Alexander-Lin twisted polynomials

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Abstract

X.S. Lin's original definition of twisted Alexander knot polynomial is generalized for arbitrary finitely presented groups. J. Cha's fibering obstruction theorem is generalized. The group of a nontrivial virtual knot shown by L. Kauffman to have trivial Jones polynomial is seen also to have a faithful representation that yields a trivial twisted Alexander polynomial.

Keywords: Knot, twisted Alexander polynomial.[1]

1 Introduction.

Twisted Alexander polynomials were introduced by X.S. Lin in a 1990 Columbia University preprint [**?**]. Lin's invariant, defined in terms of Seifert surfaces, incorporated information from linear representations of the knot group π. While the classical Alexander polynomial is an invariant of the $\mathbb{Z}[t^{\pm 1}]$-module π'/π'', its twisted counterpart can incorporate information from any term $\pi^{(k)}$ of the derived series of π'. As a result, it is often a more sensitive invariant.

In [**?**], M. Wada modified Lin's original polynomial. He developed a general invariant for any finitely presented group provided with an epimorphism to a free abelian group and also a linear representation. The definition is given below.

When applied to knot groups, Wada's invariant is often called the "twisted Alexander polynomial" (Twisted Reidemeister torsion is a more appropriate

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label. See [**?**].) A few years later, P. Kirk and C. Livingston, working from an algebraic topological perspective, offered another candidate for the title (see [**?**]).

We return to Lin's original invariant, putting it in Wada's general context. It is a "pointed invariant," depending on a group and distinguished conjugacy class of elements. We generalize a fibering obstruction theorem of J. Cha. We also give an example of a virtual knot group and nonabelian faithful representation with trivial twisted invariants.

For background on twisted Alexander invariants, see the comprehensive survey paper of S. Friedl and S. Vidussi [**?**].

2 Alexander-Lin polynomials.

For any positive integer d, we regard $\mathbb{Z}^d$ as a multiplicative group $\langle t_1, \ldots, t_d \mid t_i t_j = t_j t_i \ (1 \le i, j \le d)\rangle$. The ring $\mathbb{Z}[\mathbb{Z}^d]$ of Laurent polynomials in variables $t_1, \ldots, t_d$ will be denoted by Λ. When $d = 1$, we write t instead of t_1.

Definition 2.1. An *augmented group system* is a triple $\mathcal{A} = (G, \epsilon, x)$ consisting of a finitely presented group G, epimorphism $\epsilon : G \to \mathbb{Z}^d$, and conjugacy class of an element $x \in G$ such that $\epsilon(x)$ is nontrivial.

Augmented group systems are objects of a category. A morphism from $\mathcal{A} = (G, \epsilon, x)$ to $\bar{\mathcal{A}} = (\bar{G}, \bar{\epsilon}, \bar{x})$ is defined to be a group homomorphism $f : G \to \bar{G}$ such that $f(x) = \bar{x}$ and $\epsilon = \bar{\epsilon} f$. When f can be made an isomorphism, we regard $\mathcal{A}$ and $\bar{A}$ as the same. If f can be made an epimorphism, we we write $\mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$.

Remark 2.2. If we restrict ourselves to groups that are hopfian (that is, every epimorphism from the group to itself is an automorphism), then the relation $\twoheadrightarrow$ is a partial order on augmented group systems (see [**?**]). Knot and link groups are hopfian, as are any finitely generated, residually finite groups.

Augmented group systems were introduced in [**?**]. Definition **??** is a mild generalization.

Let $\mathcal{A} = (G, \epsilon, x)$ be an augmented group system, and let $\langle x_0, x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$ be a presentation of G. Without loss of generality we can assume that $x = x_0$ and also $m \ge n$.

Throughout, R is assumed to be a Noetherian unique factorization domain.

Given a linear representation $\gamma : G \to \mathrm{GL}_N R$, mapping $x_i \mapsto X_i$, we define a representation $\gamma \otimes \epsilon : \mathbb{Z}[G] \to M_N\Lambda$ by sending $x_i \mapsto \epsilon(x_i)X_i$, and extending. Here $M_N\Lambda$ denotes the ring of $N \times N$ matrices with entries in Λ. Pre-compose with the natural projection $F = \langle x_0, x_1, \ldots, x_n \mid \rangle \to G$ to obtain a homomorphism

$$\Phi : \mathbb{Z}[F] \to M_N\Lambda.$$

Let $\mathcal{M}_\gamma$ be the $m \times n$ matrix with i,jth entry equal to the image under Φ of the Fox partial derivative $\partial r_i/\partial x_j$, denoted by

$$\left(\frac{\partial r_i}{\partial x_j}\right)^{\Phi} \in M_N\Lambda,$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Regard $\mathcal{M}_\gamma$ as an $mN \times nN$ matrix over Λ by ignoring inner parentheses.

Definition 2.3. The *Alexander-Lin polynomial* $D_\gamma(\mathcal{A})$ of (G, ϵ, x) is the greatest common divisor of all $nN \times nN$ minors of $\mathcal{M}_\gamma$.

A priori, $D_\gamma(\mathcal{A})$ depends on the presentation that we used. However, it is a direct consequence of [**?**] that up to multiplication by a unit $\pm t_1^{n_1} \cdots t_d^{n_d} \in \Lambda$, the polynomial $D_\gamma(\mathcal{A})$ depends only on $\mathcal{A}$. (Furthermore, the sign $\pm$ is $+$ if N is even.) Such invariance was proven in [**?**] for the quotient

$$W_\gamma = \frac{D_\gamma(\mathcal{A})}{\det(I - x)^{\Phi}},$$

now known as *Wada's invariant*. Since the denominator depends only on the conjugacy class of x, invariance holds for $D_\gamma(\mathcal{A})$ as well.

In a similar way, the divisibility results of [**?**] proved for Wada's invariant give the following.

Proposition 2.4. *Let $\mathcal{A} = (G, \epsilon, x)$ and $\bar{\mathcal{A}} = (\bar{G}, \bar{\epsilon}, \bar{x})$ be augmented group systems such that $\mathcal{A} \twoheadrightarrow \mathcal{A}$ via $f : G \to \bar{G}$. Then for any representation $\gamma : \bar{G} \to \mathrm{GL}_N R$, $D_{\gamma f}(\mathcal{A})$ is divisible by $D_\gamma(\bar{\mathcal{A}})$.*

Example 2.5. If k is an oriented knot with group π_k, then there is a well-defined epimorphism $\epsilon : \pi_k \to \mathbb{Z} = \langle t \mid \rangle$ taking each oriented meridian to t. We associate to k an augmented group system $\mathcal{A}_k = (\pi_k, \epsilon, x)$, where x represents the class of an oriented meridian. If $\gamma : \pi \to \mathrm{GL}_N\mathbb{C}$, then $D_\gamma(\mathcal{A}_k)$ agrees with the invariant originally defined by Lin [**?**]. Lin used a Seifert surface S that is *free* in the sense that $\mathcal{S}^3$ split along S is a handlebody H

with fundamental group $\langle a_1, \ldots, a_{2g} \mid \rangle$. He then described the knot group π_k as

$$\langle x, a_1, \ldots, a_{2g} \mid x u_i x^{-1} = v_i \ (i = 1, \ldots, 2g) \rangle,$$

where u_i, v_i are words in $a_1^{\pm 1}, \ldots, a_{2g}^{\pm 1}$ describing positive and negative push-offs of simple closed curves representing a basis for $\pi_1 S$. Lin's invariant is the determinant of the matrix M_γ computed as above.

The following is a group-theoretic version of 3-manifold fibering results proved in different degrees of generality by Cha [?], Kitano and Morifuji [?], Goda, Kitano and Morifuji [?], Friedl and Kim [?], Pajitnov [?], and Kitayama [?].

Theorem 2.6. *Let $\mathcal{A} = (G, \epsilon, x)$ be an augmented group system with $d = 1$ and $\epsilon(x) = t$. Assume such that the kernel of $\epsilon : G \to \mathbb{Z}$ is a finitely generated free group of rank n. Then for any representation $\gamma : \pi \to \mathrm{G}_N R$,*

1. *$\deg D_\gamma(\mathcal{A}) = nN$*
2. *the leading and trailing coefficients of $D_\gamma(\mathcal{A})$ are units.*

Proof. Let $a_1, \ldots, a_n$ be generators for the kernel of ϵ. Since G is a semi-direct product of $\mathbb{Z} \cong \langle x \mid \rangle$ and the free group $F = \langle a_1, \ldots, a_n \mid \rangle$, it has a presentation of the form

$$G = \langle x, a_1, \ldots, a_n \mid x a_1 x^{-1} = \mu(a_1), \ldots, \ x a_n x^{-1} = \mu(a_n) \rangle,$$

for some automorphism μ of F. The matrix $\mathcal{M}_\gamma$ in the definition of $D_\gamma(\mathcal{A})$ is equal to

$$\mathcal{M}_\gamma = t \ \mathrm{diag}(X, \ldots, X) - \left(\frac{\partial \mu(a_i)}{\partial a_j} \right)^{\Phi},$$

where $X = \gamma(x)$. It follows that the leading and constant terms of $D_\gamma(\mathcal{A}) = \det \mathcal{M}_\gamma$ are, respectively,

$$t^{nN} (\det X)^n, \quad \pm \det \left(\frac{\partial \mu(a_i)}{\partial a_j} \right)^{\Phi}.$$

It suffices to show that the constant term is a unit. By a theorem of Birman [?], elements $y_1, \ldots, y_n$ in F generate if and only if the $n \times n$ Jacobian matrix with ijth entry $\partial y_i / \partial x_j$ has a right inverse in $M_n(\mathbb{Z}[F])$. Hence

$$J = \left(\frac{\partial \mu(a_i)}{\partial a_j} \right)$$

has a right inverse $B \in M_n(\mathbb{Z}[F])$. Replacing each occurrence of $a_i^{\pm 1}$ in B by the matrix $A_i^{\pm 1}$, for $i = 1, \ldots, n$, produces a right inverse for J^Φ. Hence its determinant is a unit. □

Example 2.7. The group π_k of the trefoil has presentation

$$\langle x, a \mid x^2ax^{-2} \cdot xa^{-1}x^{-1} \cdot a\rangle.$$

A presentation for the commutator subgroup π'_k can easily be found. Denoting x^jax^{-j} by a_j,

$$\pi'_k = \langle a_j \mid a_{j+2}a_{j+1}^{-1}a_j \ (j \in \mathbb{Z})\rangle.$$

From this one sees that $\pi'_k \cong \langle a_0, a_1 \mid \ \rangle$ is a finitely generated free group.

Consider the related group G with presentation

$$G = \langle x, a \mid x^2ax^{-2} \cdot a^2 \cdot xa^{-1}x^{-1} \cdot a^{-1}\rangle.$$

The $\mathbb{Z}[t^{\pm 1}]$-modules π'_k/π''_k and G'/G'' are isomorphic. In particular, the Alexander polynomial is $t^2 - t + 1$ in both cases. However, using Magnus's Freiheitsatz as in [**?**], one sees that G' is not a finitely generated free group. Theorem **??** provides a more elementary way of seeing this.

Consider the augmented group system $\mathcal{A} = (G, \epsilon, x)$, where $\epsilon : G \to \mathbb{Z}$ is the abelianization map sending x to 1. One checks that there exists a representation $\gamma : G \to \mathrm{GL}_3(\mathbb{Z})$ such that

$$\gamma(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \gamma(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

A simple calculation shows that $D_\gamma(\mathcal{A}) = (t^2 - 4)(t^2 - 1)(t^2 - t + 1)$. Since the constant term is not a unit, G' is not a finitely generated free group.

3 Virtual knot example.

Virtual knots were introduced by L. Kauffman in 1997. They are described by diagrams that can have "virtual crossings" (indicated by circles) as well as classical over/under crossings. A virtual knot is an equivalence class of diagrams under generalized Reidemeister moves. Details about virtual knots and their groups, a generalization of the classical notions, can be found in [**?**].

Many invariants of classical knots, such as the knot group and Jones polynomial, are defined in a natural way for virtual knots. Details can be found in [**?**].

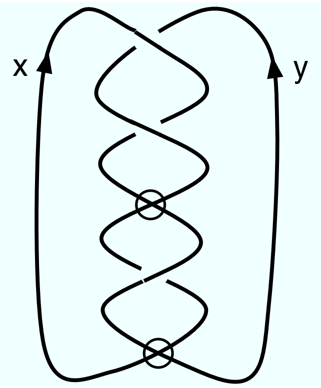

Figure 1: Virtual knot and Wirtinger generators

Example 3.1. Consider the virtual knot k in Figure **??** with generators for a Wirtinger presentation indicated. In [**?**] Kauffman gave it as an example of a virtual knot with trivial Jones polynomial. We show that it also has a faithful representation with trivial Wada invariant.

The group π of k has presentation

$$\pi = \langle x, y \mid xyx = yxy, xy^{-1}x = yxy^{-1} \rangle.$$

Substituting $y = ax$ yields

$$\pi = \langle x, a \mid x^2ax^{-2} = a^{-1} \cdot xax^{-1}, xax^{-1} = a^2 \rangle,$$

from which we can compute a presentation of the commutator subgroup:

$$\pi' = \langle a_j \mid a_{j+2} = a_j^{-1}a_{j+1}, a_{j+1} = a_j^2 \ (j \in \mathbb{Z}) \rangle,$$

where $a_j = x^j a x^{-j}$. It follows immediately that $\pi' \cong \langle a_0 \mid a_0^3 \rangle$ is a cyclic group of order 3. Hence π is a semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$:

$$\pi = \langle x, a \mid xa = a^2x, a^3 = 1 \rangle.$$

Consider the representation $\gamma : \pi \to \mathrm{GL}_2\mathbb{C}$ sending x and a to

$$X = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix},$$

where α is a nonzero complex number while ω is a primitive 3rd root of unity. If α is not a root of unity, then the matrix X has infinite order, and the representation γ is faithful. A straightforward calculation shows that

$$\mathcal{M}_\gamma = \begin{pmatrix} tX - I - A \\ I + A + A^2 \end{pmatrix} = \begin{pmatrix} -\omega - 1 & \alpha t \\ \alpha t & -\bar{\omega} - 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Alexander-Lin polynomial is $D_\gamma = 1 - \alpha^2 t^2$. Since this is the same as $\det(I - tX)$, the Wada invariant W_γ is equal to 1.

Remark 3.2. In [**?**], M. Suzuki showed that the braid group B_4 with the faithful Lawrence-Krammer representation also has trivial Wada invariant. The group in Example **??** is well known to be isomorphic to the group of a 2-knot, the 2-twist-spun trefoil. However, neither B_4 nor the group in Example **??** has a presentation with deficiency 1 (that is, one more generator than the number of relators).

Question: Does there exist an infinite nonabelian group G of deficiency 1 and abelianization $\mathbb{Z}$ together with a faithful representation $\gamma : G \to \mathrm{GL}_n\mathbb{C}$ such that the Wada invariant W_γ is equal to 1?

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