442.

[8] T. Kitayama, *Refinement of twisted Alexander invarariants and sign-determined Reidemeister torsions*, preprint, 2009, arXiv:0705.2371.v4.

[9] I. Kovacs, D. S. Silver, S. G. Williams, *Determinants of commuting-block matrices*, Amer. Math. Monthly **106** (1999), 950–952.

[10] X.S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (English Series) **17**(2001), 361–380.

[11] J. Milnor, *A Duality theorem for Reidemeister torsion*, Ann. of Math. **76** (1962), 137–147.

[12] J. Milnor, *Two complexes which are homeomorphic but combinatorially distinct*, *Ann. of Math.* **74** (1961), 575–590.

[13] D. Rolfsen, Knots and links, Mathematical Lecture Series **7**, Publish of Perish, Inc., 1976.

[14] D.S. Silver and S.G. Williams, *Dynamics of Twisted Alexander Invariants*, preprint, 2009, arXiv:0801.2118v3

[15] G. Torres and R.H. Fox, *Dual presentations of the group of a knot*, *Annals of Math.* **59** (1959), 211–218.

[16] M. Wada, *Twisted Alexander polynomial for finitely presented groups*, *Topology* **33** (1994), 241–256.

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