On the Component Number of Links from Plane Graphs

Daniel S. Silver  Susan G. Williams

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Abstract

Using the Dehn presentation of a link group, a short elementary proof is given of the result: The number of components of a link arising from a medial graph $M(\Gamma)$ by resolving vertices is equal to the nullity of the mod-2 Laplacian matrix of $\Gamma$.

1 Introduction

Let $\Gamma$ be a plane graph. Its medial graph $M(\Gamma)$ is obtained from the boundary of a regular neighborhood of $\Gamma$ by pinching each edge to create a vertex of degree 4. An example is given in Figures 1 and 2.

The construction is important in knot theory, since any diagram of a link can obtained from a suitable plane graph $\Gamma$ by resolving each vertex of $M(\Gamma)$ in one of two ways so that one arc of the diagram appears to pass over the other. (See Figure 3.) In this way, questions about knots and links can be converted into the language of graph theory. Indeed, many invariants of the link can be computed directly from $\Gamma$ (see, for example, [3], [4]).

The most basic invariant of a link $L$ is the number of its components, denoted here by $\mu(L)$. Notice that $\mu(L)$ is the same for all links associated to a medial graph $M(\Gamma)$, regardless of how the vertices are resolved. Calculation of $\mu(L)$ for links arising from special families of graphs have appeared in various places (for example, [8], [9]).

In [6] it is shown that the Tutte polynomial evaluation $T_{\Gamma}(-1,-1)$ determines $\mu(L)$. A more intrinsic formula appears in [1]:

The mod-2 Laplacian matrix of a graph $\Gamma$ is the matrix $Q_2(\Gamma) = (q_{ij})$, where $q_{ii}$ is the degree of the $i$th vertex, while for $i \neq j$, $q_{ij}$ is the number

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Figure 1: Plane Graph $\Gamma$

Figure 2: Medial graph $M(\Gamma)$

Figure 3: Link $L$ associated to $M(\Gamma)$
of edges between the $i$th and $j$th vertices. All entries are taken modulo 2. The following theorem is proved in [1].

**Theorem 1.1.** Let $L$ be a link arising from a medial graph $M(\Gamma)$ by resolving vertices. The number $\mu(L)$ of components of $L$ is the nullity of the mod-2 Laplacian matrix $Q_2(\Gamma)$.

**Remark 1.2.** As observed in [6], both the Tutte polynomial and the Laplacian matrix determination of $\mu(\Gamma)$ show that $\mu(L)$ is independent of the planar embedding of $\Gamma$.

Theorem 1.1 can be proved using standard results of algebraic topology. $Q_2(\Gamma)$ is the “unreduced mod-2 Goertiz matrix” of the link $L$, which is a presentation matrix for $H_1(M_2;\mathbb{Z}_2)$, where $M_2$ is the 2-fold cyclic cover of $S^3$ branched over $L$ (see, for example, [7]). In [2] the rank of $H_1(M_2;\mathbb{Z}_2)$ is shown to be $\mu(L) - 1$. Hence the nullity of $Q_2(\Gamma)$ is $\mu(L)$.

A combinatorial proof of Theorem 1.1 is given in Chapter 17 of [1]. It draws on substantial material about cut, flow and bicycle spaces contained in earlier chapters.

Our purpose is to give a proof of Theorem 1.1 that is very short. It relies on only a small amount of algebraic topology, reviewed in the next section.

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## 2 Link groups and the Dehn presentation

A link $L$ is a finite collection of embedded mutually disjoint circles in 3-space $\mathbb{R}^3$. Two links are regarded as the same if one can be deformed isotopically into the other.

The fundamental group $G_L = \pi_1(\mathbb{R}^3 \setminus L)$ is a link invariant. Its abelianization is isomorphic to $\mathbb{Z}^{\mu(L)}$. Hence the $\mathbb{Z}_2$-vector space $\text{Hom}(G_L, \mathbb{Z}_2)$ has dimension $\mu(L)$.

There are two well-known presentations of $G_L$ derived from a diagram $D$ for $L$. The first, the Wirtinger presentation, has generators corresponding to the arcs of the diagram, an arc being a maximal connected piece of $D$.

The second, the Dehn presentation, associates generators to the bounded regions of $D$. Relations correspond to crossings of the diagram, as in Figure 4. The unbounded region is labeled with the identity element. We make use of the Dehn presentation.

It is immediate that elements of $\text{Hom}(G_L, \mathbb{Z}_2)$ are simply labelings of the regions of $D$ by “colors” $\alpha, \beta, \gamma, \ldots \in \mathbb{Z}_2$ with the unbounded region receiving 0 and such that the sum of the four colors at any crossing vanishes.
3 Proof of Theorem 1.1

Let $\Gamma$ be a plane graph, $M(\Gamma)$ its medial graph, and $\mathcal{D}$ a diagram for any link $L$ obtained from $M(\Gamma)$ by resolving vertices.

Checkerboard shade the regions of $\mathcal{D}$ with the unbounded region un-shaded. Then $\Gamma$ can be identified with the graph having a vertex in each shaded region and an edge running through each crossing.

By shaded (resp. unshaded) generators we mean Dehn generators corresponding to shaded (resp. unshaded) regions of $\mathcal{D}$. We identify shaded generators with vertices of $\Gamma$, and the unshaded generators with faces of $\Gamma$. Since the unbounded region corresponds to the trivial element, it is easy to see from the Dehn relations that the shaded generators alone generate $G_L$.

A vertex coloring is an assignment of colors in $\mathbb{Z}_2$ to the shaded generators. It is conservative if it extends over unshaded regions to give an element of $\text{Hom}(G_L, \mathbb{Z}_2)$. If a vertex coloring extends, then it does so uniquely. Hence we can count the elements of $\text{Hom}(G_L, \mathbb{Z}_2)$ by counting conservative vertex colorings.

Assume that $v$ is a vertex of $\Gamma$ with valence $d$. Let $w_1, \ldots, w_d$ be the vertices adjacent to $v$. (We allow repetition in the case of multiple edges.) Let $\alpha_v$ (resp. $\alpha_{w_i}$) be colors assigned to to $v$ (resp. $w_i$). One checks that if the vertex coloring is conservative, then the $d$ Dehn relations involving $v$ imply:

$$\alpha_{w_1} + \ldots + \alpha_{w_d} = 0, \quad (d \text{ even}) \tag{3.1}$$

$$\alpha_v + \alpha_{w_1} + \ldots + \alpha_{w_d} = 0, \quad (d \text{ odd}) \tag{3.2}$$

Conversely, if the above conditions hold at every vertex, then the coloring is a conservative vertex coloring. To prove this, it suffices to show that the
coloring extends to the unshaded regions in a way that is consistent with all Dehn relations. For this, consider a path from the unbounded region to any unshaded region $R$, intersecting edges of $\Gamma$ transversely. Determine colors for the unshaded regions inductively: as the path leaves a region labeled $\gamma$ and crosses an edge of $\Gamma$ with vertices labeled $\alpha$ and $\beta$, assign $\alpha + \beta + \gamma$ to the unshaded region that the path enters. Then the Dehn relation holds at the crossing corresponding to that edge. We call $\alpha + \beta + \gamma$ the result of integrating along the path. The conditions (3.1) and (3.2) imply that if we integrate around a small closed path surrounding any vertex, then initial and final colors agree. Consequently, integrating along any two paths from the unbounded region to $R$ yields the same result. Since the path can be taken to cross any given edge, all Dehn relations are satisfied.

In view of the equations that must hold at each vertex, conservative vertex colorings are the elements of the null space of the mod-2 Laplacian matrix $Q_2(\Gamma)$. Hence Theorem 1.1 is proved.

References
