CROWELL'S DERIVED GROUP AND TWISTED POLYNOMIALS

Daniel S. Silver and Susan G. Williams

Dedicated to Louis H. Kauffman on the occasion of his sixtieth birthday.

ABSTRACT: The twisted Alexander group of a knot is a special case of Crowell's derived group of a permutation representation. We use it to extend a theorem of Neuwrith and Stallings, giving new necessary and sufficient conditions for a knot to be fibered.

Virtual Alexander polynomials were introduced by the authors in a previous paper. They provide obstructions for a virtual knot that must vanish if the knot has a diagram with an Alexander numbering. The extended group of a virtual knot is introduced, and using it a more sensitive obstruction is obtained. Twisted virtual Alexander polynomials are defined via Crowell's construction.

1. Introduction. The Alexander group $\mathcal{A}_\ell$ of an oriented virtual d-component link ℓ was defined in [SW01]. (The definition is reviewed below.) The group π_ℓ of the link acts by permutations on its abelianization $\mathbb{Z}^d$ in a standard way. The Alexander group can be obtained from π_ℓ together with its action. As such $\mathcal{A}_\ell$ is an example of R.H. Crowell's "derived group of a permutation representation" [C84]. When ℓ is classical, the abelianization of $\mathcal{A}_\ell$ is the Alexander module of ℓ, thereby justifying the name that we have given it.

The main objective of [SW01] was to introduce the extended Alexander group $\tilde{\mathcal{A}}_\ell$ (see §3) and its associated sequence of "ith virtual Alexander polynomials" $\Delta_{\ell,i} = \Delta_{\ell,i}(u_1, \ldots, u_d, v),\ 0 \le i < \infty$. (While some normalization is possible [SW03], virtual Alexander polynomials here are defined only up to multiplication by units $\pm u_i, \pm v$.) When ℓ is classical, a change of generators for $\tilde{\mathcal{A}}_\ell$ reduces each $\Delta_{\ell,i}$ to the usual ith Alexander polynomial of ℓ in the variables $u_1 v, \ldots, u_d v$. In that case, $\Delta_{\ell,0}$ vanishes, and each $\Delta_{\ell,i}$ is reciprocal. Consequently, the polynomials $\Delta_{\ell,i}$ can be used to show that a virtual link is not classical (that is, every diagram contains at least one virtual crossing). The 0th virtual Alexander polynomial $\Delta_{\ell,0}$ was independently derived by J. Sawollek [S99], using an invariant of links in thickened surfaces due to F. Jaeger, L. Kauffman and H. Saleur [JKS94].

The extended Alexander group $\tilde{\mathcal{A}}_\ell$ did not appear as a derived group. Here we explain its relationship with a certain derived group, uniting the mysterious foundling with its parent. In order to do this, we introduce the "extended link group" $\tilde{\pi}_\ell$, a group extension of π_ℓ that is natural and useful. Its generators are those of π_ℓ together with an additional generator x. We prove that the derived group of the permutation representation of $\tilde{\pi}_\ell$ on its abelianization, taken modulo the normal subgroup generated by x, is isomorphic to $\tilde{\mathcal{A}}_\ell$.

The extended group gives rise to new nonabelian invariants for virtual links. In particular, twisted virtual Alexander polynomials can be defined from representations of $\tilde{\pi}_\ell$ (see §3).

The authors are grateful to J.S. Carter and A. Champanerkar for comments and suggestions.

2. Crowell's derived group. The derived group [C84] represents an attempt to unify into one theory the concepts of knot group, Alexander matrix and covering space, three concepts that Crowell and Fox believed were central to knot theory (see preface to [CF77]). We review the idea using notation that is convenient for our purposes.

Both authors partially supported by NSF grant DMS-0304971.

2000 *Mathematics Subject Classification.* Primary 57M25; Secondary 20F05, 20F34.

Let G be a multiplicative group acting on the right of a nonempty set Γ. The action $\Gamma \times G \to \Gamma$, denoted by $(\gamma, g) \mapsto \gamma g$, corresponds to a representation $\rho : G \to S_\Gamma$, where S_Γ denotes the group of permutations of Γ.

Definition 2.1. [C84] The *derived group (of the permutation representation ρ)* is the free group with basis $\Gamma \times G = \{g^\gamma \mid g \in G, \gamma \in \Gamma\}$ modulo the relations $(gh)^\gamma = g^\gamma h^{\gamma g}$, for all $g, h \in G$ and $\gamma \in \Gamma$. We denote it by G_ρ.

It is helpful to regard the exponent γ in g^γ as a coordinate. For any $g \in G$ and $\gamma \in \Gamma$, one easily sees that $(g^\gamma)^{-1} = (g^{-1})^{\gamma g}$; also, $g^\gamma = 1$ if and only if g is trivial in G (see [C84]).

The derived group was motivated by path-lifting in covering spaces. Crowell defined it as a universal object in a category of groups and crossed-products.

Operator groups are both natural and useful when considering fundamental groups of covering spaces. The notion, reviewed here, can be found in [R96]. An Ω-*group* is a group K together with a set Ω (*operator set*) and a function $\phi : K \times \Omega \to K,\ (g, \omega) \mapsto g^\omega$, such that for each fixed ω, the restricted map $\phi_\omega : K \to K$ is an automorphism.

If K is an Ω-group and Ω_0 is a subset of Ω, then we can regard K as an Ω_0-group by restricting the action.

An Ω-group is *finitely generated* (respectively, *finitely presented*) if it is generated (resp. presented) by finitely many Ω-orbits of generators (resp. generators and relators). The commutator subgroup of a knot group is an example of a finitely presented $\mathbb{Z}$-group. Free Ω-groups are defined in the standard way.

Derived groups are often operator groups. We mention some important examples.

Assume that a group G acts on a multiplicative semigroup Γ so that $(\gamma\gamma')g = \gamma(\gamma' g)$ for all $\gamma, \gamma' \in \Gamma$ and $g \in G$. Let G_ρ be the associated derived group. We can define a map $\phi : G_\rho \times \Gamma \to G_\rho$ on generators of G_ρ by $\phi_\gamma(g^{\gamma'}) = g^{\gamma\gamma'}$, extending multiplicatively. One checks that G_ρ is then a Γ-group. Its abelianization G_ρ/G'_ρ is a left $\mathbb{Z}[\Gamma]$-module with $(\sum n_i \gamma_i) g^\gamma = \sum n_i g^{\gamma_i \gamma}$ for $\sum n_i \gamma_i \in \mathbb{Z}[\Gamma]$. If G has presentation $\langle g_1, \ldots, g_m \mid r_1, \ldots, r_n\rangle$, then G_ρ is finitely presented as a Γ-group by the orbits of $g_1^1, \ldots, g_m^1$ and $r_1^1, \ldots, r_n^1$. With a slight abuse of notation we denote g_i^1 by g_i. The relators r_j^1 are easily written in terms of generators g_i^γ of G_ρ (see Example 2.3).

Definition 2.2. [C84] Let $P : G \to \Gamma$ be a group homomorphism. Let $\Gamma \times G \to \Gamma$ be the G-action on Γ given by $(\gamma, g) \mapsto \gamma P(g)$, and let $\rho : G \to S_\Gamma$ be the associated permutation representation. The *derived module* of P is the $\mathbb{Z}[\Gamma]$-module G_ρ/G'_ρ.

If G acts on a left R-module V then, regarding R as a multiplicative semigroup, G_ρ has the structure of an R-group. The requisite function $G_\rho \times R \to G_\rho$ is defined on generators by $(g^v, r) \mapsto g^{rv}$ and extended multiplicatively.

The following example illustrates how the usual abelian invariants of a knot as well as their twisted counterparts come naturally from the Alexander group.

Example 2.3. The group π_k of the figure-eight knot $k = 4_1$ has presentation

$$\pi_k = \langle a, b \mid \bar{a}ba\bar{b}ab\bar{a}\bar{b}a\bar{b}\rangle, \tag{2.1}$$

where $\bar{\ }$ denotes inverse. Let $P : \pi_k \to \langle t \mid \ \rangle \cong \mathbb{Z}$ be the abelianization homomorphism mapping a and b to t. The derived group of the associated permutation representation $\rho : \pi_k \to S_\mathbb{Z}$ has $\mathbb{Z}$-group presentation

$$\pi_{k,\rho} = \langle a, b \mid \bar{a}^{t^{-1}} b^{t^{-1}} a\bar{b}ab^t\bar{a}^t\bar{b}a\bar{b}\rangle.$$

The generators a and b represent orbits $\{a^{t^i}\}_{i\in\mathbb{Z}}$ and $\{b^{t^i}\}_{i\in\mathbb{Z}}$ of group generators, while the single relator $\delta(r)$ represents an orbit $\{r^{t^i}\}_{i\in\mathbb{Z}}$ of relators in $\pi_{k,\rho}$. The relator $\delta(r)$ is a rewrite of r^1, where r is the relator in (2.1). It is easily computed by a variant of Fox calculus, using the axioms:

$$\delta(g) = g, \quad \delta(\bar{g}) = \bar{g}^{P(\bar{g})}$$

$$\delta(wg) = \delta(w)g^{P(w)}, \quad \delta(w\bar{g}) = \delta(w)\bar{g}^{P(w\bar{g})}.$$

Here g is any generator while w is a word in generators and their inverses.

For this particular representation ρ, we refer to the group $\pi_{k,\rho}$ as the *Alexander group* of the knot, and denote it by $\mathcal{A}_k$. Abelianizing $\mathcal{A}_k$ gives the derived module of the abelianization homomorphism. It is the Alexander module of the knot, a $\mathbb{Z}[t^{\pm 1}]$-module with matrix presentation described by the usual Fox partial derivatives (see [BZ03]):

$$\left(\left(\frac{\partial r}{\partial a}\right)^P \left(\frac{\partial r}{\partial b}\right)^P \right) = (-t^{-1} + 3 - t \qquad t^{-1} - 3 + t).$$

The superscript indicates that each term of the formal sum is to be replaced by its P-image.

The 0th characteristic polynomial of the matrix, $t^2 - 3t + 1$ (well defined up to multiplication by $\pm t^i$), is the (1st) Alexander polynomial $\Delta_k(t)$ of the knot. The index shift is a consequence of the fact that we removed a redundant relator from the Wirtinger presentation for π_k when obtaining (2.1).

Twisted Alexander polynomials, introduced for knots in [L90] and defined for finitely presented groups in [W94], can be obtained by replacing ρ with more general representations. Let $P_1 : \pi_k \to \langle t \mid \rangle$ be the abelianization homomorphism considered above, and let $P_2 : \pi_k \to \mathrm{SL}_2(\mathbb{C})$ be the discrete faithful representation

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix},$$

where $\omega = \dfrac{1 - \sqrt{-3}}{2}$ (see Chapter 6 of [T78]). Together, P_1 and P_2 determine a homomorphism $P : \pi_k \to GL_2(\mathbb{C}[t^{\pm 1}])$ described by:

$$a \mapsto \begin{pmatrix} t & t \\ 0 & t \end{pmatrix} \quad b \mapsto \begin{pmatrix} t & 0 \\ -t\omega & t \end{pmatrix}.$$

One can regard $P(g)$ as $P_1(g) \otimes P_2(g)$, the product of the scalar monomial $P_1(g)$ with $P_2(g)$ in $\mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}} \mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C}[t^{\pm 1}])$. Denote the image of P by Γ.

The derived module of $P : \pi_k \to \Gamma$ has a presentation matrix given by a 1×2 Jacobian:

$$\left(\left(\frac{\partial r}{\partial a}\right)^P \left(\frac{\partial r}{\partial b}\right)^P \right).$$

Each entry is a formal sum of matrices. Add its terms to form a single 2×2-matrix. The result is:

$$\left(\begin{pmatrix} -\bar{\omega}t + 1 + \bar{\omega} - \frac{1}{t} & t - 2 - \omega + \frac{1}{t} \\ i\sqrt{3}\bar{\omega}(t-1) & -\bar{\omega}t + 3 - \frac{1}{t} \end{pmatrix} \begin{pmatrix} -2 + \frac{1}{t} & -\omega t + 1 + \omega - \frac{1}{t} \\ \bar{\omega}t - 1 & 2t - 3 + \frac{1}{t} \end{pmatrix} \right). \tag{2.2}$$

Remove the inner parentheses to form a 2×4 *twisted Alexander matrix* A_ρ. Delete the first two columns (corresponding to the generator a), take the determinant, and divide by $\mathrm{Det}(tI - P(a))$. The resulting polynomial is Wada's invariant,

$$W_{k,\rho} = \frac{(t-1)^2(t^2 - 4t + 1)}{(t-1)^2} = t^2 - 4t + 1.$$

The result is the same if we reverse the roles of a by b. (See [**W94**].)

Adding terms and removing inner parentheses to obtain B appears *ad hoc.* The following alternative approach eliminates the need.

The group π_k acts on the free $\mathbb{C}[t^{\pm 1}]$-module $V = \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}[t^{\pm 1}]$ via $P : \pi_k \to \mathrm{GL}_2(\mathbb{C}[t^{\pm 1}])$: for any $v \in V$ and $g \in \pi_k$, we define vg to be $vP(g)$. A permutation representation ρ_V of π_k on V is defined. By remarks above, the derived group π_{π_k, ρ_V} is a $\mathbb{C}[t^{\pm 1}]$-group, and the abelianization $\pi_{k,\rho_V}/\pi'_{k,\rho_V}$ becomes a left $\mathbb{C}[t^{\pm 1}]$-module if we impose the extra relations $(v_1 + v_2)g = v_1 g + v_2 g$ for every $v_1, v_2 \in V$ and $g \in \pi_k$. It is generated by $e_1 a, e_2 a, e_1 b, e_2 b$, where $e_1 = (1, 0), e_2 = (0, 1)$ is the standard basis for V, and one easily checks that A_ρ is a presentation matrix for it.

Remark 2.4. (i) Twisted Alexander polynomials of any d-component link ℓ can be found in a similar way, using the abelianization $P_1 : \pi_\ell \to \mathbb{Z}^d$.

(ii) The greatest common divisor of the six 2×2 minors of A is an invariant of k and the representation P. It is the 1st twisted Alexander polynomial Δ_1, defined by P. Kirk and C. Livingston [KL99]. The greatest common divisor of the 1×1 minors (that is, the entries of A) is the 0th twisted polynomial Δ_0. The relationship $W_{k,\rho} = \Delta_1/\Delta_0$ holds generally.

The reader is cautioned that an invariant does not result by simply taking the greatest common divisor of the two 2×2 minors corresponding to the entries of (2.2), contrary to Proposition 1.3 of [JW93]. Such a quantity is not preserved when the presentation of π_k is altered by a Tietze move, in contradiction to the assertion on page 213 of [JW93]. Indeed, in the above example, the greatest common divisor of the two minors is $(t-1)^2(t^2-4t-1)$. However, if in the presentation of π_k we replace a by the new generator $c = \bar{a}b$, accomplished by a sequence of Tietze moves, then the polynomial $t^2 - 4t + 1$ is obtained.

We recall from [SW01] that for any knot k, the Alexander group $\mathcal{A}_k$ is the free product of the commutator subgroup $[\pi_k, \pi_k]$ with a countable-rank free group F. One can choose F to be the free group generated by any $\mathbb{Z}$-group family $\{x^{t^i}\}$ of group generators, where x is a Wirtinger generator ([C84]; see [SW01] also). Notice, in particular, that $\mathcal{A}_k/\langle\langle x \rangle\rangle$ is isomorphic to $[\pi_k, \pi_k]$.

The commutator subgroup $[\pi_k, \pi_k]$ of any knot $k \subset \mathbb{S}^3$ is a finitely generated $\mathbb{Z}$-group. As an ordinary group, $[\pi_k, \pi_k]$ need not be finitely generated. A theorem of J. Stallings [S62] states that if it is, then k is fibered, and $[\pi_k, \pi_k]$ is free of rank $2g_k$, where g_k denotes the genus of k. The converse is also true: if k is fibered, then $[\pi_k, \pi_k]$ is a free group of rank $2g_k$ [N63].

A consequence is that if k is fibered, then its Alexander polynomial is monic with degree equal to twice the genus of k. H. Goda, T. Kitano and T. Morifuji [GTM03] extended this classical result by showing that if k is fibered, then Wada's invariant $W_{k,\rho}(t)$ corresponding to any homomorphism $P_2 : \pi_k \to \mathrm{SL}_{2n}(\mathbb{F})$, where $\mathbb{F}$ is a field and P_1 the abelianization homomorphism, is a rational function of monic polynomials. We prove a companion result for Alexander groups, generalizing the theorem of Neuwirth and Stallings.

Consider a knot $k \subset \mathbb{S}^3$ with abelianization homomorphism $P_1 : \pi_k \to \langle t \mid \rangle$ and $P_2 : \pi_k \to \mathrm{SL}_n(R)$ any homomorphism. We assume that R is a unique factorization domain. Let $P : \pi_k \to \mathrm{GL}_n(R[t^{\pm 1}])$ be the product homomorphism, as above, with image Γ. Let $\rho : \pi_k \to S_\Gamma$ be the associated permutation representation. We denote by $\mathcal{R}(\pi)$ the collection of permutation representations that arise this way.

The derived group $\pi_{k,\rho}$ is the *ρ-twisted Alexander group* of k, and we denote it by $\mathcal{A}_{k,\rho}$. When P_2 is trivial, $\pi_{k,\rho}$ reduces to the Alexander group of k. The twisted Alexander group is a Γ-group. The image Γ_2 of P_2 is a subgroup of Γ. We can also regard $\mathcal{A}_{k,\rho}$ as a Γ_2-group, letting only elements of Γ_2 act on it. Throughout the remainder or the section, x will denote a Wirtinger generator of π_k corresponding to a meridian of k.

Theorem 2.5. If a knot k is fibered, then, for any representation $\rho \in \mathcal{R}(\pi)$, the twisted Alexander group $\mathcal{A}_{k,\rho}$ modulo $\langle\langle x \rangle\rangle$ is a free Γ_2-group of rank $2g_k$. Conversely, if $\mathcal{A}_{k,\rho}$ modulo $\langle\langle x \rangle\rangle$ is a finitely generated Γ_2-group, for some $\rho \in \mathcal{R}(\pi)$, then k is fibered.

Proof of Theorem 2.5. Assume that k is fibered. Then π_k is an HNN extension of a free group $F = F(a_1, \ldots, a_{2g_k})$, where $g = g_k$ is the genus of k, and it has a presentation of the form

$$\langle x, a_1, \ldots, a_{2g} \mid x a_1 x^{-1} = \phi(a_1), \ldots, x a_{2g} x^{-1} = \phi(a_{2g}) \rangle \tag{2.3}$$

for some automorphism ϕ of F (see [BZ03], for example). The twisted Alexander group $\mathcal{A}_{k,\rho}$ has a Γ-group presentation

$$\langle x, a_1, \ldots, a_{2g} \mid x a_1^{tP_2(x)} \bar{x}^{P_2(x a_1 \bar{x})} = \widetilde{\phi(a_1)}, \ldots,\ x a_1^{tP_2(x)} \bar{x}^{P_2(x a_{2g} \bar{x})} = \widetilde{\phi(a_{2g})} \rangle,$$

where $\widetilde{\ }$ denotes the δ-rewrite, explained above. The quotient Γ-group $\mathcal{A}_{k,\rho}/\langle\langle x \rangle\rangle$ has presentation

$$\langle a_1, \ldots, a_{2g} \mid a_1^{tP_2(x)} = \widetilde{\phi(a_1)}, \ldots,\ a_{2g}^{tP_2(x)} = \widetilde{\phi(a_{2g})} \rangle.$$

Since $P_2(x)$ is invertible, each a_i^t can be expressed as $\widetilde{\phi(a_i)}^{P_2(x)^{-1}}$. Generally, the relators can be used one at a time to express each $a_i^{t^j}$, $j \neq 0$, as a word in the Γ_2-orbits of $a_1, \ldots, a_{2g}$. Hence $\mathcal{A}_{k,\rho}/\langle\langle x \rangle\rangle$ is isomorphic to the free Γ_2-group generated by $a_1, \ldots, a_{2g}$.

Conversely, assume that $\mathcal{A}_{k,\rho}/\langle\langle x \rangle\rangle$ is a finitely generated Γ_2-group, for some $\rho \in \mathcal{R}(\pi)$. The knot group π_k is generated by $x, a_1, \ldots, a_m$, where $a_1, \ldots, a_m \in [\pi_k, \pi_k]$. Then $\mathcal{A}_{k,\rho}/\langle\langle x \rangle\rangle$ is generated by elements of the form $a_i^{t^j M}$, where $j \in \mathbb{Z}$ and M ranges over matrices in Γ_2, while $[\pi_k, \pi_k]$ is generated by the elements $a_i^{t^j}$. The mapping $a_i^{t^j M} \mapsto a_i^{t^j}$ determines a surjection $f : \mathcal{A}_{k,\rho}/\langle\langle x \rangle\rangle \to [\pi_k, \pi_k]$. By assumption, we can find generators $a_i^{t^j M}$ of $\mathcal{A}_{k,\rho}/\langle\langle x \rangle\rangle$ such that the values of j are bounded. Consequently, the image of f is finitely generated. Hence k is fibered. ■

3. Extended group of a virtual link. As in the classical case, virtual knots and links are equivalence classes of diagrams, the equivalence relation generated by Reidemeister moves. However, in the case of virtual knots and links, we also allow virtual crossings (indicated by a small circle about the crossing), and the set of Reidemeister moves is suitably generalized. The notion is due to Kauffman, and the reader is referred to [K98] or [K99] for details. A result of M. Goussarov, M. Polyak and O. Viro [GPV00] assures us that if two classical diagrams—that is, diagrams without virtual crossings—are equivalent by generalized Reidemeister moves, then they are equivalent by the usual classical Reidemeister moves. In this sense, virtual knot theory extends the classical one.

Many invariants of classical knots and links are also defined in the virtual category. The knot group π_ℓ is such an invariant. Given any diagram of a virtual knot or link ℓ, a Wirtinger presentation is obtained by assigning a generator to each arc. (An arc is a maximal connected component of the diagram containing no classical under-crossing.) A relation is associated to each classical crossing in the usual way.

Let $\ell = \ell_1 \cup \cdots \cup \ell_d$ be an oriented virtual link of d components with group π_ℓ. Regard $\mathbb{Z}^d$ as a multipicative group freely generated by $u_1, \ldots, u_d$. Let $P : \pi_\ell \to \mathbb{Z}^d$ be the abelianization homomorphism mapping the class of the ith oriented meridian to u_i. Let ρ be the associated permutation representation $\rho : \pi_\ell \to S_{\mathbb{Z}^d}$. The derived group is the Alexander group $\mathcal{A}_\ell$ introduced in [SW01] .

Let $\ell = \ell_1 \cup \cdots \cup \ell_d$ be an oriented virtual link with diagram D. An *edge* of D is a maximal segment of an arc going from one classical crossing to the next. The *extended group* $\tilde{\pi}_\ell$ has generators $a, b, c, \ldots$ corresponding to edges together with an additional generator x not associated with any edge. Relations come in pairs, corresponding to classical crossings: $ab = cd$, $\bar{x}bx = c$, if the crossing is positive, and $ab = cd$, $\bar{x}dx = a$, if the crossing is negative (see Figure 1). Here and throughout $^-$ denotes inverse. It is a straightforward matter to check that the group so defined is unchanged if a generalized Reidemeister move is applied to the diagram, and we leave this to the reader.

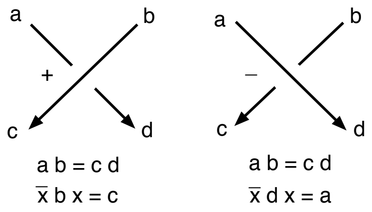

Figure 1: Relations for extended group $\tilde{\pi}_\ell$.

The extended Alexander group $\tilde{\mathcal{A}}_\ell$ of the link ℓ was defined in [SW01] to be the $\mathbb{Z}^{d+1}$-group with generators $a, b, c, \ldots$, corresponding to edges as above, and relations at each classical crossing: $ab^{u_i} = cd^{u_j}$, $c^v = b$, if the crossing is positive; $ab^{u_i} = cd^{u_j}$, $a^v = d$, if the crossing is negative. We assume that generators a, d correspond to edges on the ith component of the link while b, c correspond to edges on the jth component. One regards each of $a, b, c, \ldots$ as families of generators indexed by monomials in $u_1, \ldots, u_d, v$, generators for the group $\mathbb{Z}^{d+1}$ written multiplicatively. Similarly, each relation r is a family of relations indexed by monomials.

Let $P : \tilde{\pi}_\ell \to \mathbb{Z}^{d+1}$ be the abelianization homomorphism mapping generators corresponding to edges on the ith component to u_i, and mapping x to v, and let $\rho : \tilde{\pi}_\ell \to S_{\mathbb{Z}^{d+1}}$ be the associated permutation representation. Consider the derived group $\tilde{\pi}_{\ell,\rho}$. It is generated by $a, b, c, \ldots$ and x. Modulo x, the δ-rewrite of the relations for $\tilde{\pi}_{\ell,\rho}$ are easily seen to be those of the extended Alexander group. Indeed, the δ-rewrite of $ab = cd$ is $ab^{u_i} = cd^{u_j}$, while the rewrite of $xc\bar{x} = b$ is $xc^v\bar{x}^{u_j} = b$. Killing x, that is, killing its $\mathbb{Z}^{d+1}$-orbit, yields the relations for $\tilde{\mathcal{A}}_\ell$ at a positive classical crossing. A similar argument applies to negative crossings. Summarizing:

Proposition 3.2. Let $\ell = \ell_1 \cup \cdots \cup \ell_d$ be an oriented virtual link of d components, and let $\tilde{\pi}_\ell$ be its extended group. Let $\tilde{\pi}_{\ell,\rho}$ be the derived group of the permutation representation associated to the abelianization homomorphism $P : \tilde{\pi}_\ell \to \mathbb{Z}^{d+1}$. Then as $\mathbb{Z}^{d+1}$-groups, the extended Alexander group $\tilde{\mathcal{A}}_\ell$ is isomorphic to $\tilde{\pi}_{\ell,\rho}/\langle\langle x \rangle\rangle$, the derived group modulo the normal subgroup generated by x.

Proposition 3.3(ii) yields new invariants for virtual links.

Proposition 3.3. (i) Let ℓ be an oriented virtual link. Then

$$1 \to \langle\langle x \rangle\rangle \to \tilde{\pi}_\ell \xrightarrow{p} \pi_\ell \to 1$$

is an exact sequence of groups, where $\langle\langle\ \rangle\rangle$ denotes normal closure and p is the quotient map sending $x \mapsto 1$.

(ii) If ℓ is a classical link, then

$$1 \to \mathcal{A}_\ell \to \tilde{\pi}_\ell \xrightarrow{\chi} \langle t \mid \ \rangle \to 1$$

is an exact sequence of groups, where χ is the homomorphism mapping $x \mapsto t$ and $a, b, c, \ldots$ to 1.

Proof. Killing x converts the relations for $\tilde{\pi}_\ell$ into the Wirtinger relations for π_ℓ. Hence (i) is proved.

In order to prove (ii), assume that D is a classical diagram for ℓ; that is, a diagram without virtual crossings. By the Seifert smoothing algorithm, we can label the edges of D with integers $\nu(a), \nu(b), \nu(c), \ldots$ such that at any crossing such as in Figure 1, whether positive or negative, edges corresponding to a, c receive the same label, say $\nu \in \mathbb{Z}$, while edges corresponding to b, d receive $\nu + 1$. Such an assignment of integers will be called an *Alexander numbering.* (See [SW01] for details.)

After replacing each generator $a, b, c, \ldots$ with $x^{\nu(a)} a x^{-\nu(a)}, x^{\nu(b)} b x^{-\nu(b)}, x^{\nu(c)} c x^{-\nu(c)} \ldots$, the relation at a positive crossing becomes $axb\bar{x} = cxd\bar{x}$ and $c = b$. At a negative crossing, the relation becomes $axb\bar{x} = cxd\bar{x}$ and $a = d$. Notice in particular that $\tilde{\pi}_\ell$ is generated by x and symbols corresponding to the *arcs* rather than the edges of D. Moreover, the form of the relations allows us to describe the kernel of χ using the Reidemeister-Schreier method: it is generated as a $\langle t \mid \rangle$-group by symbols corresponding to the arcs of D together with relations $ab^t = bd^t$ at a positive crossing and $ab^t = ca^t$ at a negative crossing. This is also a description of the Alexander group $\mathcal{A}_\ell$ of the link. ∎

Proposition 3.3 motivates the following.

Definition 3.4. A virtual link is *almost classical* if some diagram for the link admits an Alexander numbering.

Remark 3.5. (i) Reassuringly, classical implies almost classical. However, we see in Example 3.10 that the converse does not hold.

(ii) The conclusion of Proposition 3.2(ii) holds for almost classical links. The proof is similar.

Example 3.6. The virtual knot k in Figure 2 appears in [K02]. Its extended group has generators $a, \ldots, h$. Using the relations corresponding to the classical crossings, one finds that $\tilde{\pi}_k$ has group presentation

$$\langle x, a, d \mid axd\bar{a}\bar{x}\bar{a}xa^2xa = dxaxd, \quad dxaxd = xadxa \rangle.$$

The kernel of the homomorphism $\chi : \tilde{\pi}_k \to \langle t \mid \rangle$ mapping $a, d \mapsto 1$ and $x \mapsto t$ is a $\langle t \mid \rangle \cong \mathbb{Z}$-group. The following $\mathbb{Z}$-group presentation can be found using the Reidemeister-Schreier method:

$$\ker(\chi) = \langle a, d \mid ad^t \bar{a}^t \bar{a} (a^2)^t a^{t^2} = da^t d^{t^2},\ da^t d^{t^2} = a^t d^t a^{t^2} \rangle \qquad (3.1).$$

If k were almost classical, then the group presented by (3.1) would be isomorphic to the Alexander group $\mathcal{A}_k$ of k. We compute a presentation of $\mathcal{A}_k$ directly from Figure 2, and find

$$\mathcal{A}_k = \langle c, d \mid cd^t c^{t^2} = dc^t d^{t^2},\ cd^t c^{t^2} = dc^t d^{t^2} \rangle \qquad (3.2).$$

Each presentation (3.1), (3.2) determines a $\mathbb{Z}[t^{\pm 1}]$-module with 2×2-relation matrix. Both matrices have trivial determinant, which is the 0th characteristic polynomial. However, the 1st characteristic polynomial corresponding to (3.1) is 1 while that corresponding to (3.2) is $t^2 - t + 1$. Since these are unequal, the two groups are not isomorphic, and hence k is not almost classical.

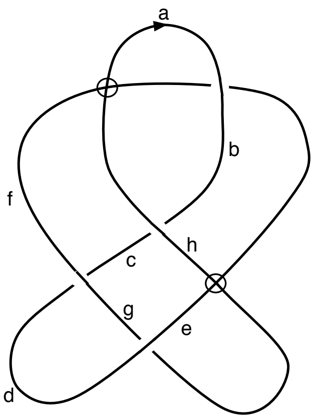

Figure 2: Virtual knot k

.

Remark 3.7. It follows that the knot in Example 3.6 is not classical. Heather Dye has shown us an alternative proof using minimal surface representations, as in [DK05].

The proof of Theorem 4.1 of [SW01] shows that if ℓ is an almost classical oriented virtual link of d components, then a presentation for the Alexander group $\mathcal{A}_\ell$ obtained from any diagram of the link can be converted into a presentation for the extended Alexander group $\tilde{\mathcal{A}}_\ell$ by replacing each occurrence of u_i by $u_i v,\ 1 \le i \le d$. As a consequence, each virtual Alexander polynomial $\Delta_i(u_1, \ldots, u_d, v)$ is a polynomial in the d variables $u_1 v, \ldots, u_d v$.

The arguments are similar for twisted groups and virtual Alexander polynomials. Let $P_1 : \tilde{\pi}_\ell \to \mathbb{Z}^{d+1}$ be the abelianization representation, and let $P_2 : \tilde{\pi}_\ell \to \mathrm{GL}_n(\mathbb{C})$ be a linear representation. Together, P_1 and P_2 determine a representation $P : \tilde{\pi}_\ell \to \mathrm{GL}_n(\mathbb{C}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}, v^{\pm 1}])$. As above, we denote the image of P by Γ and the induced permutation representation of π_ℓ on Γ by ρ. A twisted Alexander matrix A_ρ and associated polynomials $\Delta_{\ell,\rho,i} = \Delta_{\ell,\rho,i}(u_1, \ldots, u_d, v)$ can now be defined as in Example 2.3.

If the diagram for ℓ has N classical crossings, A_ρ is an $Nn \times Nn$-matrix. The greatest common divisor of the determinants of all $(Nn-i) \times (Nn-i)$ minors produces the *ith twisted virtual Alexander polynomial* $\Delta_{\ell,\rho,i}$. As is the case with $\Delta_{\ell,0}$, the twisted polynomial $\Delta_{\ell,\rho,0}$ must vanish if ℓ is classical, and the argument is similar (see [SW01]).

Theorem 3.8. *If ℓ is almost classical, then a presentation for the twisted Alexander group $\mathcal{A}_{\ell,\rho}$ obtained from any diagram of ℓ can be converted into a presentation for the extended Alexander group $\tilde{\mathcal{A}}_{\ell,\rho}$ by replacing each occurrence of u_i by $u_i v,\ 1 \le i \le d$.*

Corollary 3.9. *Assume that ℓ is almost classical. Then each twisted virtual Alexander polynomial $\Delta_{\ell,\rho,i}(u_1, \ldots, u_d, v)$ is a polynomial in the d variables $u_1 v, \ldots, u_d v$.*

Example 3.10. Consider the diagram D for a virtual knot k in Figure 3. An Alexander numbering for D is shown, and hence k is almost classical. The 1st virtual Alexander polynomial $\Delta_1(u, v)$ is easily computed and seen to be $2 - uv$. If k were classical, then its Alexander polynomial would be $2 - t$. However, $2 - t$ is not reciprocal, and hence it cannot be the Alexander polynomial of any classical knot. From this we conclude that k is not classical.

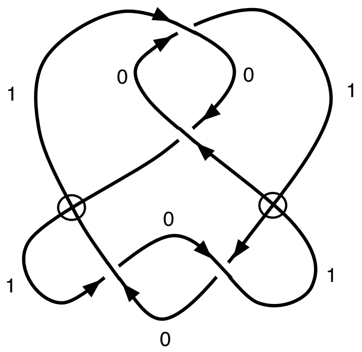

Figure 3: Non-classical knot that is almost classical.

References.

[BZ03] G. Burde and H. Zieschang, Knots, 2nd ed., Walter de Gruyter, Berlin, 2003.

[CF63] R.H. Crowell and R.H. Fox, Introduction to Knot Theory, Graduate Texts in Mathematics, No. 57, Springer-Verlag, New York, 1977.

[C84] R. H. Crowell, The derived group of a permutation representation, *Advances in Math.* **53** (1984), 88–124.

[DK05] H. Dye and L.H. Kauffman, Minimal surface representations of virtual knots and links, arXiv math.GT/0401035

[GTM03] H. Goda, T. Kitano and T. Morifuji, Reidemeister torsion, twisted Alexander polynomial and fibered knots, preprint, arXiv math.GT/0311155

[GPV00] M. Goussarov, M. Polyak and O. Viro, Finite type invariants of classical and virtual knots, *Topology* **39** (2000), 1045–1068.

[JKS94] F. Jaeger, L.H. Kauffman, H. Saleur, The Conway polynomial in $\mathbb{R}^3$ and in thickened surfaces: a new determinant formulation, *J. Combin. Theory Ser. B* **61** (1994), 237–259.

[JW93] B. Jiang and S. Wang, Twisted topological invariants associated with representations, in Topics in Knot Theory (Erzurum, 1992), 211–227, Nato Adv. Sci. Inst. Ser. C., Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, 1993.

[K96] T. Kitano, Twisted Alexander polynomial and Reidemeister torsion, *Pacific J. Math.* **174** (1996), 431–424.

[K98] L.H. Kauffman, A survey of virtual knot theory, in Knots in Hellas '98 (Delphi), 143–202, Ser. Knots Everything **24**, World Sci. Publ, River Edge, NJ, 2000.

[K99] L.H. Kauffman, Virtual knot theory, *European J. Comb.* **20** (1999), 663–690.

[KL99] P. Kirk and C. Livingston: *Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants*, *Topology* **38** (1999), 635–661.

[K02] T. Kishino, Master's Thesis (in Japanese), Osaka City University, 2002.

[L90] X.-S. Lin: *Represenations of knot groups and twisted Alexander polynomials*, preprint, 1990. [Later published in *Acta Math. Sinica*, **17** (2001), 361–380.]

[N63] L.P. Neuwirth, On Stallings fibrations, *Proc. Amer. Math. Soc.,* **59** (1963), 359–365.

[R96] D. Robinson, A Course in Group Theory, Springer-Verlag, New York, 1997.

[S99] J. Sawollek, On Alexander-Conway polynomials for virtual knots and links, preprint, arXiv math.GT/9912173.

[SW01] D.S. Silver and S.G. Williams, Alexander groups and virtual links, *J. Knot Theory and its Ramifications* **10** (2001), 151–160.

[SW03] D.S. Silver and S.G. Williams, Polynomial invariants of virtual links, *J. Knot Theory and its Ramifications* **12** (2003), 987–1000.

[S62] J. Stallings, On fibering certain 3-manifolds, in Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Institute, 1961), 95–100, Prentice-Hall, Englewood Cliffs, NJ.

[T78] W.P. Thurston, The Geometry and Topology of 3-Manifolds, Lecture Notes, Princeton Univ., 1978.

[W94] M. Wada: *Twisted Alexander polynomial for finitely presentable groups*, *Topology* **33** (1994), 241–256.

Address: Department of Mathematics and Statistics, University of South Alabama, Mobile AL 36688 USA
E-mail: silver@jaguar1.usouthal.edu; swilliam@jaguar1.usouthal.edu