

# Virtual Knot Invariants from Group Biquandles and Their Cocycles

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## Abstract

A group-theoretical method, via Wada's representations, is presented to distinguish Kishino's virtual knot from the unknot. Biquandles are constructed for any group using Wada's braid group representations. Cocycle invariants for these biquandles are studied. These invariants are applied to show the non-existence of Alexander numberings and checkerboard colorings.

## 1 Introduction

The purposes of this paper include defining biquandle structures on groups, and giving a group-theoretic proof that Kishino's virtual knot is non-trivial. A biquandle structure or a birack structure is related to solutions to the set-theoretic Yang-Baxter equation (SYBE). Given an invertible solution that satisfies an additional condition (corresponding to a Reidemeister type I move), we obtain a biquandle, and every biquandle gives a solution to the SYBE. Most examples that were known up to this point came from generalizations of the Burau representation. The principal examples that we consider in this paper come from Wada's representations of braid groups as free group automorphisms. Our first example indicates that one of these representations can be used to distinguish Kishino's virtual knot from the unknot. By abelianizing such groups, we recover an analog of Burau matrix. Using these examples, we construct and calculate cocycle invariants that come from the homology theory of biquandles. As applications we give obstructions to checkerboard colorability and (mod 2)-Alexander numberings of arcs of virtual knots.

In Section 2 we examine Wada's group invariants for virtual knots. The biquandle structures are defined on any group in Section 3 using Wada's representations, and colorings of virtual knot diagrams by such biquandles are studied. Cocycle invariants are constructed and applied in Section 4.

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## 2 Wada's group invariants for virtual knots

Wada [28] considered representations of braid groups to the automorphism groups of free groups of the following type. Fix finite words  $u(x, y)$  and  $v(x, y)$  in  $x$  and  $y$ . Assume that the map  $x \mapsto u(x, y)$ ,  $y \mapsto v(x, y)$  is an automorphisms of the free group  $F_2 = \langle x, y \rangle$ . Wada's representation,  $\rho : B_n \rightarrow \text{Aut}(F_n)$  of the  $n$ -string braid group,  $B_n$ , to the automorphism group of the free group  $F_n = \langle x_1, \dots, x_n \rangle$  is defined as follows. The standard generator  $\sigma_i$  is sent to the automorphism

$$\begin{aligned} x_i \rho(\sigma_i) &= u(x_i, x_{i+1}), \\ x_{i+1} \rho(\sigma_i) &= v(x_i, x_{i+1}), \\ x_j \rho(\sigma_i) &= x_j, \quad (j \neq i, i+1). \end{aligned}$$

Such representations are also studied independently by A. J. Kelly [20] as mentioned in the review article by Przytycki [26] of Wada's paper.

A set of necessary conditions for the words  $u(x, y)$  and  $v(x, y)$  to define braid group representations was given by Wada. These are labeled  $T$ ,  $M$ ,  $B$  below, to indicate the conditions that come from the top, middle, and bottom arcs, respectively, in the type III Reidemeister move:

$$\begin{aligned} \text{T:} & \quad u(u(x, y), u(v(x, y), z)) = u(x, u(y, z)) \\ \text{M:} & \quad v(u(x, y), u(v(x, y), z)) = u(v(x, u(y, z)), v(y, z)) \\ \text{B:} & \quad v(v(x, y), z) = v(v(x, u(y, z)), v(y, z)) \end{aligned}$$

It is interesting to compare these to the conditions for a birack [3, 19, 25]. The conditions  $T, M, B$  are called *Wada's conditions*.

Figure 8 below indicates the correspondence with the Reidemeister type III move using the biquandle notation  $(R_1(A, B), R_2(A, B))$  introduced in Section 3.

The last two types of representation that Wada considered (types 6 and 7 in his notation) are depicted and named  $W_1$  and  $W_2$  in Figs. 1 and 2, respectively. In the type  $W_1$  representation  $u(x, y) = y^{-1}$  and  $v(x, y) = yxy$  while in the  $W_2$  representation  $u(x, y) = x^{-1}y^{-1}x$  and  $v(x, y) = y^2x$ . The left-hand side of each figure shows the words corresponding to the inverse automorphism of  $F_2$ . Another braid group representation that Wada considered yields the *the core group*,  $\text{Core}(K)$ , of a knot  $K$ . In this representation  $u(x, y) = y$  and  $v(x, y) = yx^{-1}y$ . If  $K$  is a classical knot (one without virtual crossings, see the next paragraph), then Kelly showed these three representations give rise to the same knot invariants.

We generalize Kelly's result in Theorem 2.5 below. It is a pleasant exercise to show that the core group representation,  $W_1$ , and  $W_2$  satisfy Wada's conditions.

Virtual knots and links were introduced by Kauffman (see [18] for example). A *virtual knot or link diagram* consists of a 4-valent graph embedded in the plane with two types of vertices called *classical crossings* and *virtual crossings*. A classical crossing has two non-adjacent edges at a vertex being distinguished as an *over-crossing arc* with the remaining two forming a pair of *under-crossing arcs*. The under-crossing arcs are indicated in a drawing by not connecting them to the vertex. In a drawing of a virtual crossing, the vertex is encircled. Two non-adjacent edges are regarded as a single edge. Thus the *edges* in a virtual diagram start and end at classical crossings and may pass

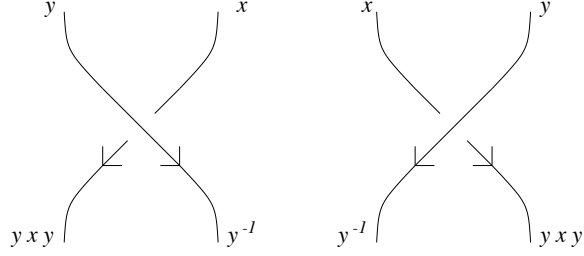


Figure 1: Wada's braid representation, type  $W_1$

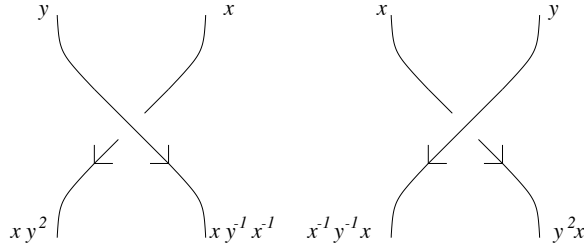


Figure 2: Wada's braid representation, type  $W_2$

through a virtual crossing. A *virtual knot* or *virtual link* is an equivalence class of virtual diagrams modulo the equivalence relation generated by the virtual Reidemeister moves.

Let  $\mathcal{E}$  denote the set of edges of a given diagram. Then Wada's group invariants are defined by assigning generators to elements of  $\mathcal{E}$  and assigning relations to crossings. Specifically, let  $\mathcal{E} = \{x_1, \dots, x_m\}$ . These letters will be also used as generators. At a positive crossing as depicted in the right of Fig. 1 or Fig. 2 denote the (generators corresponding to) top edges  $x, y \in \mathcal{E}$  as depicted. Let  $u, v \in \mathcal{E}$  be the bottom edges. Then the relations  $r_j, r'_j$  corresponding to this crossing are  $u = u(x, y)$ ,  $v = v(x, y)$ , respectively, where  $u(x, y)$  and  $v(x, y)$  are words in  $x, y$  that define Wada's representation. For example, for type  $W_1$ , the relations are  $u = y^{-1}$  and  $v = yxy$ , cf. the right of Fig. 1. At negative crossings, the words on the left are used. The following is a straightforward calculation.

**Lemma 2.1** *Wada's group invariants are well-defined for virtual knots and links.*

Denote by  $W_i(K)$ ,  $i = 1, 2$ , the group invariants that are defined by using the type  $W_i$  representation. Note that these groups are free groups of rank  $n$  for the unlink of  $n$  components for any positive integer  $n$ .

As an example we compute  $W_2(K)$  for a virtual knot  $K$  widely known as Kishino's knot, depicted in Fig. 3. It has attracted attention for its remarkable property that it is a connected sum of two diagrams of the trivial knot; it has trivial Jones polynomial, trivial fundamental group and quandle. Furthermore, it can be seen by computing presentations that both its core group and its Wada group of type  $W_1$  are infinite cyclic.

Kishino's knot was first proved to be non-trivial in [21] by using the Jones polynomial of the 3-strand parallel of the virtual knot. Later other methods were discovered, including a computation

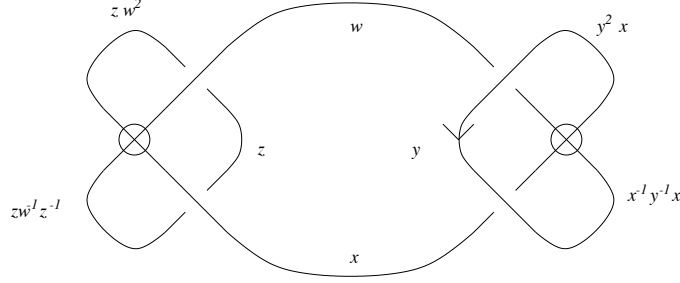


Figure 3: A group invariant for Kishino's knot

of the minimal genus of the surface on which the virtual knot lies [13], biquandles [1], and the bracket on surfaces [7]. Creel and Nelson [6] used Yang-Baxter cocycles [3] to distinguish Kishino's example from the unknot. See also [8].

**Proposition 2.2** *For the Kishino knot  $K$ , the group invariant  $W_2(K)$  is not infinite cyclic.*

*Proof.* In Fig. 3, Kishino's knot is depicted with labels on four of the edges induced by the  $W_2$  representation. The first two relations below are induced at the top right crossing, and the last two are induced at the bottom left crossing:

$$w = (x^{-1}y^{-1}x)(y^2x)^2, \quad (1)$$

$$y = (x^{-1}y^{-1}x)(y^2x)^{-1}(x^{-1}y^{-1}x)^{-1}, \quad (2)$$

$$x = (zw^{-1}z^{-1})^{-1}(zw^2)^{-1}(zw^{-1}z^{-1}), \quad (3)$$

$$z = (zw^2)^2(zw^{-1}z^{-1}). \quad (4)$$

These relations are induced from the following considerations. The edge with label  $w$  (on the upper right of the diagram) is the target of the under-crossing edge, and the arc labeled  $y$  is the target of the over-crossing edge. On the upper right crossing the edges are labeled  $y^2x$  and  $x^{-1}y^{-1}x$  since these are the target edges of the lower right crossing. These give the first two relations above. Similar considerations hold on the lower left for the arcs with labels  $z$  and  $x$ .

The relation (4) simplifies to (4') :  $zw = (w^2z)^2$ . Using this, (3) simplifies to (3') :  $x = (w^2z)^{-1}$ , or  $z = (xw^2)^{-1}$ , which simplifies (4') to (4'') :  $xw^2 = wx^2$ . The relation (2) simplifies to (2') :  $x^{-1}yx = y^3xy$ . Repeated use of (2') simplifies (1) to (1') :  $w = y^{-3}x^{-1}y^{-3}$ . Hence we obtain

$$W_2(K) = \langle x, y \mid x^{-1}yx = y^3xy, x(y^{-3}x^{-1}y^{-3})^2 = (y^{-3}x^{-1}y^{-3})x^2 \rangle.$$

Under the substitution  $a = xy^3$ , the first relation becomes  $ya = a^2y$ , and the second relation is rewritten as  $y^3a^{-1}y^3a^2 = a^2y^3ay^3$  (take the inverse of both sides of the relation and simplify after substitution). The first relation can be seen as a commutation relation, and in general  $ya^m = a^{2m}y$  for any positive integer  $m$ . It follows that  $y^3a^{15} = a^{10}y^3$ . Hence the group has presentation

$$W_2(K) = \langle a, y \mid ya = a^2y, y^3a^{15} = a^{10}y^3 \rangle.$$

The first relation implies that  $y^3ay^{-3} = a^8$ . Hence the second relation can be replaced by  $a^{120}y^3 = a^{10}y^3$  which is equivalent to  $a^{110} = 1$ . Using the first relation again, we can write

$a^{110} = (a^{55})^2 = ya^{55}y^{-1}$ . Therefore we can replace  $a^{110} = 1$  by the conjugate relation  $a^{55} = 1$ . Thus

$$W_2(K) = \langle a, y \mid yay^{-1} = a^2, a^{55} \rangle.$$

Regarded additively, this group is simply the semidirect product of  $\mathbb{Z}_{55}$  by an infinite cyclic group  $\langle y \mid \rangle$ , the action of  $y$  being multiplication by 2. In particular,  $W_2(K)$  is not cyclic.  $\square$

It is known [28, 20] that if  $K$  is classical, then the groups  $W_i(K)$ ,  $i = 1, 2$ , are isomorphic to the core group  $\text{Core}(K)$ . Kelly's result can be extended in some cases to the virtual category. In this way a necessary criterion for Alexander numbering is formulated.

**Definition 2.3** [27] An *Alexander numbering* of edges  $\mathcal{E}$  of an oriented virtual link diagram is an assignment of integers on edges such that at every crossing the following condition is satisfied. When a (positive or negative) crossing is placed so that both edges are oriented downward, the left edges receive the same integer  $i$ , and then the right edges receive the integer  $i + 1$  (see Fig. 4).

The numbering of an edge remains unchanged when it passes through a virtual crossing. A *(mod 2)-Alexander numbering* is defined similarly, but the indices on the edges are integers modulo 2.

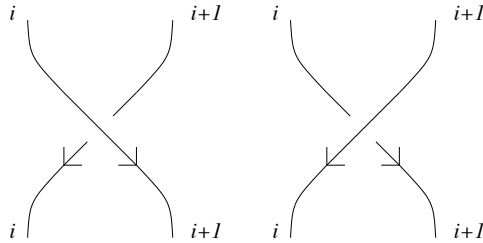


Figure 4: Alexander numbering

**Definition 2.4** [27] An oriented virtual link is *almost classical* if it has a diagram that admits an Alexander numbering. It is *almost classical (mod 2)* if it admits a (mod 2)-Alexander numbering.

An example is given in [27] of a virtual knot that is almost classical but is shown by Alexander groups to be non-classical.

**Proposition 2.5** *If an oriented diagram for a virtual knot  $K$  has a (mod 2)-Alexander numbering, then the group  $W_1(K)$ , is isomorphic to the core group of  $K$ .*

*Proof.* We can write a presentation for  $W_1(K)$  with a generator for each edge of the diagram and a pair of relations at each crossing. These are of the form  $c = u(a, b)$ ,  $d = v(a, b)$ , with  $u(a, b) = b^{-1}$  and  $v(a, b) = bab$ . If the crossing is positive then  $a$  and  $b$  are the generators corresponding to the left and right top edges, and  $c$  and  $d$  correspond to the bottom left and right edges, respectively. For a negative crossing,  $a$  and  $b$  correspond to the bottom left and right edges, and  $c$  and  $d$  are the left and right top edges, respectively. The core group has a similar presentation, but there  $u$  and  $v$  are replaced by  $u'(a, b) = b$  and  $v'(a, b) = ba^{-1}b$ .

Given a (mod 2)-Alexander numbering for the diagram  $K$ , we obtain an isomorphism from  $W_1(K)$  to the core group by sending each generator  $x$  to  $x^{-\text{ind}}$  where ind denotes the index on the corresponding edge. If  $a$  and  $c$  (which are on the left) have index 0, then the crossing relations are sent to  $c = u(a, b^{-1}) = u'(a, b)$  and  $d^{-1} = v(a, b^{-1}) = (v'(a, b))^{-1}$ . Similarly if  $a$  and  $b$  have index 1, then the relations are sent to  $c^{-1} = u(a^{-1}, b) = u'(a, b)^{-1}$  and  $d = v(a^{-1}, b) = v'(a, b)$ . Thus in either case the image of the pair of  $W_1$  relations at a crossing is the pair of core relations at the same crossing, up to a trivial rewriting. The (mod 2)-Alexander numbering allows us to make a global choice for the images of the generators.  $\square$

**Example 2.6** If a virtual knot or link  $K$  does not have a diagram admitting a (mod 2)-Alexander numbering, then  $W_1(K)$  need not be isomorphic to the core group of  $K$ .

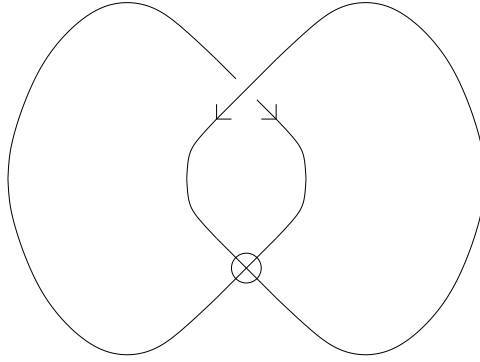


Figure 5: A virtual link for which  $W_1(K)$  and  $\text{Core}(K)$  are distinct

The simplest example is provided by the virtual Hopf link in Fig. 5. An easy computation shows that  $W_1(K)$  is isomorphic to  $\langle x, y \mid xy = yx, y^2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$ . On the other hand, the core group of  $K$  is isomorphic to  $\langle x, y \mid xy^{-1} = yx^{-1} \rangle$ . Letting  $z = yx^{-1}$ , we immediately see that the latter group is isomorphic to  $\langle x, z \mid z^2 \rangle$ , the free product of  $\mathbb{Z}$  and  $\mathbb{Z}_2$ .

An example for which  $K$  is a virtual knot is given here. Consider the knot  $K$  in Fig. 6, a connected sum of two virtual trefoils. A straightforward calculation, similar to that above, shows that  $W_1(K)$  is the classical trefoil knot group  $\langle x, y \mid xyx = yxy \rangle$  while the core group  $\text{Core}(K)$  is the free product of  $\mathbb{Z}$  and  $\mathbb{Z}_3$ .

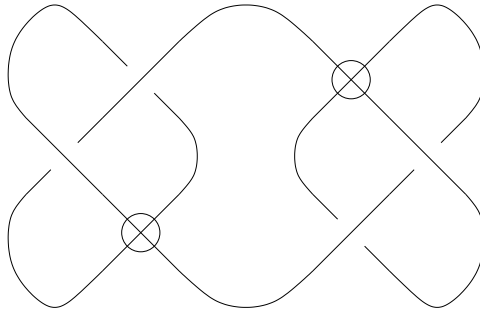


Figure 6: A virtual knot for which  $W_1(K)$  and  $\text{Core}(K)$  are distinct

We conclude this section with:

**Conjecture 2.7** *The free part of the abelianization of a Wada group is  $\mathbb{Z}$ .*

### 3 Biquandles from Wada's representations

In this section we define biquandle structures on any group using Wada's representations and study colorings of virtual knots by such biquandles.

A *quandle*,  $X$ , is a set with a binary operation  $(a, b) \mapsto a * b$  such that

- (I) For any  $a \in X$ ,  $a * a = a$ .
- (II) For any  $a, b \in X$ , there is a unique  $c \in X$  such that  $a = c * b$ .
- (III) For any  $a, b, c \in X$ , we have  $(a * b) * c = (a * c) * (b * c)$ .

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [2, 10, 12, 17, 23]. Generalizations of racks and quandles have been studied in several papers. Here we follow descriptions in [1, 9].

**Definition 3.1** [1, 9] A *birack* is a set  $X$  together with a mapping  $R : X \times X \rightarrow X \times X$  that has the following properties.

1. The map  $R$  is invertible. The inverse of  $R$  is denoted by  $\bar{R} : X \times X \rightarrow X \times X$ . We will write

$$R(A_1, A_2) = (R_1(A_1, A_2), R_2(A_1, A_2)) = (A_3, A_4),$$

where  $A_i \in X$  for  $i = 1, 2, 3, 4$ .

2. For any  $A_1, A_3 \in X$  there is a unique  $A_2 \in X$  such that  $R_1(A_1, A_2) = A_3$ . We say that  $R_1$  is *left-invertible*.
3. For any  $A_2, A_4 \in X$  there is a unique  $A_1 \in X$  such that  $R_2(A_1, A_2) = A_4$ . We say that  $R_2$  is *right-invertible*.
4.  $R$  satisfies the set-theoretic Yang-Baxter equation:

$$(R \times 1)(1 \times R)(R \times 1) = (1 \times R)(R \times 1)(1 \times R),$$

where 1 denotes the identity mapping.

**Definition 3.2** [1, 9] A biquandle  $(X, R)$  is a birack with the following property, called the *type I condition*: For any  $a \in X$  there are unique elements  $x_a$  and  $y_a$  such that  $x_a = R_1(x_a, a)$ ,  $a = R_2(x_a, a)$ ,  $y_a = R_2(a, y_a)$ , and  $a = R_1(a, y_a)$ .

Direct calculations imply the following.

**Lemma 3.3** *Suppose that  $u(x, y)$  and  $v(x, y)$  are words in  $F_2$  such that  $x \mapsto u(x, y)$  and  $y \mapsto v(x, y)$  define isomorphisms of the free group that satisfy Wada's conditions. Then for any group  $G$ , the binary operations  $R_i : G \times G \rightarrow G$ ,  $i = 1, 2$ , given by*

$$R_1(x, y) = u(x, y), \quad R_2(x, y) = v(x, y)$$

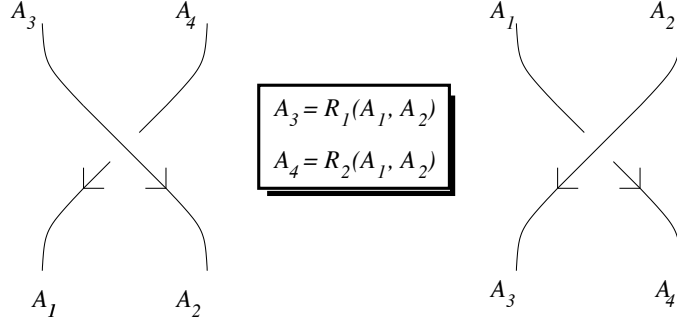


Figure 7: A coloring by birack elements

define a birack structure on  $G$ . The biracks corresponding to the type  $W_1$  and  $W_2$  representations,

$$R_1(x, y) = y^{-1}, \quad R_2(x, y) = yxy,$$

$$\text{or } R_1(x, y) = x^{-1}y^{-1}x, \quad R_2(x, y) = y^2x,$$

respectively, are biquandles.

Call the first and second, a group biquandle of type  $W_1$  and  $W_2$ , respectively. Note that if the group  $G$  is abelian, then both types define the same map  $R(x, y) = (-y, 2y + x)$ . We call this an *abelian Wada biquandle*.

Let  $(X, R)$  be a biquandle and  $\mathcal{E}$  be the set of edges of a virtual knot diagram  $K$  (an edge runs from classical crossing to classical crossing).

**Definition 3.4** [1, 9] A coloring  $\mathcal{C} : \mathcal{E} \rightarrow X$  is a map such that at every crossing, the conditions depicted in Fig. 7 are satisfied where  $(A_3, A_4) = R(A_1, A_2)$ , and  $A_1, \dots, A_4 \in X$ .

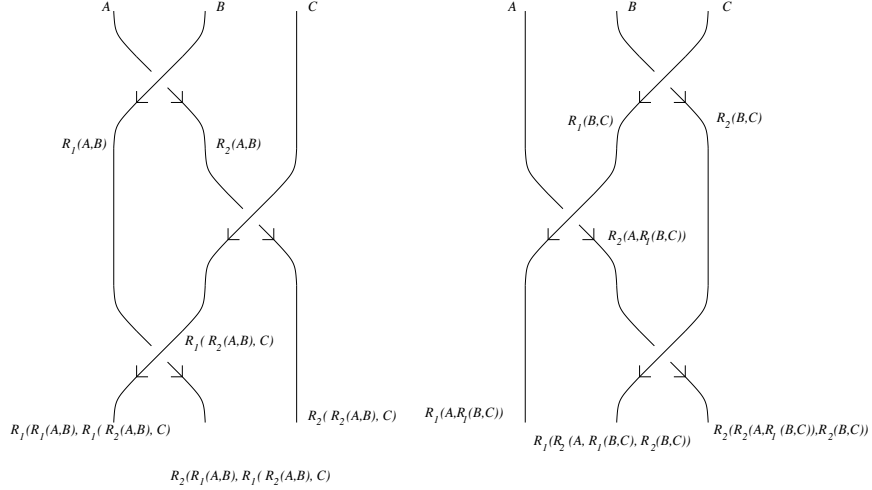
Let  $\text{Col}_X(K)$  denote the set of colorings of a virtual knot diagram  $K$  by a biquandle  $X$ . If two diagrams are related by a generalized Reidemeister move, there is a one-to-one correspondence between their sets of colorings, so that the cardinality of colorings  $|\text{Col}_X(K)|$  is an invariant of virtual knots (see [1, 9], for example).

**Remark 3.5** If the Wada group invariant of  $K$  associated to  $R$  is  $\Gamma$ , then  $\text{Col}_X(K)$  can be identified with  $\text{Hom}(\Gamma, G)$ .

**Definition 3.6** A coloring of a knot  $K$  by the biquandle  $(G, R)$  with  $G = \mathbb{Z}_n$  or  $G = \mathbb{Z}$  — in this case  $R(x, y) = (-y, 2y + x)$  — is called a *cyclic coloring*.

Note that any virtual diagram has a cyclic coloring by  $\mathbb{Z}_n$ , for any  $n$ , such that every arc receives the color 0. On the other hand, if a diagram has a cyclic coloring with a single color  $m \neq 0 \in \mathbb{Z}_n$ , then either the diagram is the diagram of the unknot without a real crossing, or  $n = 2m$ , as  $m = x = y = -y = x + 2y$  must be satisfied at every real crossing.

**Proposition 3.7** Any virtual link has a coloring by the infinite cyclic abelian Wada biquandle  $\mathbb{Z}$  such that at least one color is not 0.



$R_1(R_1(A,B), R_1(R_2(A,B), C)) = R_1(A, R_1(B,C))$ $R_2(R_1(A,B), R_1(R_2(A,B), C)) = R_1(R_2(A, R_1(B,C)), R_2(B,C))$ $R_2(R_2(A,B), C) = R_2(R_2(A, R_1(B,C)), R_2(B,C))$
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Figure 8: The set-theoretic Yang-Baxter equation

*Proof.* Any virtual link is represented as the closure of a virtual braid [16] of  $n$  strings for some positive  $n$ , that is, a sequence of ordinary braid generators and virtual crossings. Denote by  $\sigma_i$  and  $v_i$  the standard braid generator and the virtual crossing, respectively, at the  $i$ th and  $(i + 1)$ st strings,  $i = 1, \dots, n - 1$ . Let  $\beta$  be an  $n$ -string virtual braid whose closure represents a given virtual link. Colors at the top strings are represented by a vector  $w \in \mathbb{Z}^n$ , and subsequent colors are computed by matrices. The matrix corresponding to  $\sigma_i$  is  $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$  placed at the  $(i, i + 1)$  block of the identity matrix of size  $n$ , and for  $v_i$ , the transposition matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is placed. Hence the vector  $w'$  representing the colors of the bottom strings of the braid is computed by  $Mw$ , where  $M$  is the matrix representing  $\beta$ , which is a product of the above described matrices. Note that each matrix above has the property that  $w_1 = [1, \dots, 1]$  (the row vector with every entry 1) is a left eigenvector of eigenvalue 1, so that  $M$  has the same property. Since  $M$  has the eigenvalue 1, there is a right eigenvector of  $M$  over the field of rational numbers, with eigenvalue 1. Multiplying by a common multiple of the denominators of the entries of such an eigenvector, we obtain an integral eigenvector. Using the entries to color the top strands of  $\beta$ , we obtain a coloring that extends to the closure.  $\square$

**Corollary 3.8** *For any oriented virtual link  $L$  and for  $i = 1, 2$ , there are epimorphisms from the Wada groups  $G_i(L) \rightarrow \mathbb{Z}$  for  $i = 1, 2$ .*

In [14] checkerboard colorable virtual knots were defined, and the properties of their Jones polynomials were studied. In [15] an abstract link diagram is constructed from a virtual diagram as follows. A surface with boundary,  $F$ , is constructed as a handlebody. The 0-handles correspond to the classical crossings in the virtual diagrams; the 1-handles correspond to the edges in the diagram and four such 1-handles are attached in the natural fashion to a 0-handle as the edges approach

the vertices. The resulting diagram represents a link in the 3-manifold that is  $F \times I$ . A virtual link diagram is *checkerboard colorable* if the graph of the virtual link in  $F$  is checkerboard colorable. Equivalently, the 1-cycle represented by the link is null-homologous modulo 2. If a virtual link possesses such a diagram, it is called *checkerboard colorable*. The relation between checkerboard colorability and Alexander numberings is as follows. A virtual link is checkerboard colorable if and only if it has an oriented diagram that admits a  $(\text{mod } 2)$ -Alexander numbering. For we can color the regions to the left of the edges that are labeled 0 by black, right by white, and use the opposite colors for those edges that are labeled 1.

**Lemma 3.9** *If virtual link  $L$  is checkerboard colorable, then there is a one-to-one correspondence between the set of colorings of any given diagram of  $L$  by an abelian Wada biquandle  $\mathbb{Z}_n$  and the set of Fox  $n$ -colorings.*

*Proof.* Let  $L$  be an oriented virtual knot diagram with a  $(\text{mod } 2)$ -Alexander numbering, so that every edge  $\alpha \in \mathcal{E}$  is assigned 0 or 1, denoted by  $\epsilon(\alpha)$ . Let  $\mathcal{C}$  be a cyclic coloring by an abelian Wada biquandle  $\mathbb{Z}_n$ . Define a Fox coloring  $\mathcal{C}'(\alpha)$ ,  $\alpha \in \mathcal{A}$ , by  $\mathcal{C}'(\alpha) = (-1)^{\epsilon(\alpha)}\mathcal{C}(\alpha)$ . Then it is checked that this provides a one-to-one correspondence.  $\square$

We remark that this change of basis is the abelian version of Proposition 2.5.

**Corollary 3.10** *If a virtual knot  $K$  is checkerboard colorable, then  $K$  is Fox  $n$ -colorable (non-trivially) if and only if  $|\text{Col}_X(K)| > n$ , where  $X = \mathbb{Z}_n$  denotes a cyclic abelian Wada biquandle.*

**Remark 3.11** If an oriented diagram of a virtual link is checkerboard colorable, then there is a coloring by the cyclic Wada biquandle  $\mathbb{Z}$  using only  $\pm 1$ . Let  $L$  be a  $(\text{mod } 2)$ -Alexander numbering, and simply assign 1 for an arc  $\alpha$  with  $L(\alpha) = 0$  and  $-1$  if  $L(\alpha) = 1$ . Thus there is a non-zero coloring by  $\mathbb{Z}$  such that the span (the largest integer minus the smallest integer that appear in the colors) of the coloring is 2.

For example, from Fig. 3, by computing the abelianized elements, we see that this diagram of Kishino's knot does not have a non-zero coloring by  $\mathbb{Z}$  whose span is 2. The smallest span is 4. This does not, however, prove that Kishino's knot is not checkerboard colorable, as there might be another diagram with span 2.

The *span* of an oriented virtual link is the minimal span among all diagrams and all non-zero colorings by  $\mathbb{Z}$ . We conjecture that the span of Kishino's knot is 4.

## 4 Cocycle invariants

A homology theory and cocycle invariants for biquandles were defined in [3]. In this section, we study constructions of cocycles for group biquandles. For the purposes of this paper, we review definitions of cocycles only as functional equations, instead of going through homology theories, in order to use them to define cocycle invariants.

Let  $(X, R)$  be a biquandle and  $A$  be an abelian group. A function  $g : X \rightarrow A$  is called a *Yang-Baxter 1-cocycle* if it satisfies:

$$(\delta_1 g)(x, y) := g(x) + g(y) - g(R_1(x, y)) - g(R_2(x, y)) = 0.$$

A function  $f : X^2 \rightarrow A$  is called a *Yang-Baxter 2-cocycle* if it satisfies:

$$\begin{aligned} (\delta_2 f)(x, y, z) &:= f(x, y) + f(R_2(x, y), z) + f(R_1(x, y), R_1(R_2(x, y), z)) \\ &\quad - \{f(y, z) + f(x, R_1(y, z)) + f(R_2(x, R_1(y, z)), R_2(y, z))\} = 0. \end{aligned}$$

The 2-cocycle condition corresponds to the Reidemeister type III move. See Fig. 8. The ordered arguments of a function,  $f$ , are the left and right in-coming labels on the arcs near a crossing  $A, B, R_2(A, B), C$  etc.. The three positive terms in the cocycle condition, then, correspond to the crossings in the left-hand side of the type III move, and the negative terms correspond to the crossings on the right-hand side. The type II move is handled by assigning the inverse value of  $f$  at negative crossings. To define knot invariants, the following condition becomes necessary for invariance under the type I Reidemeister move.

**Definition 4.1** [3] Let  $(X, R)$  be a biquandle, and  $A$  be an abelian group. Recall that for any  $a \in X$  there are unique elements  $x_a$  and  $y_a$  such that  $x_a = R_1(x_a, a)$ ,  $a = R_2(x_a, a)$ ,  $y_a = R_2(a, y_a)$ , and  $a = R_1(a, y_a)$ .

A Yang-Baxter 2-cocycle  $f$  is said to satisfy the *type I condition* if  $f(x_a, a) = 0$  and  $f(a, y_a) = 0$  for every  $a \in X$ .

Cocycles of Alexander quandles that are in polynomial form were considered by Mochizuki [24] and have been used to define cocycle knot invariants. These have been shown to have a wide range of applications. Let  $G = \mathbb{Z}_n$  or  $\mathbb{Z}$ . (The case  $G = \mathbb{Z}$  is referred to as the case  $n = 0$ .) Recall that the abelian Wada biquandle is given by  $R(x, y) = (-y, 2y + x)$ . Then the 2-cocycle condition with the coefficient group  $A = \mathbb{Z}_n$  reads

$$f(x, y) + f(2y + x, z) + f(-y, -z) = f(y, z) + f(x, -z) + f(x - 2z, 2z + y).$$

In this case, the type I condition for 2-cocycles becomes  $f(x, -x) = 0$ .

**Lemma 4.2** For any non-negative integer  $n$ , the function  $f(x, y) = x + y$  is a 2-cocycle of the abelian biquandle  $G = \mathbb{Z}_n$  with values in the coefficient group  $A = \mathbb{Z}_n$ . It satisfies the type I condition and is not a coboundary that is, there is no function  $g : G \rightarrow A$  such that  $\delta_1 g = f$ .

*Proof.* It is a direct calculation to check the 2-cocycle condition and the type I condition formulated above. To determine that  $f$  is not a coboundary, we find a 2-chain that is a cycle and upon which  $f$  evaluates non-trivially. The set of 2-chains is the free abelian group generated by pairs of elements in the biquandle  $X$ . The boundary of a generating chain is given as  $\partial_2(x, y) = \{x\} + \{y\} - \{R_1(x, y)\} - \{R_2(x, y)\}$  — the braces indicate that the elements of  $X$  are considered as generators of a free abelian group on  $X$ . In the current context,  $\partial_2(x, y) = \{x\} + \{y\} - \{-y\} - \{2y + x\}$ .

Consider the 2-chain  $c = (1, 0) \in \mathbb{Z}_n \times \mathbb{Z}_n$ . Then  $c$  is a 2-cycle, and  $f(c) = 1 \neq 0$ , so that  $f$  is not a coboundary.  $\square$

**Corollary 4.3** The two-dimensional homology group  $H_2^{\text{YB}}(G; A)$  of the abelian Wada biquandle  $G = \mathbb{Z}_n$  with the coefficient group  $A = \mathbb{Z}_n$  is non-trivial for any non-negative integer  $n$ .

**Remark 4.4** Let  $p$  be an odd prime and consider cocycles of the abelian biquandle  $G = \mathbb{Z}_p$  that take values in  $A = \mathbb{Z}_p$ . Consider the expression  $h(x, y) = (1/p)[(x^p + 2y^p) - (x + 2y)^p]$  that is inspired by Mochizuki's cocycle. Since the numerator is divisible by  $p$ ,  $h(x, y)$  takes integral values, which are then reduced modulo  $p$ . A direct calculation gives that  $h$  is a 2-cocycle. For  $p = 3$  and  $5$ , respectively,  $h$  evaluates non-trivially on the 2-cycles  $(1, 1) + 2(2, 2)$ ,  $(1, 1) + 2(2, 2) + 4(3, 3)$ . From the proof of the above lemma, we see that the cocycles  $f(x, y) = x + y$  and  $h(x, y) = (1/p)[(x^p + 2y^p) - (x + 2y)^p]$  are linearly independent and not coboundaries. Hence  $\text{rank}(H_2^{\text{YB}}(G; A))$  is at least two for  $p = 3, 5$ .

We recall the definition of *biquandle 2-cocycle invariants* from [3]. Let  $K$  be a classical or virtual knot or link diagram. Let  $(X, R)$  denote a finite biquandle, and let  $\phi : X^2 \rightarrow A$  denote 2-cocycle that satisfies the type I condition where  $A$  is an abelian group which is written multiplicatively. Let  $\mathcal{C}$  denote a coloring  $\mathcal{C} : \mathcal{E} \rightarrow X$ , where  $\mathcal{E}$  denotes the set of arcs of  $K$ . For a positive crossing  $\tau$  as depicted in the right of Fig. 7, let  $\alpha$  and  $\beta$  be the top left and right arcs labeled by  $A_1$  and  $A_2$ , respectively, and  $\mathcal{C}(\alpha) = A_1$  and  $\mathcal{C}(\beta) = A_2$  as depicted. For a negative crossing at the left of Fig. 7, let  $\alpha$  and  $\beta$  be the bottom left and right arcs, also labeled by  $A_1$  and  $A_2$  respectively, so that the coloring is also given by  $\mathcal{C}(\alpha) = A_1$  and  $\mathcal{C}(\beta) = A_2$ . A (*Boltzmann*) *weight*  $B(\tau, \mathcal{C})$  (that depends on  $f$ ) at a crossing  $\tau$  is defined by  $B(\tau, \mathcal{C}) = u^{\epsilon(\tau)m}$ , where  $\epsilon(\tau) = 1$  or  $-1$  if  $\tau$  is positive or negative, respectively.

The (*Yang-Baxter*) *cocycle knot invariant* is defined by the state sum expression

$$\Phi_{\text{YB}}(K) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}).$$

The product is taken over all crossings of the given diagram  $K$ , and the sum is taken over all possible colorings. The values of the state sum are taken to be in the group ring  $\mathbb{Z}[A]$  where  $A$  is the coefficient group written multiplicatively. The state sum depends on the choice of 2-cocycle  $f$ . The cocycle invariant  $\Phi_{\text{YB}}(K)$  does not depend on the choice of a diagram for  $K$ , and thus is an invariant for virtual knots and links. Note that the image of  $\Phi_{\text{YB}}(K)$  under the map  $\mathbb{Z}[A] \rightarrow \mathbb{Z}$  sending all elements of  $A$  to 1 is equal to the number of colorings  $|\text{Col}_X(K)|$ .

We investigate the cocycle invariant for  $G = \mathbb{Z}_n = A$  with the 2-cocycle  $f(x, y) = x + y$ .

**Theorem 4.5** *For the cocycle  $f(x, y) = x + y$ , suppose that  $n$  is odd or  $n = 0$ . If a virtual knot,  $K$ , is checkboard colorable, then  $\sum_{\mathcal{C}} B(\tau, \mathcal{C}) = 1$  for any coloring  $\mathcal{C}$ .*

*Proof.* Let  $K$  be an oriented virtual link diagram that has a (mod 2)-Alexander numbering, and a coloring  $\mathcal{C}$  be given by a cyclic Wada biquandle  $\mathbb{Z}_n$  with  $n$  odd. Consider the corresponding Fox coloring  $\mathcal{C}_F$  that is given by Lemma 3.9. Let  $L : \mathcal{E} \rightarrow \mathbb{Z}_2$  be a (mod 2)-Alexander numbering. Let  $\mathcal{A}$  be the set of arcs in  $K$ . (An arc is broken neither at an over-crossing, nor at a virtual crossing). To the initial and terminal points  $s(\alpha)$ ,  $t(\alpha)$ , respectively, of every arc  $\alpha$ , we make the following assignment  $g$  of elements of  $\mathbb{Z}_n$ : If the Fox coloring  $\mathcal{C}_F(\alpha) = x \in \mathbb{Z}_n$ , then  $g(s(\alpha)) = -x$  and  $g(t(\alpha)) = x$ . Since these have opposite signs, the sum over all assignments on initial and terminal points of all arcs is zero. On the other hand, we show that the sum is equal to twice the weight at each crossing, and the result follows. Consider a positive crossing formed by an incoming arc  $\alpha$  and the over-arc  $\beta$  with  $\mathcal{C}(\alpha) = x$  and  $\mathcal{C}(\beta) = y$ . Let  $\gamma$  be the outgoing under-arc, so that  $\mathcal{C}(\gamma) = x + 2y$ .

Then  $g(t(\alpha)) = (-1)^{L(\alpha)}x$  and  $g(s(\gamma)) = -(-1)^{L(\gamma)}(x + 2y)$ , so the sum is  $(-1)^{L(\alpha)}2(x + y)$  since  $L(\gamma) = L(\alpha) + 1$  which is the desired value (twice the weight of the crossing). The weight at a negative crossing is checked similarly.  $\square$

As an application, let  $VT(2, k)$  be an oriented virtual  $(2, k)$ -torus knot or link represented by the closure of virtual 2-braid  $(\sigma_1)^k v_1$ , where  $\sigma_1$  and  $v_1$  denote the standard and virtual braid generators, respectively. The orientations are chosen to be downward.

**Proposition 4.6** *For any integer  $k \neq 0$ ,  $VT(2, k)$  is not checkerboard colorable.*

*Proof.* Assume  $k$  to be positive, as the negative case is similar. Assume that  $VT(2, k)$  is almost classical (mod 2) to derive a contradiction to Theorem 4.5. Suppose the top left and right arcs receive colors  $(x, y)$ ,  $x, y \in \mathbb{Z}_n$ . Then below  $(\sigma_1)^k$ , the colors are  $(-((k-1)x + ky), kx + (k+1)y)$ . After the virtual crossing  $v_1$ , the bottom colors are  $(kx + (k+1)y, -((k-1)x + ky))$ , which must be equal to  $(x, y)$  for this to color the closure. Thus  $(x, y)$  colors if and only if  $(k-1)x + (k+1)y \equiv 0 \pmod{n}$ . For a given  $k$ , choose  $n$  such that  $\gcd(k-1, n) = 1$ , then there are  $n$  solutions  $(x, y)$  to the above equation. This implies  $|\text{Col}_X(K)| = n$ , where  $X = \mathbb{Z}_n$ . One solution is  $(x, y) = (k+1, 1-k)$ , and the weight at the top crossing (therefore at all the other crossings) is 2. Hence the contribution of this particular coloring to the invariant is  $2(k-1)$ , which is not zero in  $\mathbb{Z}_n$ , giving a non-trivial contribution to the invariant.  $\square$

**Remark 4.7** Theorem 4.5 can be stated more generally for possibly infinite quandles: If  $K$  is checkerboard colorable, then for any coloring, the product  $\prod_{\tau} B(\tau, \mathcal{C})$  is the identity element of  $A$ . Then in Proposition 4.6, the biquandle  $\mathbb{Z}$  can be used, as the color vector  $(k+1, 1-k) \in \mathbb{Z} \times \mathbb{Z}$  extends as well, and gives the non-trivial product  $2k$ . Note that  $|2k|$  is the span of this coloring (Remark 3.11). Thus we conjecture that the non-zero minimum of the product  $\prod_{\tau} B(\tau, \mathcal{C})$  for the biquandle  $\mathbb{Z}$  gives a lower bound of the span, and that the span of  $VT(2, k)$  is  $|2k|$ .

## References

- [1] Bartholomew, A.; Fenn, R., *Quaternionic invariants of virtual knots and links*, preprint, arXiv:math.GT/0610484.
- [2] Brieskorn, E., *Automorphic sets and singularities*, Contemporary math., 78 (1988), 45–115.
- [3] Carter, J.S.; Elhamdadi, M.; Saito, M., *Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles*, Fund. Math. 184 (2004), 31-54.
- [4] Carter, J.S.; Jelsovsky, D.; Kamada, S.; Langford, L.; Saito, M., *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947–3989.
- [5] Carter, J.S.; Saito, M., *Set-theoretic Yang-Baxter equations via Fox calculus*, Journal of Knot Theory and Its Ramifications, Vol. 15, No. 8 (2006) 949-956.
- [6] Creel, C.; Nelson, S., *Symbolic computation with finite biquandles*, J. Symbolic Comput. 41 (2006) 811-817.
- [7] Dye, H.; Kauffman, L.H., *Minimal surface representations of virtual knots and links*, Algebr. Geom. Topol. 5, (2005,) 509-535.

- [8] Dye, H.; Kauffman, L.H., *Virtual Homotopy*, arXiv:math.GT/0611076.
- [9] Fenn, R.; Jordan-Santana, M.; Kauffman, L.H., *Biquandles and virtual links*, Topology Appl. 145 (2004), no. 1-3, 157–175.
- [10] Fenn, R.; Rourke, C., *Racks and links in codimension two*, Journal of Knot Theory and Its Ramifications Vol. 1 No. 4 (1992), 343–406.
- [11] Fox, R. H., *Free differential calculus I. Derivation in the free group ring*, Ann. of Math. (2) 57, (1953), 547–560.
- [12] Joyce, D., *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg., 23, 37–65.
- [13] Kadokami, T., *Detecting non-triviality of virtual links*, J. Knot Theory Ramifications 12 (2003), 781–803.
- [14] Kamada, N., *On the Jones polynomials of checkerboard colorable virtual links*, Osaka J. Math. 39 (2002), 325–333.
- [15] Kamada, N.; Kamada, S., *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications 9 (2000), 93–106.
- [16] Kamada, S., *Braid representation of virtual knots and welded knots*, Preprint, arXiv:math.GT/0008092.
- [17] Kauffman, L.H., *Knots and Physics*, World Scientific, Series on knots and everything, vol. 1, 1991.
- [18] Kauffman, L.H., *Virtual Knot Theory*, European J. Combin. **20** (1999), 663–690.
- [19] Kauffman, L.H.; Radford, D. *Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links*, Diagrammatic morphisms and applications (San Francisco, CA, 2000), 113–140, Contemp. Math., 318, Amer. Math. Soc., Providence, RI, 2003.
- [20] Kelly, A. J., *Groups from link diagrams*, Ph.D. Thesis, Warwick Univ., Coventry, 1991.
- [21] Kishino, T.; Satoh, S., *A note on non-classical virtual knots*, J. Knot Theory Ramifications 13 (2004), 845–856.
- [22] X.-S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Mathematica Sinica, English series 17 (2001), 361–380.
- [23] Matveev, S., *Distributive groupoids in knot theory*, (Russian) Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78–88, 160.
- [24] T. Mochizuki, *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra, 179 (2003), 287–330.
- [25] Nelson, S.; Vo, J. *Matrices and Finite Biquandles*, Homology, Homotopy and Applications 8 (2006) 51-73.
- [26] Przytycki, J. H., Math. Review article, MR1167178(94e : 57014).
- [27] Silver, D.; Williams, S., *Crowell’s derived group and twisted polynomials*, J. Knot Theory and its Ramifications 15 (2006), 1079-1094.
- [28] Wada, M., *Group invariants of links*, Topology, 31 (1992), 399-406.
- [29] M. Wada, *Twisted Alexander polynomials for finitely presented groups*, Topology 33 (1994), 241–256.