Knot Group Epimorphisms, II

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Abstract

We consider the relations $\geq$ and $\geq_p$ on the collection of all knots, where $k \geq k'$ (respectively, $k \geq_p k'$) if there exists an epimorphism $\pi k \to \pi k'$ of knot groups (respectively, preserving peripheral systems). When k is a torus knot, the relations coincide and k' must also be a torus knot; we determine the knots k' that can occur. If k is a 2-bridge knot and $k \geq_p k'$, then k' is a 2-bridge knot with determinant a proper divisor of the determinant of k; only finitely many knots k' are possible.

Keywords: Knot group, peripheral structure[1]

1 Introduction

In recent years, numerous papers have investigated epimorphisms between knot groups and non-trivial maps between knot exteriors (or compact, orientable 3-manifolds with boundary); see [2], [9], [18], [22], [16], [27], [28], for example. We consider the first of these problems, which we formulate as follows (cf. [22], [23]).

1. Given a nontrivial knot $k \subset \mathbb{S}^3$, classify the collections of knots K for which there exists an epimorphism of knot groups $\pi K \to \pi k$, perhaps one preserving peripheral structure.

2. For k fixed, classify those knots K for which there exists an epimorphism $\pi k \to \pi K$.

Let k be a knot in $\mathbb{S}^3$, and let $E(k)$ denote the exterior of k. Orient both $\mathbb{S}^3$ and k. Choose and fix a point $*$ on $\partial E(k)$, and set $\pi k = \pi_1(\mathbb{S}^3 \setminus k, *)$. Also, choose oriented curves m and l in $\partial E(k)$ meeting transversely at $*$

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and representing a meridian-longitude system for πk; we use m and l to represent their classes in πk as well. We consider knot group epimorphisms $\phi : \pi K \to \pi k$ defined up to automorphisms of πk. Since the type of a knot is determined by its complement [12], an automorphism of πk necessarily sends m to a conjugate of m or m^{-1} [33]. (We recall that knots have the *same type* if there exists an autohomeomorphism of $\mathbb{S}^3$ taking one knot to the other.) Hence we will call an element of πk that is conjugate to m or m^{-1} a *meridian* of k, and we say that such an element is *meridional*.

Recall that if k is nontrivial, then the inclusion-induced homomorphism $i_* : \pi_1(\partial E(k)) \to \pi_1(E(k))$ is injective and defines a conjugacy class of subgroups of πk – the so-called peripheral subgroups of πk, each member isomorphic to $\mathbb{Z} \times \mathbb{Z}$. A homomorphism of knot groups *preserves peripheral structure* if it takes peripheral subgroups into peripheral subgroups. Recall also that for knots k and k' and epimorphism $\phi : \pi k \to \pi k'$, we always have $\phi[(\pi k)'] = [(\pi k')']$, $\phi^{-1}[(\pi k')'] = (\pi k)'$, and $\ker(\phi) \subset (\pi k)'$. Here $(\ )'$ denotes commutator subgroup.

We write $k \geq k'$ whenever there exists an epimorphism $\phi : \pi k \to \pi k'$. If an epimorphism exists that preserves peripheral structure, then we write $k \geq_p k'$. The relation $\geq$ is a partial order on prime knots, while $\geq_p$ is a partial order on the collection of all knots [27]. (In [27] and [28] a slightly different notation is used.) We write $k > k'$ if $k \geq k'$ but $k \neq k'$. The expression $k >_p k'$ has a similar meaning.

In Section 2, we prove that if $\phi : \pi k \to \pi k'$ is an epimorphism taking a meridian of k to a meridian of k' and if k' is prime, then ϕ preserves peripheral structure. We prove several results about the relations $k \geq k'$ and $k \geq_p k'$ when k is either a torus knot or a 2-bridge knot. For a given torus knot k, Proposition 2.4 classifies those knots k' for which there exists an epimorphism $\phi : \pi k \to \pi k'$, while Proposition 2.5 describes ϕ up to an automorphism of $\pi k'$. If k is a (p_1, q_1) 2-bridge knot (with p_1, q_1 relatively prime odd integers, $p_1 \geq 3$ and $-p_1 < q_1 < p_1$) and if $k >_p k'$ with k' nontrivial, then Proposition 2.10 asserts that k' is a (p_2, q_2) 2-bridge knot such that p_2 properly divides p_1.

As a corollary to Proposition 2.10, we show that given any 2-bridge knot k, there are only finitely many knots k' for which a meridian-preserving epimorphism $\pi k \to \pi k'$ exists. This is a partial answer to a problem of J. Simon (Problem 1.12 of [17]). We close Section 2 with an example of two knots k, k' for which there exists an epimorphism $\pi k \to \pi k'$ preserving meridians but taking the longitude of k to 1. Such epimorphisms correspond to zero-degree maps $E(k) \to E(k')$.

In Section 3 we introduce the notions of minimal and p-minimal knots.

We prove that twist knots are p-minimal, while a (p_1, p_2)-torus knot is minimal if and only if both p_1 and p_2 are prime. Section 4 comprises a list of open questions.

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2 Epimorphisms and partial orders

The followng proposition and its corollary give the useful fact that if m is a meridian for a knot k, then an epimorphism $\phi : \pi k \to \pi k'$ such that $\phi(m)$ is a meridian of k' also preserves peripheral structure, provided that k' is a prime knot.

Proposition 2.1. *Let k be a prime knot with meridian-longtitude pair (m, l) Then $Z(m) \cap (\pi k)'' = \langle l \rangle$ (= subgroup of πk generated by l), where $Z(m)$ is the centralizer of m in πk.*

Proof. Suppose that $g \in (\pi k)''$, $g \neq 1$, and $mg = gm$. If $g \in \langle m, l \rangle$, then $g = l^d$, for some $d \neq 0$, since $g \in (\pi k)''$. We therefore assume that $g \notin \langle m, l \rangle$. By Theorem 1 of [29], k is either a torus knot or a nontorus cable knot, since k is prime.

Assume first that k is a torus knot, and set $P = \langle m, l \rangle$. By Theorem 2 of [29], $g^{-1}Pg \cap P$ is infinite cyclic (since $mg = gm$). Since $g^{-1}Pg \cap P$ contains m, we have $g^{-1}Pg \cap P = \langle m \rangle$. But $g^{-1}Pg \cap P$ also contains a generator of the center of πk. Since this is a contradiction, k must be a nontorus, cable knot.

We have now that $E(k) = E(k_0) \cup_{T_0} S$, where S is a cable space and k is a cable about a nontrivial knot k_0. We can assume that S is a component of the characteristic submanifold of $E(k)$. Note that S is a small Seifert fibered manifold having an annulus with exactly one cone point as its base orbifold. Since m and l can be considered as elements of $\pi_1 S$ (well defined up to conjugation in πk), it follows from Theorem VI 1.6 (i) of [13] that $Z(m)$ is a subgroup of $\pi_1 S$. Therefore, $g \in \pi_1 S$ (along with m and l), and hence g commutes with a generator of the center of $\pi_1 S$, which of course belongs to P.

As in the case of a torus knot, $g^{-1}Pg \cap P$ (as a subgroup of πk) is neither trivial nor infinite cyclic, which yields a contradiction. ∎

Remark 2.2. 1. It is easy to see that the proposition does not hold if k is composite.

2. If $\phi : \pi k \to \pi k'$ is an epimorphism, then $\phi(l) \in Z(\phi(m)) \cap (\pi k')''$ $(= Z(\phi(m) \cap (\pi k')')$ [14]. In fact, given k' and elements $\mu, \lambda \in \pi k'$, there exists a knot k with meridian-longitude pair (m, l) and an epimorphism $\phi : \pi k \to \pi k'$ such that $\phi(m) = \mu$ and $\phi(l) = \lambda$ if and only if μ normally generates $\pi k'$and $\lambda \in Z(\mu) \cap (\pi k')''(see[14])$.

Corollary 2.3. *Let k be a knot and k' a prime knot. Let (m, l) and (m', l') be meridian-longitude pairs for k and k', respectively. If there exists an epimorphism $\phi : \pi k \to \pi k'$ with $\phi(m) = m'$, then ϕ preserves peripheral structure; in fact, $\phi(l) = (l')^d$, for some $d \in \mathbb{Z}$.*

Proof. As noted in Remark 2.2 above, we must have $\phi(l) \in Z(\phi(m)) \cap (\pi k')''$ $(= Z(m') \cap (\pi k')'')$. Since k' is prime, $Z(m') \cap (\pi k')'' = \langle l \rangle$. Thus $\phi(l) = (l')^d$, for some $d \in \mathbb{Z}$. □

When we say that a knot is a (p, q)-torus knot, we will always assume that $p, q \geq 2$ and that $(p, q) = 1$. Such a knot is necessarily nontrivial.

Proposition 2.4. *Let k be a (p_1, p_2)-torus knot, and let k' be a nontrivial knot. The following statements are equivalent.*

(1) $k \geq_p k'$,

(2) $k \geq k'$,

(3) k' is an (r_1, r_2)-torus knot, for some $r_1, r_2 \geq 2$, such that $r_1 | p_1$ and $r_2 | p_2$, or $r_1 | p_2$ and $r_2 | p_1$.

Proof. Obviously, statement (1) implies statement (2). Assume that $k \geq k'$. Then there exists an epimorphism $\phi : \pi k \to \pi k'$. If k' is not a torus knot, then ϕ must kill the center of πk, since the only knots with groups having nontrivial centers are torus knots [6], and thus ϕ factors through the free product $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$ of cyclic groups. But no knot group can be a homomorphic image of $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$, since knot groups contain no nontrivial elements of finite order. Therefore, there exist integers $r_1, r_2 \geq 2$ with $(r_1, r_2) = 1$ such that k' has the type of an (r_1, r_2)-torus knot.

We have the following commutative diagram of epimorphisms

$$\begin{array}{ccc} \pi k & \longrightarrow & \mathbb{Z}_{p_1} * \mathbb{Z}_{p_2} \\ \phi \downarrow & & \downarrow \psi \\ \pi k' & \longrightarrow & \mathbb{Z}_{r_1} * \mathbb{Z}_{r_2} \end{array}$$

in which the horizontal maps are canonical, and ψ is the diagram-filling homomorphism. Let t_1, t_2 generate $\mathbb{Z}_{p_1}, \mathbb{Z}_{p_2}$, respectively. Since ψ is an epimorphism, $\psi(t_1), \psi(t_2)$ generate $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$. Moreover, each of $\psi(t_1)$ and

$\psi(t_2)$ has finite order in $\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$. It follows from the torsion theorem for free products (see Theorem 1.6 of [15], for example) that there are generators s_1 and s_2 of $\mathbb{Z}_{r_1}$ and $\mathbb{Z}_{r_2}$, respectively, such that either $\psi(t_1) = u_1 s_1 u_1^{-1}$ and $\psi(t_2) = u_2 s_2 u_s^{-1}$ or else $\psi(t_1) = u_1 s_2 u_1^{-1}$ and $\psi(t_2) = u_2 s_1 s_2^{-1}$, for some $u_1, u_2 \in \mathbb{Z}_{p_1} * \mathbb{Z}_{p_2}$. Hence either $r_1 | p_1$ and $r_2 | p_2$ or else $r_1 | p_2$ and $r_2 | p_1$. Hence statement (2) implies statement (3).

Finally, assume statement (3). Let T be a standardly embedded, unknotted torus in $\mathbb{S}^3$ with complementary solid tori V_1 and V_2 such that $V_1 \cap V_2 = T$. Assume that C_i is an oriented core of V_i, for $i = 1, 2$, serving as an axis for periodic rotations of $\mathbb{S}^3$, each taking T and the other axis to itself. Moreover, let k be a (p_1, p_2)-torus knot in T with $|\mathrm{lk}(k, C_i)| = p_i$, for $i = 1, 2$, and such that periodic rotations of $\mathbb{S}^3$ of appropriate orders about each C_i take k to itself (see Proposition 14.27 [7]). Assume that $r_1 | p_1$ and $r_2 | p_2$, and let $n_i r_i = p_i$, for $i = 1, 2$. A rotation of $\mathbb{S}^3$ of order n_2 about C_1 then yeilds a (p_1, r_2)-torus knot k'' as a factor knot. Similary, a rotation of $\mathbb{S}^3$ of order n_1 about the image axis of C_2 under the first rotation yields the (r_1, r_2)-torus knot k' as a factor knot. Thus we have $k \geq_p k'' \geq_p k'$; that is, k' is obtained from k by at most two periodic rotations, each of which preserves peripheral structure. If $r_1 | p_2$ and $r_2 | p_1$, then the proof is similar. □

For a given torus knot k, Proposition 2.4 classifies those nontrivial knots k' for which there exists an epimorphism $\phi : \pi k \to \pi k'$. The next result describes ϕ up to an automorphism of $\pi k'$. We recall from [26] that an automorphism of the (p, q)-torus knot group $\langle x, y \mid x^p = y^q \rangle$, with $p, q > 1$ and $(p, q) = 1$, has the form $x \mapsto w^{-1} x^{\epsilon} w,\ y \mapsto w^{-1} y^{\epsilon} w$, for $\epsilon \in \{-1, 1\}$.

Proposition 2.5. *If k and k' are nontrivial torus knots with groups $\pi k = \langle u, v \mid u^{p_1} = v^{p_2} \rangle$ and $\pi k' = \langle a, b \mid a^{r_1} = b^{r_2} \rangle$ such that $r_i | p_i$ $(i = 1, 2)$, and if $\phi : \pi k \to \pi k'$ is an epimorphism, then up to an automorphism of $\pi k'$, we have $\phi(u) = a^{n_2}$ and $\phi(v) = c^{-1} b^{n_1} c$, where $n_i r_i = p_i$ $(i = 1, 2)$ and $c = b^s a^t$, for some $s, t \in \mathbb{Z}$.*

Proof. The element $(\phi(u))^{p_1}$ is in the center $Z(\pi k')$. Hence $\phi(u) = c_1^{-1} a^{\alpha_1} c_1$ or $\phi(u) = c_2^{-1} b^{\alpha_2} c_2$, for some $c_1, c_2 \in \pi k'$ and $\alpha_1, \alpha_2 \in \mathbb{Z}$ (see Lemma II. 4.2 [13]). If $\phi(u) = c_2^{-1} b^{\alpha_2} c_2$ then $b^{\alpha_2 n_1 r_1} = b^{s r_2}$, for some s, and so $r_2 | \alpha_2$, since $(r_2, n_1 r_1) = 1$. But then $\phi(u) = c_2^{-1} b^{\alpha_2} c_2 \in Z(\pi k')$, which is a contradiction, since ϕ is an epimorphism. Thus $\phi(u) = c_1^{-1} a^{\alpha_1} c_1$ (and $\phi(v) = c_2^{-1} b^{\alpha_2} c_2$, for some $c_2 \in \pi k'$ and $\alpha_2 \in \mathbb{Z}$).

Now $(\phi(u))^{p_1} = (\phi(v))^{p_2} \in \mathbb{Z}(\pi k')$, and so $a^{\alpha_1 p_1} = b^{\alpha_2 p_2}$; that is, $a^{r_1 (n_1 \alpha_1)} = b^{r_2 (n_2 \alpha_2)}$. Since $a^{r_1} = b^{r_2}$ in $\pi k'$, it follows that $n_1 \alpha_1 = n_2 \alpha_2$;

hence $n_d|\alpha_e$ for $d, e \in \{1,2\}$ and $d \neq e$, as $(n_1, n_2) = 1$. Thus we can write $\alpha_2 = n_1\alpha_1 n_2^{-1}$ and get $\phi(u) = c_1 a^{\alpha_1} c_1$ and $\phi(v) = c_2^{-1} b^{n_1\alpha_1 n_2^{-1}} c_2$, where α_1 is a multiple of n_2. Setting $\alpha_1 n_2^{-1} = n$, we have $\phi(u) = c_1^{-1} a^{nn_2} c_1$ and $\phi(v) = c_2^{-1} b^{nn_1} c_2$, for $0 < n \leq \alpha_1$.

We show that $n = 1$. Since $(p_1, p_2) = 1$ and $ip_1 + jp_2 = (in_1)r_1 + (jn_2)r_2 = 1$, for some i and j, the element $u^j v^i$ can be taken as a meridian of k and the normal closure of $\phi(u^j v^i)$ is $\pi k'$. As a convenience, after conjugation of $\pi k'$ by c_1, we assume that $\phi(u) = a^{nn_2}$ and $\phi(v) = c^{-1} b^{nn_1} c$, where $c = c_2 c_1^{-1}$. So

$$\begin{aligned}\phi(u^j v^i) &= a^{(jn_2)n} c^{-1} b^{(in_1)n} c \\ &= (a^{(jn_2)n} b^{(in_1)n})(b^{-(in_1)n} c^{-1} b^{(in_1)n} c).\end{aligned}$$

Since $\phi(u^j v^i)$ normally generates $\pi k'$, we have $|\mathrm{lk}(k', m)| = 1$, where m represents $\phi(u^j v^i)$. Since $\phi(u^j v^i)$ and $a^{(jn_2)n} b^{(in_1)n}$ have the same abelianizations, this linking number is $r_1(in_1) + r_2(jn_2)$. Thus $\phi(u) = a^{n_2}$ and $\phi(v) = c^{-1} b^{n_1} c$.

Now $A = \{a^{n_2}, c^{-1} b^{n_1} c\}$ generates $\pi k'$ (by assumption), and since $B = \{a, c^{-1} b^{n_1} c\}$ generates A, then B generates $\pi k'$. Similarly, $C = \{a, c^{-1} bc\}$ generates B and hence $\pi k'$. Taking C as the generating set of $\pi k'$, it is now an exercise to show that c (in $\mathbb{Z}_{r_1} * \mathbb{Z}_{r_2}$) has the form $b^s a^t$ ($1 \leq s \leq r_2, 1 \leq t \leq r_1$). (In fact, such an exercise appears as Excercise 15, page 194, of [19].) Thus $\psi^{-1}(b^s a^t) = (b^s a^t) Z(\pi k')$, where $\psi : \langle a, b \mid a^{r_1} = b^{r_2} \rangle \to \langle a, b \mid a^{r_1}, b^{r_2} \rangle = \mathbb{Z}_{r_1} * \mathbb{Z}_{r_2}$ is defined by $a \mapsto a$ and $b \mapsto b$ so that $\{a, c^{-1} bc\}$ generates $\mathbb{Z}_{r_1} * \mathbb{Z}_{r_2}$.

□

Corollary 2.6. *Torus-knot group epimorphisms preserve peripheral structure.*

Proof. Let k be a (p_1, p_2)-torus knot, and let k' be a nontrivial knot. Suppose that there exists an epimorphism $\phi : \pi k \to \pi k'$. By Proposition 2.4, k' is an (r_1, r_2)-torus knot, and we can assume that $n_i r_i = p_i$, $(i = 1, 2)$. We have $\pi k = \langle u, v \mid u^{p_1} = v^{p_2} \rangle$, $\pi k' = \langle a, b \mid a^{r_1} = b^{r_2} \rangle$, and $ip_1 + jp_2 = (in_1)r_1 + (jn_2)r_2 = 1$, for some i, j. The element $u^j v^i$ is a meridian of k, and according to Proposition 2.5, we can assume that $\phi(u) = a^{n_2}$ and $\phi(v) = c^{-1} b^{n_1} c$, where $c = b^s a^t$ $(s, t \in \mathbb{Z})$. Thus $u^j v^i \mapsto a^{jn_2}(a^{-t} b^{-s}) b^{in_1}(b^s a^t) = a^{-t}(a^{jn_2} b^{in_1}) a^t$, which is clearly a meridian. It follows from Corollary 2.3 that ϕ preserves peripheral structure. In fact, if l_1 and l_2 are the (appropriate) longitudes of k and k', respectively, then $\phi(l_1) = a^{-t} l_2^{n_1 n_2} a^t$. □

Corollary 2.7. *If k is a torus knot and if $k \geq k'$, then $\pi k'$ embeds in πk.*

Proof. If k is a (p_1, p_2)-torus knot, then k' is an (r_1, r_2)-torus knot, for some $r_1, r_2 \geq 2$ with $(r_1, r_2) = 1$, and either $r_1 | p_1$ and $r_2 | p_2$ or else $r_1 | p_2$ and $r_2 | p_1$. It follows immediately that $\pi k'$ embeds in πk (see Theorem 5.1 [11]). $\square$

Remark 2.8. 1. If k is a (p_1, p_2)-torus knot, then there may well exist an (r_1, r_2)-torus knot k' such that $\pi k'$ embeds in πk but it is not the case that $k \geq k'$. For example, let $p_1 = 2$ and $p_2 = 3 \cdot 5$, and take $r_1 = 3, r_2 = 5$. Then $\pi k'$ embeds in πk by [11], but it is not the case that $k \geq k'$ by Proposition 2.4.

2. By Corollary 2.7, we know that if k is a torus knot, then $k \geq k'$ implies that $\pi k'$ is a subgroup of πk. The index of this embedding is finite, however, if and only if $k' = k$ (see Remark 3, page 42 of [11]).

Corollary 2.9. *If k is a torus knot and $k \geq k'$, then the crossing number of k is no less than that of k'.*

Proof. This follows from Proposition 2.4 and the fact that [20] the crossing number of a (p, q)-torus knot is $\min\{p(q-1), q(p-1)\}$. $\square$

We are particularly interested in the relative strengths of the two relations $\geq$ and $\geq_p$. Stated in general terms, our inquiry takes the form:

Q1. Given knots k and k', when does $k \geq k'$ imply $k \geq_p k'$?

For knots k and k' with at most 10 crossings $k \geq k'$ implies $k \geq_p k'$ by [18]. Question 1 generates a number of related questions. One of them is:

Q2. For which pairs of knots k and k' does there exist an epimorphism $\pi k \to \pi k'$ but no epimorphism preserving meridians?

For the present, we will consider the case $k \geq_p k'$ with k a 2-bridge knot. When we say that k is a (p, q) 2-bridge knot, we assume that p, q are relatively prime odd integers, $p \geq 3$ and $-p < q < p$. Recall that p is $\det(k)$, the determinant of k.

A representation $\pi \to \mathrm{SL}_2\mathbb{C}$ is *parabolic* if if projects to a parabolic representation $\pi \to \mathrm{PSL}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}/\langle -I \rangle$ sending some and thus every meridian to a parabolic element.

Proposition 2.10. *Let k be a (p_1, q_1) 2-bridge knot and let k' be a nontrivial knot. If $k >_p k'$, then k' is a (p_2, q_2) 2-bridge knot such that p_2 properly divides p_1 (and hence $\Delta_{k_2}(t)$ properly divides $\Delta_{k_1}(t)$.)*

Proof. Let m_1 be a meridian of k and m_2 a meridian of k'. Since $k \geq_p k'$, we have an epimorphism $\phi : \pi k \to \pi k'$ with $\phi(m_1) = m_2$, which induces an epimorphism $\pi k/\langle\langle m_1^2 \rangle\rangle \to \pi k'/\langle\langle m_2^2 \rangle\rangle$ of π-orbifold groups. Since k is a (p_1, q_1) 2-bridge knot, $\pi k/\langle\langle m_1^2 \rangle\rangle$ is the dihederal group D_{p_1} of order $2p_1$. Hence $\pi k'/\langle\langle m_2^2 \rangle\rangle$ is (finite) dihedral or $\mathbb{Z}_2$. By the Smith Conjecture [4], the group $\pi k'/\langle\langle m_2^2 \rangle\rangle$ is not $\mathbb{Z}_2$ (since by hypothesis, k' is not trivial) and thus it is isomorphic to D_{p_2}, for some p_2. Hence p_2 divides p_1, and therefore p_2 is odd. It follows that k' is a (p_2, q_2) 2-bridge knot, for some q_2; see Proposition 3.2 of [5]. Note that Proposition 3.2 of [5] depends on Thurston's orbifold geometrization theorem; see [3], for example).

To see that $p_1 > p_2$, we examine two cases. First assume that k is a 2-bridge torus knot (a $(p_1, 2)$-torus knot). By Proposition 2.4, k' is a $(p_2, 2)$-torus knot and $p_1 > p_2$, since $k >_p k'$.

For the second case, we assume that k is hyperbolic, and we apply Riley's parabolic representation theory [23]. Accordingly, if k is a (p, q) 2-bridge knot, then there are exactly $(p-1)/2$ conjugacy classes of nonabelian parabolic $\mathrm{SL}_2\mathbb{C}$ representations, corresponding to the roots of a monic polynomial $\Phi_{p,q}(w)$. As $\phi : \pi k \to \pi k'$ preserves peripheral structure, each parabolic representation $\theta' : \pi k' \to \mathrm{SL}_2\mathbb{C}$ induces a parabolic representation $\theta : \pi k \to \mathrm{SL}_2\mathbb{C}$, and since ϕ is an epimorphism, ϕ induces a one-to-one function of conjugacy classes. When $p_2 = p_1$, the function is a bijection, and hence some representation θ' must induce an injection $\theta : \pi k \to \mathrm{SL}_2\mathbb{C}$, a lift of the faithful discrete representation $\pi k \to \mathrm{PSL}_2\mathbb{C}$ corresponding to the hyperbolic structure of $\mathbb{S}^3 \setminus k$ (see [32]). Since $\theta = \theta' \circ \phi$, the epimorphism ϕ is in fact an isomorphism, a contradiction as k and k' have different types. Hence $p_1 > p_2$.

From the fact that $p_2 = |\Delta_{k'}(-1)|$ and $p_1 = |\Delta_k(-1)|$, it follows that $\Delta_{k'}(t)$ properly divides $\Delta_k(t)$. □

The following corollary provides a partial answer to a problem of J. Simon (see Problem 1.12 of [17]).

Corollary 2.11. *Let k be a 2-bridge knot. There exist only finitely many knots k' for which a meridian-preserving epimorphism $\pi k \to \pi k'$ exists.*

Proof. Assume that a meridian-preserving epimorphism $\phi : \pi k \to \pi k'$ exists. Since πk is generated by two elements, the same is true of $\pi k'$. By [21], k' is a prime knot. Corollary 2.3 implies that ϕ preserves peripheral systems. By Proposition 2.10, the knot k' is 2-bridge. The Alexander polynomial of k' must divide that of k, and by [24], only finitely many possible such knots k' exist. □

Remark 2.12. Given a (p, q) 2-bridge knot k, one can use the Riley polynomial $\Phi_{p,q}$ to determine all knots k' such that $k \geq_p k'$. Properties and applications of Riley polynomials will be discussed in a forthcoming paper.

Corollary 2.13. *If k is a nontrivial 2-bridge knot, then any meridian-preserving epimorphism $\phi : \pi k \to \pi k'$ maps the longitude nontrivially.*

Proof. By Proposition 2.10, k' is a 2-bridge knot. Assume that ϕ maps the longitude of k trivially. Let $\theta' : \pi k' \to \mathrm{SL}_2\mathbb{C}$ be any nonabelian parabolic representation. Then $\theta' \circ \phi$ is a nonabelian parabolic representation of πk sending the longitude to the identity matrix, contradicting Lemma 1 of [25]. Thus ϕ maps the longitude of k nontrivially.

□

Remark 2.14. 1. In [22], the authors give a sufficient condition for the existence of a peripheral-structure preserving epimorphism between 2-bridge link groups. The condition is in fact a very efficient machine for generating many such epimorphisms. For example, one can use it to show that $k >_p k'$ for k the $(175, 81)$ 2-bridge knot and k' the $(7, 3)$ 2-bridge knot, as pointed out by K. Murasugi.

2. F. Gonzalez-Acuña and A. Ramirez [9] proved that $k \geq_p \tau_{a,b}$, where $\tau_{a,b}$ is some torus knot, if and only if k has property Q. (A knot k has property Q if there is a closed surface F in $\mathbb{S}^3 = X \cup_F Y$ such that $k \subset F$ and k is imprimative in each of $H_1(X)$ and $H_1(Y)$. Basic examples of such knots are torus knots.) In [10], they determined the 2-bridge knots k such that $k \geq_p \tau_{a,2}$ for some odd $a \geq 3$.

3. Define a *knot manifold* to be a compact, connected, orientable, irreducible 3-manifold with boundary an incompressible torus. Such a manifold is said to be *small* if it contains no closed essential surface. A 3-manifold M *dominates* another 3-manifold N if there is a continuous, proper map $f : M \to N$ of nonzero degree. (Here proper means that $f^{-1}(\partial N) = \partial M$.) A knot manifold is *minimal* if it dominates only itself. In [2], Boileau and Boyer show that twist knots and $(-2, 3, n)$-pretzel knots (n not divisible by 3) are minimal.

Suppose that $k >_p k'$ with k' nontrivial, and let $\phi : \pi k \to \pi k'$ be an epimorphism that preserves peripheral structure. If (m_1, l_1) and (m_2, l_2) are fixed meridian-longitude pairs for πk and $\pi k'$, respectively, then we can assume that $\phi(m_1) = m_2$ and $\phi(l_1) = l_2^d$, for some $d \in \mathbb{Z}$. Then ϕ is induced by a proper map $f : E(k) \to E(k')$, and the absolute value of the degree of f is $|d|$, since $f_* : H_3(E(k), \partial E(k)) \to H_3(E(k'), \partial E(k'))$ takes the top class

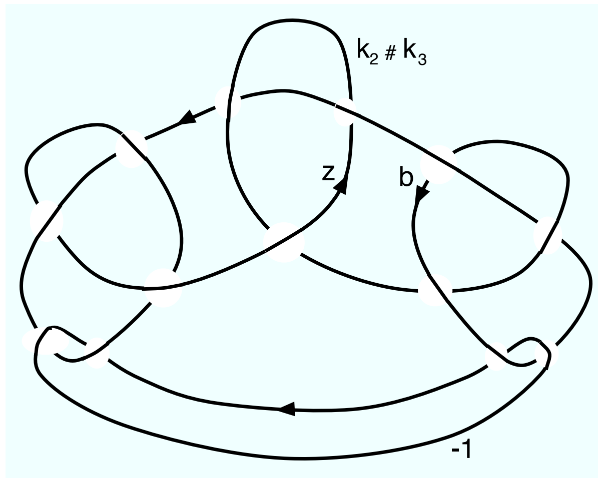

Figure 1: Surgery description of Riley's knot

of $H_3(E(k), \partial E(k))$ to $\deg(f)$ times the top class of $H_3(E(k'), \partial E(k'))$ (see Proposition 6.2 of [22]).

Example 2.15. From the proof of Corollary 2.6, it follows that an epimorphism $\pi k \to \pi k'$ in which each of k and k' is a nontrivial torus knot is induced by a nonzero-degree map; that is, a map sending $m_1 \mapsto m_2$ and $l_1 \mapsto l_2^d$ for some nonzero d. It is easy, however, to find knots k and k' and an epimorphism $\pi k \to \pi k'$ with $m_1 \mapsto m_2$ but $l_1 \mapsto 1$. For this, one can choose k to be the square knot and k' the trefoil.

As a second example, let R_y denote the knot of [23] termed his "favorite knot." A surgery description of R_y appears in Figure 1. It is not difficult to find an epimorphism $\pi R_y \to \pi(k_2 \sharp k_3)/\langle\langle zb^{-1} \rangle\rangle$, where $k_2 \sharp k_3$ is the square knot indicated in Figure 1. Since the longitude of $k_2 \sharp k_3$ goes to 1 (as does zb^{-1}) under the appropriate epimorphism $\pi(k_2 \sharp k_3) \to \pi 3_1$, we have a meridian-preserving, longitude-killing epimorphism $\pi R_y \to \pi 3_1$. A similar argument shows that $8_{20} > 3_1$ (with longitude sent to 1), but neither is $R_y > 8_{20}$ nor is $8_{20} > R_y$, since R_y and 8_{20} are prime fibered knots of the same genus. Other such examples can be found in [9].

3 Minimality

Definition 3.1. 1. A knot k is *minimal* if $k \geq k'$ implies that $\pi k \cong \pi k'$ or else k' is trivial.

2. k is *p-minimal* if $k \geq_p k'$ implies that $k = k'$ or else k' is trivial.

(Compare these definitions of "minimality" with that given in the second part of Remark 2.14 above.)

Recall that the Alexander polynomial of a nontrivial 2-bridge knot is not equal to 1. The next result follows immediately from Proposition 2.10.

Corollary 3.2. *If k is a nontrivial (p_1, q_1) 2-bridge knot and if no proper nontrivial factor of $\Delta_k(t)$ is a knot polynomial, then k is p-minimal. In particular, k is p-minimal if $\Delta_k(t)$ is irreducible or if p_1 is prime.*

Remark 3.3. A nontrivial (p_1, q_1) 2-bridge knot k can have both $\Delta_k(t)$ reducible and p_1 composite but still be p-minimal. The simplest example is $k = 6_1$, the $(9, 4)$ 2-bridge knot.

Corollary 3.4. *1. Every nontrivial twist knot is p-minimal.*

2. For each $n \geq 3$, there is a p-minimal knot with crossing number n.

Proof. In view of Corollary 3.2, we need only note that the Alexander polynomial of a nontrivial twist knot is quadratic and that for each $n \geq 3$ there is a twist knot with crossing number n. □

Corollary 3.5. *Every genus-one 2-bridge knot is p-minimal.*

Proof. Let k be a genus-one 2-bridge knot. Since k is alternating, the degree of its Alexander polynomial $\Delta_k(t)$ is 2. If k' is a nontrivial knot and $k >_p k'$, then k' is a 2-bridge knot by Proposition 2.10, and $\Delta_{k'}(t)$ divides $\Delta_k(t)$ properly. Since the degree of a knot polynomial is even, the degree of $\Delta_{k'}(t)$ must be 0. This is impossible, however, since k' is nontrivial and alternating. □

From Proposition 2.4 we have:

Corollary 3.6. *If k is a (p_1, p_2)-torus knot, then k is minimal if and only if both p_1 and p_2 are prime.*

Corollaries 3.2-3.6 should be compared with theorems 3.16, 3.19 and corollaries 3.17, 3.18, 3.20 of [2]

4 Questions.

If $k \geq_p k'$, then what properties of k' can we deduce from k? For example, if k is fibered, then so is k'.

Not all properties of k are inherited by k'. For example, if k is prime, then k' need not be [27].

Q3: If k is alternating, must k' also be alternating? [Yes, if k is 2-bridge. This follows from Proposition 2.10 together with the fact that 2-bridge knots are alternating [1].]

Q4: Must the genus of k' be less than or equal to that of k? [Yes, if k is a 2-bridge knot or a fibered knot. In these cases k' is a knot with the same property and $\Delta_{k'}(t)$ divides $\Delta_k(t)$. However, the genus of a 2-bridge or fibered knot equal to half the degree of its Alexander polynomial [8], [7]. See also Proposition 3.7 of [27].]

Q5: Must the crossing number of k' be less than or equal to that of k? [Yes, if k is a torus knot. This follows from Corollary 2.9.]

Q6: Must the Gromov invariant of k' be less than or equal to that of k? [Yes, if k is a 2-bridge knot or a torus knot. If k is a 2-bridge knot, then any nontrivial epimorphism $\phi : \pi k \to \pi k'$ maps the longitude of k nontrivially, by Corollary 2.13. There exists a map $E(k) \to E(k')$ of degree $d > 0$, and hence the Gromov invariant of k is at least d times that of k'. If k is a torus knot, then so is k', by Proposition 2.4. Both k and k' have vanishing Gromov invariant [31].]

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