

On Reciprocity of Twisted Alexander Invariants

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Abstract

Given a knot and an $SL_n\mathbb{C}$ representation of its group that is conjugate to its dual, the representation that replaces each matrix with its inverse-transpose, the associated twisted Reidemeister torsion is reciprocal. An example is given of a knot group and $SL_3\mathbb{Z}$ representation that is not conjugate to its dual for which the twisted Reidemeister torsion is not reciprocal.

Keywords: Knot, twisted Reidemeister torsion, twisted Alexander polynomial¹

1 Introduction

The Alexander polynomial $\Delta(t)$ of a knot k can be computed from a diagram of k or from a presentation of the knot group (see [?], for example). It is an integral Laurent polynomial, well defined up to multiplication by units $\pm t^i \in \mathbb{Z}[t^{\pm 1}]$, and it is usually normalized to be a polynomial with nonzero constant coefficient.

It is well known that $\Delta(t)$ is *reciprocal* in the sense that

$$\Delta(t^{-1}) \doteq \Delta(t), \tag{1.1}$$

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where \doteq indicates equality up to multiplication by units. This is a consequence of Poincaré duality of the knot exterior (see [?] for an alternative approach based on duality in the knot group).

In 1990 X.S. Lin introduced a more sensitive invariant using information from nonabelian representations of the knot group [?]. Later, refinements were described by M. Wada [?] and others including P. Kirk and C. Livingston [?], J. Cha [?], and others. These twisted Alexander invariants have proven to be useful for a variety of questions about knots including questions about concordance [?], knot symmetry [?] and fibrations [?]. See [?] for a survey.

We briefly review the definition of perhaps the best-known twisted Alexander invariant. Let k be a knot with exterior X , endowed with the structure of a CW complex. We fix a Wirtinger presentation $\langle x_0, x_1, \dots, x_k \mid r_1, \dots, r_k \rangle$ for the knot group $\pi = \pi_1(X)$. Let $\phi : F_k \rightarrow \pi$ be the associated projection of the free group $F_k = \langle x_0, x_1, \dots, x_k \mid \rangle$ to π . It induces a ring homomorphism $\tilde{\phi} : \mathbb{Z}[F_k] \rightarrow \mathbb{Z}[\pi]$.

Let $\epsilon : \pi \rightarrow H_1(X; \mathbb{Z}) \cong \langle t \mid \rangle$ be the abelianization mapping each x_i to t . It induces a ring homomorphism $\tilde{\epsilon} : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[t^{\pm 1}]$.

Assume that $\gamma : \pi \rightarrow \mathrm{SL}_n \mathbb{C}$ is a linear representation. Let $\tilde{\gamma} : \mathbb{Z}[\pi] \rightarrow M_n(\mathbb{C})$ be the associated ring homomorphism to the algebra of $n \times n$ matrices over \mathbb{C} . We obtain a homomorphism

$$\tilde{\gamma} \otimes \tilde{\epsilon} : \mathbb{Z}[\pi] \rightarrow M_n(\mathbb{C}[t^{\pm 1}]), \quad (1.2)$$

mapping g to $t^{\epsilon(g)}\gamma(g)$, that we denote more simply by Φ .

Let $M_{\gamma \otimes \epsilon}$ denote the $k \times (k+1)$ matrix with (i, j) -component equal to the $n \times n$ matrix $\Phi(\frac{\partial r_i}{\partial x_j}) \in M_n(\mathbb{C}[t^{\pm 1}])$. Here $\frac{\partial r_i}{\partial x_j}$ denotes Fox partial derivative. Let $M_{\gamma \otimes \epsilon}^0$ denote the $k \times k$ matrix obtained by deleting the column corresponding to x_0 . We regard $M_{\gamma \otimes \epsilon}^0$ as a $kn \times kn$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$.

Definition 1.1. The *Wada invariant* $W_\gamma(t)$ is

$$\frac{\det M_{\gamma \otimes \epsilon}^0}{\det \Phi(x_0 - 1)}.$$

When γ is the trivial 1-dimensional representation, $M_{\gamma \otimes \epsilon}^0$ is a matrix $M(t)$ that we call the *Alexander matrix* of k . (This terminology is used, for example, in [?], but it is not standard.) The determinant of $M(t)$ is the (untwisted) Alexander polynomial $\Delta(t)$ of k .

Remark 1.2. The rational function $W_\gamma(t)$ need not be a polynomial. See [?].

The matrix $M_{\gamma \otimes \epsilon}$ represents a boundary homomorphism for a twisted chain complex

$$C_*(X; V[t^{\pm 1}]_\gamma) = (\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V) \otimes_\gamma C_*(\tilde{X}). \quad (1.3)$$

Here $V = \mathbb{C}^n$ is a vector space on which π acts via γ , while $C_*(\tilde{X})$ denotes the cellular chain complex of the universal cover \tilde{X} with the structure of a CW complex that is lifted from X . The group ring $\mathbb{Z}[\pi]$ acts on the left via deck transformations. On the other hand, $\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V$ has the structure of a right $\mathbb{Z}[\pi]$ -module via

$$(p \otimes v) \cdot g = (pt^{\epsilon(g)}) \otimes (v\gamma(g)), \text{ for } \gamma \in \pi.$$

Remark 1.3. The homology group $H_1(X; V[t^{\pm 1}])$ of the chain complex (??) is a finitely generated $\mathbb{C}[t^{\pm 1}]$ -module. Its 0th elementary divisor, $\Delta_\gamma(t)$, lately competes with $W_\gamma(t)$ for the name ‘‘twisted Alexander polynomial.’’ In many cases they are equal; generally, $\Delta_\gamma(t)$ is $\det M_{\gamma \otimes \epsilon}^0$ divided by a factor of $\det \Phi(x_0 - 1)$. See [?] or [?] for details.

Let $\mathbb{C}(t)$ denote the field of rational functions. When $\det M_{\gamma \otimes \epsilon}^0 \neq 0$, the chain complex

$$C_*(X; V(t)) = (\mathbb{C}(t) \otimes_{\mathbb{C}} V) \otimes_\gamma C_*(\tilde{X}) \quad (1.4)$$

is acyclic [?], and hence the (Reidemeister) torsion $\tau_\gamma(t)$ is defined. In [?] it is shown that $\tau_\gamma(t)$ coincides with the Wada invariant $W_\gamma(t)$.

Remark 1.4. Conjugating the representation γ corresponds to a change of basis for V . It is well known that the invariants $W_\gamma(t)$, $\Delta_\gamma(t)$ and $\tau_\gamma(t)$ are unchanged.

T. Kitano used Poincaré duality to prove in [?] that for orthogonal representations $\gamma : \pi \rightarrow \text{SO}_n(\mathbb{R})$, the torsion $\tau_\gamma(t)$ is reciprocal; that is, $\tau_\gamma(t^{-1}) \doteq \tau_\gamma(t)$. He asked whether reciprocity holds for general representations $\gamma : \pi \rightarrow \text{SL}_n(\mathbb{C})$. The question appeared more recently in [?].

Several years later, Kirk and Livingston showed in [?] that reciprocity holds whenever γ is unitary. In particular, it holds for all representations with finite image.

It is not difficult to find representations $\gamma : \pi \rightarrow \text{GL}_n \mathbb{C}$ such that $\tau_\gamma(t)$ is non-reciprocal. For example, consider the Wirtinger presentation

$\langle x_0, x_1, x_2 \mid x_0x_1 = x_2x_0, x_1x_2 = x_0x_1 \rangle$ of the trefoil knot group π . The assignment $x_i \mapsto X_i \in \mathrm{GL}_2\mathbb{C}$, such that

$$X_0 = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} a & -(a^2 - a + 1) \\ 0 & 1 \end{pmatrix}, \quad X_2 = X_1^{-1}X_0X_1$$

yields $\tau_\gamma(t) = at^2 + 1$. The question of reciprocity for representations in $\mathrm{SL}_n\mathbb{C}$ is more subtle.

In Section ?? we show that reciprocity need not hold for general representations in $\mathrm{SL}_n\mathbb{C}$. The representations γ that we consider have the property that the dual representation $\bar{\gamma}$, obtained by replacing each matrix $\gamma(g), g \in \pi$, by its inverse-transpose, is not conjugate to γ . We wish to thank Walter Neumann for suggesting to us that such a representation might yield non-reciprocal torsion.

In Section ?? we prove that if a representation $\gamma : \pi \rightarrow \mathrm{SL}_n\mathbb{C}$ is conjugate to its dual, then the torsion $\tau_\gamma(t)$ is reciprocal.

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2 Examples

Any reciprocal even-degree integral polynomial $\Delta(t)$ such that $\Delta(1) = \pm 1$ arises as the Alexander polynomial of a knot (see [?], for example). Let $f(t)$ be any monic integral polynomial with constant coefficient -1 and $f(1) = \pm 1$. Choose a knot k with Alexander polynomial $\Delta(t) = f(t)f(t^{-1})$.

Let C be the companion matrix of $(t-1)f(t)$. Then $C \in \mathrm{SL}_n\mathbb{Z}$, where $\deg f = n-1$. Consider the cyclic representation $\gamma : \pi \rightarrow \mathrm{SL}_n\mathbb{Z}$ sending each generator x_0, x_1, \dots, x_k of a Wirtinger presentation of π to C . We have

$$\tau_\gamma(t) \doteq \frac{\det M_{\gamma \otimes \epsilon}^0}{\det \Phi(x_0 - 1)} \doteq \frac{\det M_{\gamma \otimes \epsilon}^0}{f(t^{-1})(t-1)}. \quad (2.1)$$

The matrix $M_{\gamma \otimes \epsilon}^0$ can be obtained from the $(k \times k)$ Alexander matrix $M(t)$ by replacing each polynomial entry $\sum a_i t^i$ with the $(n \times n)$ block matrix $\sum a_i (tC)^i$. Since the $n \times n$ blocks commute,

$$\det M_{\gamma \otimes \epsilon}^0 = \prod_{\lambda} \det M(t\lambda),$$

where λ ranges over the eigenvalues of C , that is, the roots of $(t-1)f(t)$ (see [?] for details). Hence

$$\det M_{\gamma \otimes \epsilon}^0 \doteq \prod_{\lambda} \Delta(t\lambda) = \Delta(t) \prod_{\lambda: f(\lambda)=0} f(t\lambda)f(t^{-1}\lambda^{-1}).$$

Since $\Delta(t)$ and $\det M_{\gamma \otimes \epsilon}^0(t)$ are integral polynomials, so is

$$g(t) = \prod_{\lambda: f(\lambda)=0} f(t\lambda)f(t^{-1}\lambda^{-1}).$$

Lemma 2.1. *If $\deg f = 2$, then $g(t)$ is reciprocal.*

Proof. Our assumptions about $f(t)$ imply that its roots have the form $\lambda, -\lambda^{-1}$, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $g(t) = f(t\lambda)f(t^{-1}\lambda^{-1})f(-t\lambda^{-1})f(-t^{-1}\lambda)$ while $g(t^{-1}) = f(t^{-1}\lambda)f(t\lambda^{-1})f(-t^{-1}\lambda^{-1})f(-t\lambda)$. Observe that $g(t)$ and $g(t^{-1})$ have the same roots:

- $f(t\lambda)$ and $f(-t^{-1}\lambda^{-1})$ have roots: $t = 1, -\lambda^{-2}$;
- $f(t^{-1}\lambda^{-1})$ and $f(-t\lambda)$ have roots: $t = -1, \lambda^{-2}$;
- $f(-t\lambda^{-1})$ and $f(t^{-1}\lambda)$ have roots: $t = 1, -\lambda^2$;
- $f(-t^{-1}\lambda)$ and $f(t\lambda^{-1})$ have roots: $t = -1, \lambda^2$.

It follows that $g(t^{-1}) = \alpha g(t)$, for some $\alpha \in \mathbb{C} \setminus \{0\}$. Letting $t = 1$, we see that $\alpha = 1$. Hence $g(t^{-1}) = g(t)$. □

Remark 2.2. The numerator $\det M_{\gamma \otimes \epsilon}^0$ of (??) is a polynomial invariant $D_\gamma(t)$ of k (see [?]). Since $\Delta(t)$ is reciprocal, Lemma ?? implies that $D_\gamma(t)$ is reciprocal whenever $\deg f = 2$. Example ?? below shows that this conclusion need not hold when $\deg f > 2$.

Proposition 2.3. *Let $f(t)$ be a polynomial as above with degree 2. If $f(t)$ is non-reciprocal, then $\tau_\gamma(t)$ is a non-reciprocal integral polynomial of the form $(t-1)h(t)$.*

Proof. From equation (??),

$$\tau_\gamma(t) \doteq \frac{f(t)f(t^{-1})g(t)}{f(t^{-1})(t-1)} \doteq \frac{f(t)g(t)}{t-1}. \quad (2.2)$$

Since $g(t)$ and $t-1$ are reciprocal but $f(t)$ is not, $\tau_\gamma(t)$ is non-reciprocal. To see that $\tau_\gamma(t)$ has the desired form, note that $(t-1)^2$ divides $g(t)$ since both factors $f(t\lambda), f(-t\lambda^{-1})$ of $g(t)$ vanish when $t = 1$. □

Example 2.4. Let $f(t) = t^2 - t - 1$. Then

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Computation shows that $g(t) = (t-1)^2(t+1)^2(t^2-3t+1)(t^2+3t+1)$. By equation (??),

$$\tau_\gamma(t) \doteq (t^2 - t + 1)(t - 1)(t + 1)^2(t^2 - 3t + 1)(t^2 + 3t + 1),$$

which is non-reciprocal.

Example 2.5. Let $f(t) = t^3 - t - 1$. Then

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Computation shows that $g(t) = (t-1)^3(t^3-t-1)^2(t^3-t^2+2t-1)(t^6+3t^5+5t^4+5t^3+5t^2+3t+1)$. The polynomial $f(t)f(t^{-1})g(t)$ is the numerator $D_\gamma(t)$ of Wada's invariant (??). It is non-reciprocal.

It is not difficult to see that for any cyclic representation, $D_\gamma(t) \doteq \Delta_\gamma(t)$ (see Section 3 of [?]) Hence this example shows that $\Delta_\gamma(t)$ can also be non-reciprocal.

3 Sufficient condition for reciprocity

If $\gamma : G \rightarrow \text{GL}_n\mathbb{F}$ is a linear representation, then the *dual* (or *contragredient*) representation $\bar{\gamma}$ is defined by

$$\bar{\gamma}(g) = {}^t\gamma(g)^{-1},$$

where t denotes transpose.

The following elementary lemma is included for the reader's convenience.

Lemma 3.1. *A representation $\gamma : G \rightarrow \text{GL}_n\mathbb{F}$ is conjugate to its dual if and only if there exists a nondegenerate bilinear form $(v, w) \mapsto \{v, w\} \in \mathbb{F}$ on V such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$ and $g \in G$.*

Proof. Assume that $\bar{\gamma}$ is conjugate to γ . Then there exists a matrix $A \in \mathrm{GL}_n \mathbb{F}$ such that $A^{-1}\gamma(g)A = {}^t\gamma(g)^{-1}$, for all $g \in G$. Define $\{v, w\} = vA {}^tw$. Since A is invertible, the bilinear form is nondegenerate. It is easy to check that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V$.

Conversely, assume that γ preserves a nondegenerate bilinear form $(v, w) \mapsto \{v, w\}$. There exists an invertible matrix $A \in \mathrm{GL}_n \mathbb{F}$ such that $\{v, w\} = vA {}^tw$. Since γ preserves the form, we have $v\gamma(g)A {}^t\gamma(g) {}^tw = \{v \cdot g, w \cdot g\} = \{v, w\} = vA {}^tw$, for all $v, w \in V, g \in G$. It follows that $\gamma(g)A {}^t\gamma(g) = A$ for all $g \in G$. Hence $A^{-1}\gamma(g)A = {}^t\gamma(g)^{-1}$, and so $\bar{\gamma}$ is conjugate to γ . \square

As before, let k be a knot with group π . Assume that $\gamma : \pi \rightarrow \mathrm{SL}_n \mathbb{F}$ is a representation, where \mathbb{F} is an arbitrary field. As above, $V = \mathbb{F}^n$ is a right $\mathbb{Z}[\pi]$ -module via $v \cdot g = v\gamma(g)$, for all $v \in V$ and $\gamma \in \pi$. Let $W = \mathbb{F}^n$ with the dual $\mathbb{Z}[\pi]$ -module structure given by $w \cdot g = w {}^t\gamma(g)^{-1}$.

Theorem 3.2. *Assume that $\det M_{\gamma \otimes \epsilon}^0 \neq 0$. If γ is conjugate to its dual representation $\bar{\gamma}$, then the torsion $\tau_\gamma(t)$ is reciprocal.*

Proof. The following argument is similar to those of [?] and [?].

Recall that X is the exterior of k , endowed with a CW cell structure. Let X' be the same space but with the dual cell structure. Let $\bar{\cdot} : \mathbb{F}(t) \rightarrow \mathbb{F}(t)$ be the involution induced by $t \mapsto t^{-1}$.

Assume that $\gamma : \pi \rightarrow \mathrm{SL}_n \mathbb{F}$ is a representation that is conjugate to its dual. By Lemma ?? there exists a nondegenerate bilinear form $(v, w) \mapsto \{v \cdot g, w \cdot g\}$ such that $\{v \cdot g, w \cdot g\} = \{v, w\}$ for all $v, w \in V, g \in \pi$. Consider the twisted chain complexes

$$C_* = (\mathbb{F}(t) \otimes V) \otimes C_*(\tilde{X}), \quad D_* = (\mathbb{F}(t) \otimes W) \otimes C_*(\tilde{X}', \partial\tilde{X}'),$$

where \tilde{X} and \tilde{X}' denote universal covering spaces of X and X' , respectively. We abbreviate these by $V_{\gamma \otimes \epsilon} \otimes C_*(\tilde{X})$ and $V_{\bar{\gamma} \otimes \epsilon} \otimes C_*(\tilde{X}')$, respectively.

Define a bilinear pairing $C_q \times D_{3-q} \rightarrow \mathbb{F}(t)$ by

$$\langle p \otimes v \otimes z_1, q \otimes w \otimes z_2 \rangle = \sum_{g \in \pi} (z_1 \cdot gz_2) p \bar{q} \{v \cdot g, w\}, \quad (3.1)$$

where $z_1 \cdot gz_2$ is the algebraic intersection number in \mathbb{Z} of cells z_1 and gz_2 . We extend linearly.

The pairing induces a $\mathbb{F}(t)$ -module isomorphism $D_{3-q} \rightarrow \overline{\mathrm{Hom}}(C_q, \mathbb{F}(t))$, where $\overline{\mathrm{Hom}}$ denotes the dual space with $(q \cdot h)(z) = \bar{q}(h(z))$, for all $q \in \mathbb{F}(t), z \in C_q$. Consequently, there exists a nondegenerate pairing $H_q(X; V(t)) \times$

$H_{3-q}(X', \partial X'; W(t)) \rightarrow \mathbb{F}(t)$. Since the torsion of C_* is defined, by our hypothesis, the torsion of D_* is too.

Choose a basis $\{v_i\}$ over \mathbb{F} for V and lifts to \tilde{X} of simplices of X . In this way, we obtain a preferred $\mathbb{F}(t)$ -basis for C_* . Basis members have the form $1 \otimes v_i \otimes z_j$. We get a natural basis over $\mathbb{F}(t)$ for D_* by picking a basis for W that is dual to the basis for V with respect to $\{, \}$, and choosing dual cells in \tilde{X}' of the fixed lifts of simplices of X . As observed in [?], the bases for C_* and D_* that we build are dual with respect to the bilinear form (??).

Let $\tau(X; V_{\gamma \otimes \epsilon})$ denote the torsion of C_* . Similarly, let $\tau(X', \partial X'; V_{\bar{\gamma} \otimes \epsilon})$ denote the torsion of D_* . Then $\tau(X; V_{\gamma \otimes \epsilon}) = \tau(X', \partial X'; V_{\bar{\gamma} \otimes \epsilon})$ by Theorem 1' of [?]. Furthermore,

$$\begin{aligned} \tau(X', \partial X'; V_{\bar{\gamma} \otimes \epsilon}) &= \tau(X, \partial X; V_{\bar{\gamma} \otimes \epsilon}) \quad (\text{by subdivision}) \\ &= \tau(X, \partial X; V_{\gamma \otimes \bar{\epsilon}}) \quad (\text{since } \gamma \text{ is conjugate to } \bar{\gamma}) \\ &= \bar{\tau}(X, \partial X; V_{\gamma \otimes \epsilon}) \\ &= \bar{\tau}(X; V_{\gamma \otimes \epsilon}). \end{aligned}$$

The last equality is a result of Lemma 2 of [?] and the fact that $\tau(\partial X; V_{\gamma \otimes \epsilon}) = 1$ (see [?]). Hence

$$\tau_\gamma(t) = \tau(X; V_{\gamma \otimes \epsilon}) = \bar{\tau}(X; V_{\gamma \otimes \epsilon}) = \bar{\tau}_\gamma(t).$$

□

Remark 3.3. If $\mathbb{F} = \mathbb{R}$, and the bilinear form in Lemma ?? is positive-definite, then by considering a basis for V that is orthonormal with respect to the form, we see that A is the identity matrix. In this case, $\gamma(g) = {}^t\gamma(g)^{-1}$ for all $g \in G$, and hence γ is conjugate to an orthogonal representation. Similarly, if $\mathbb{F} = \mathbb{C}$ and the bilinear form is hermitian and positive-definite, γ is conjugate to a unitary representation.

Corollary 3.4. *If $\gamma : \pi \rightarrow \text{Sp}_{2n}\mathbb{C}$ is a symplectic representation, then $\tau_\gamma(t)$ is reciprocal.*

Proof. The representation preserves the bilinear form given by $A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. □

Since $\text{Sp}_2\mathbb{C} = \text{SL}_2\mathbb{C}$, the following is immediate.

Corollary 3.5. *If γ is any representation of π in $\text{SL}_2\mathbb{C}$, then $\tau_\gamma(t)$ is reciprocal.*

Corollary ?? shows that Example ?? is, in a sense, the simplest possible.

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