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ALEXANDER GROUPS AND VIRTUAL LINKS

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ABSTRACT

The extended Alexander group of an oriented virtual link l of d components is defined. From its abelianization a sequence of polynomial invariants $\Delta_i(u_1, \ldots, u_d, v)$, $i = 0, 1, \ldots$, is obtained. When l is a classical link, Δ_i reduces to the well-known ith Alexander polynomial of the link in the d variables $u_1v, \ldots, u_dv$; in particular, Δ_0 vanishes.

Keywords: virtual knot, Alexander group.

1. Introduction.

Let $l = l_1 \cup \ldots \cup l_d$ be an oriented link in S^3, with complement $X = S^3 - l$. The Alexander module of l is $H_1(\tilde{X}, p^{-1}(*); \mathbf{Z})$, where $p : \tilde{X} \to X$ is the universal abelian cover and $* \in X$. The module is finitely generated over the ring $\mathcal{R}_d = \mathbf{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ of Laurent polynomials in d variables, and from it one derives a sequence $\Delta_i = \Delta_i(u_1, \ldots, u_d)$ of important invariants of l, its Alexander polynomials. (See [12], for example.)

R. Crowell [1] showed that the Alexander module is the abelianization of a certain nonabelian group, the derived group of a permutation representation of the link group on $\mathbf{Z}^d \cong H_1(X; \mathbf{Z})$. In [18] we gave a diagram-based defintion of this group, which we denoted by $\mathcal{A}$ and called the Alexander group of l. We review the definition below. Here we construct an extension $\tilde{\mathcal{A}}$ by $\mathcal{A}$, by splitting, in a sense, the action of each meridianal generator into two parts — one part depending only on the underlying 4-valent graph of the diagram, the other determined by the overcrossing arcs of the diagram. We show that both the Alexander group and the extended Alexander group can be defined equally well for virtual links, a generalization of links introduced in 1996 by L. Kauffman.

The extended Alexander group $\tilde{A}$ admits an action by $\mathbf{Z}^{d+1}$, and its abelianization can be regarded as a module over $\mathcal{R}_{d+1}$. The usual theory of elementary divisors yields a sequence $\Delta_i = \Delta_i(u_1, \ldots, u_d, v)$ of polynomial invariants for the link l. When l is classical (that is, a link in the traditional sense), we see by a change of generators that Δ_i reduces to the usual ith Alexander polynomial of l in the d variables $u_1v, \ldots, u_dv$; in particular, Δ_0 vanishes. Our interest in these polynomials is that they provide new obstructions for a virtual link to be classical. We conclude with several examples that illustrate their effectiveness. In particular, we answer a question of Kauffman [9] by showing that a certain virtual knot with infinite cyclic group and trivial Jones polynomial is in fact nontrivial.

J. Sawollek has also discovered the polynomial Δ_0, using [4]. (See Remark 4.2.)

2. Virtual links.

A **diagram** D for a classical link l is a 4-valent plane graph resulting from a regular projection of l with a *trompe l'oeil* device at each vertex conveying "over and under" information. The neighborhood of a vertex is called a **diagram crossing** (Figure 1 (a)). From such a picture one can reconstruct the link. Two links are isotopic if and only if a diagram for one can be transformed into a diagram for the other by a finite sequence of **Reidemeister moves** (Figure 2, column 1). Links are regarded as the same if they are isotopic. Hence we may think of a link as an equivalence class of diagrams.

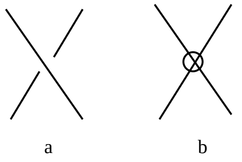

Figure 1. (a) Classical crossing; (b) Virtual crossing

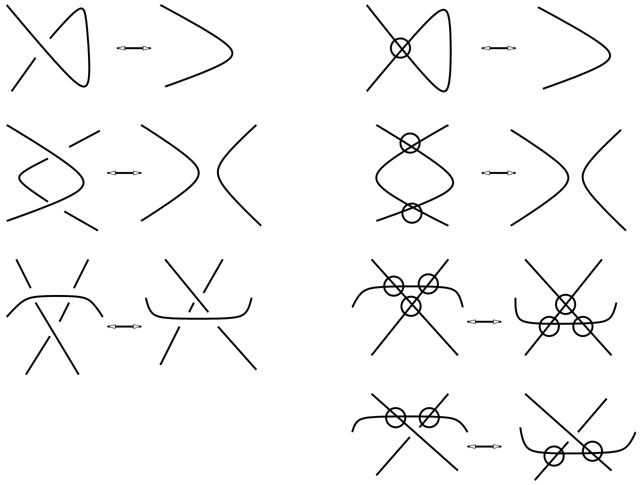

Figure 2. Generalized Reidemeister moves

In 1996 L. Kauffman introduced virtual links [6] by allowing another type of crossing, called a virtual crossing (Figure 1 (b)). A virtual link is an equivalence class of virtual link diagrams under **generalized Reidemeister moves** (Figure 2).

A theorem of M. Goussarov, M. Polyak and O. Viro states that if diagrams of a classical knot are equivalent under generalized Reidemeister moves, then they are equivalent under ordinary Reidemeister moves [2]. Although the theorem is stated only for knots, it can be proved by the same method for links as well. In this sense, the theory of virtual links extends the classical one. The reader who wishes to know more about this rapidly evolving subject is encouraged to consult [7],[8].

All virtual links considered here are oriented; in other words, they are represented by diagrams in which each component has a preferred direction. For such links one considers generalized Reidemeister moves in which arcs have all possible assigned directions. A result of V. Turaev [19] (see also [10, p. 81]) reduces the number of necessary moves.

Kauffman showed that a diagram D of an oriented virtual link l admits a well-defined group G, the **group** of l. As in the classical case, one assigns generators $a, b, c, \ldots$ to the arcs of a diagram D. (An arc of a virtual link diagram is a maximal connected component with virtual crossings ignored.) Each classical crossing determines a relation, as in Figure 3. When l is a classical link, G is isomorphic to fundamental group of the link complement, and the presentation of G that we obtain in this way is the usual Wirtinger presentation. For general virtual links the group G has no such simple topological meaning. (However, see [5] for an interpretation of G as the fundamental group of a singular 3-manifold.)

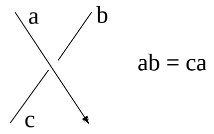

Figure 3. Relation for G associated to a crossing

Kauffman also showed that an invariant for oriented virtual links that extends the well-known Jones polynomial can be computed from a diagram. This is done by means of the Kauffman bracket. The reader is referred to [7] or [8] for details.

In classical knot theory the group is sufficient to detect knotting. More precisely, a knot $k \subset S^3$ is trivial if and only if its group G is infinite cyclic. Recently Kauffman gave the example in Figure 4 of a virtual knot that has infinite cyclic group and Jones polynomial equal to 1. Strongly expecting the knot to be nontrivial, he asked how this could be shown [9]. We will prove that the polynomial $\Delta_0(u, v)$ for k is nonzero, thereby establishing that k is nontrivial. A second, independent proof that k is nontrivial has been found by J. Scott Carter.

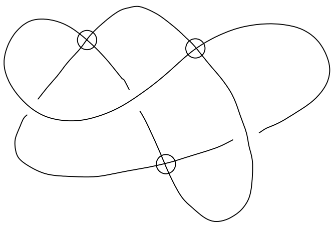

Figure 4. Nontrivial virtual knot with
infinite cyclic group and Jones polynomial 1

3. The Alexander group.

Let $l = l_1 \cup \cdots \cup l_d$ be an oriented virtual link with diagram D. We describe a group $\mathcal{A}$ by a presentation. Families of generators $a_{\mathbf{n}}, b_{\mathbf{n}}, c_{\mathbf{n}}, \ldots$, each indexed by elements $\mathbf{n} \in \mathbf{Z}^d$, correspond to the arcs of D. To each crossing we associate an indexed family of relations of the form $a_{\mathbf{n}} b_{\mathbf{n}+u_i} = c_{\mathbf{n}} a_{\mathbf{n}+u_j}$, $\mathbf{n} \in \mathbf{Z}^d$ (see Figure 5). Here $u_1, \ldots, u_d$ are the standard basis elements of $\mathbf{Z}^d$. Note that the overcrossing arc is contained on the ith component of the link, while the undercrossing arcs are contained on the jth. For notational convenience we denote generator families by $a, b, c, \ldots$, and we denote a family of relations such as $a_{\mathbf{n}} b_{\mathbf{n}+u_i} = c_{\mathbf{n}} a_{\mathbf{n}+u_j}$ by $ab^{u_i} = ca^{u_j}$.

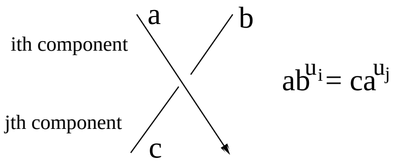

Figure 5. Alexander group relation

When $d = 1$, the group $\mathcal{A}$ has a *présentation* $\mathbf{Z}$*-dynamique finie* in the sense of [3]; for arbitrary d one might say that $\mathcal{A}$ has a finite $\mathbf{Z}^d$-dynamic presentation. However, here we will use the terminology of [13], and say that $\mathcal{A}$ is a finitely presented $\mathbf{Z}^d$-group. In particular, $\mathcal{A}$ admits a $\mathbf{Z}^d$-action by automorphisms, $\mathbf{m} \mapsto \sigma_{\mathbf{m}} \in \mathrm{Aut}(\mathcal{A})$. The automorphism $\sigma_{\mathbf{m}}$ simply adds $\mathbf{m}$ to the index of each generator of $\mathcal{A}$.

We leave to the reader the straightforward task of checking that the Alexander groups associated to equivalent diagrams are isomorphic as $\mathbf{Z}^d$-groups (that is, they admit an isomorphism that commutes with the $\mathbf{Z}^d$-action). Hence we can speak of the **Alexander group** $\mathcal{A}$ associated to a virtual link. The Alexander group of a trivial link of d components is a free $\mathbf{Z}^d$-group of rank d; that is, a group presented by d generator families with no relations.

The group of a virtual link is unchanged when the *forbidden Reidemiester move* in Figure 6 is performed on a diagram. One can easily check that the Alexander group is similarly unchanged by such a move.

The Alexander group of a virtual link is closely related to the commutator subgroup of the link group, as the following proposition shows.

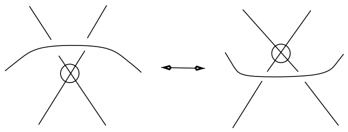

Figure 6. Forbidden Reidemeister move

Proposition 3.1. (cf. [1],[18]) The Alexander group $\mathcal{A}$ of an oriented virtual link l is isomorphic (as an ordinary group) to $[G,G]*F$, the free product of the commutator subgroup $[G,G]$ of the group of l and a free group F of countable rank. If l is a knot, then $[G,G]$ and F have natural $\mathbf{Z}$-actions, and $\mathcal{A}\cong[G,G]*F$ as $\mathbf{Z}$-groups.

Proof. Let X be the canonical 2-complex P with $\pi_1 P\cong G$, corresponding to a Wirtinger presentation. The complex P has a single 0-cell $*$, 1-cells $a,b,c,\ldots$ corresponding to the Wirtinger generators, and 2-cells with boundaries described by the Wirtinger relations. Let $\tilde{P}$ denote the universal abelian cover of P. The 1-cells of $\tilde{P}$ covering $a,b,c,\ldots$ are denoted by $\mathbf{n}\tilde{a},\mathbf{n}\tilde{b},\mathbf{n}\tilde{c},\ldots$, where as usual $\mathbf{n}$ ranges over $\mathbf{Z}^d$. A 2-cell of P corresponding to a Wirtinger relator of the form $aba^{-1}c^{-1}$ as in Figure 3 is covered by 2-cells with boundary $\mathbf{n}\tilde{a}\cdot(\mathbf{n}+u_i)\tilde{b}\cdot[(\mathbf{n}+u_j)\tilde{a}]^{-1}\cdot(\mathbf{n}\tilde{c})^{-1}$, where the product is the usual concatenation of paths.

Collapse the 0-skeleton $\tilde{P}^{(0)}$ of $\tilde{P}$ to a point. The 1-cells of $\tilde{P}$ now represent generators $a_{\mathbf{n}},b_{\mathbf{n}},c_{\mathbf{n}},\ldots$ for $\pi_1(\tilde{P}/\tilde{P}^{(0)})$ while the 2-cells represent the relations in the definition of $\mathcal{A}$. Note that $\tilde{P}/\tilde{P}^{(0)}$ is homotopy equivalent to the union of $\tilde{P}\cup_{\tilde{P}^{(0)}}\mathrm{Cone}(\tilde{P}^{(0)})$, which itself is homotopy equivalent to $\tilde{P}\cup_T[T\cup_{\tilde{P}^{(0)}}\mathrm{Cone}(\tilde{P}^{(0)})]$, where T is a maximal tree of the 1-skeleton of $\tilde{P}$. By an application of the Seifert-van Kampen theorem, $\mathcal{A}$ is isomorphic (as an ordinary group) to the free product $\pi_1(\tilde{P})*F(T)$, where $F(T)$ is a free group on the edges of T. Since $\pi_1(\tilde{P})\cong[G,G]$, the first assertion is proved.

When $d>1$ the free group $F(T)$ is not invariant under the $\mathbf{Z}^d$-action on $\mathcal{A}$. However, in the case of a knot $(d=1)$ we can choose the maximal tree T to correspond to any generator family for $\mathcal{A}$, a $\mathbf{Z}$-invariant subgroup of $\mathcal{A}$. The commutator subgroup $[G,G]$ is also invariant, and hence the second assertion is proved. $\Box$

We esablish a corollary that explains our choice of name for $\mathcal{A}$. The $\mathbf{Z}^d$-action on $\mathcal{A}$ induces an $\mathcal{R}_d$-module structure on the abelianization $\mathcal{A}_{\mathrm{ab}}$.

Corollary 3.2. (cf. [1]) If l is a classical link, then the module $\mathcal{A}_{\mathrm{ab}}$ is isomorphic to $H_1(\tilde{X},p^{-1}(*);\mathbf{Z})$, the Alexander module of l.

Proof. Let $p:\tilde{X}\to X$ be the universal abelian cover of $X=S^3-l$, and let $*\in X$. Then $\pi_1(\tilde{X}/p^{-1}(*))$ is isomorphic to $\pi_1(\tilde{P}/\tilde{P}^{(0)})$ as $\mathbf{Z}^d$-groups. From the proof of Proposition 3.1, $\mathcal{A}_{\mathrm{ab}}$ is isomorphic to $H_1(\tilde{X}/p^{-1}(*);\mathbf{Z})$, which is isomorphic to $H_1(\tilde{X},p^{-1}(*);\mathbf{Z})$. $\Box$

Example 3.3. Consider the diagram of the trefoil knot in Figure 6. The Alexander group has presentation

$$\mathcal{A} \cong \langle a, b, c \mid ab^u = bc^u, bc^u = ca^u, ca^u = ab^u \rangle.$$

(When $d = 1$ it is convenient to write u instead of u_1.) Any one relation is a consequence of the others; we eliminate the third. Moreover, the first relation enables us to express c in terms of a, b. We obtain the presentation,

$$\mathcal{A} \cong \langle a, b \mid b^{u^{-1}} ab^u = a^{u^{-1}} ba^u \rangle.$$

The proof of Proposition 3.1 (last paragraph) implies that $\mathcal{A}$ has an even simpler presentation

$$\mathcal{A} \cong \langle b \mid b = b^{u^{-1}} b^u \rangle * \langle a \mid \rangle.$$

Indeed, this can be seen directly, but we leave it to the reader. The group $\langle b \mid b = b^{u^{-1}} b^u \rangle$ is the commutator subgroup of the trefoil knot group, a free group of rank 2.

4. Extending the Alexander group.

We replace $\mathbf{Z}^d$ by $\mathbf{Z}^{d+1}$, the free abelian group generated by $u_1, \ldots, u_d, v$. Given a diagram D for an oriented virtual link l, we associate a $\mathbf{Z}^{d+1}$-group $\tilde{\mathcal{A}}$ as follows. We regard each overcrossing arc as the union of two arcs joined at the point of overcrossing; if N is the number of classical crossings in D, then the total number of arcs will be at least $2N$. As before, we associate generators $a_{\mathbf{n}}, b_{\mathbf{n}}, c_{\mathbf{n}}, \ldots$, indexed by elements of $\mathbf{Z}^{d+1}$, to arcs. However, at each classical crossing we impose *two* families of relations: $a_{\mathbf{n}} b_{\mathbf{n}+u_i} = c_{\mathbf{n}} d_{\mathbf{n}+u_j}$, $a_{\mathbf{n}+v} = d_{\mathbf{n}}$, if the crossing is negative, and $a_{\mathbf{n}} b_{\mathbf{n}+u_i} = c_{\mathbf{n}} d_{\mathbf{n}+u_j}$, $c_{\mathbf{n}+v} = b_{\mathbf{n}}$ if the crossing is positive (see Figure 7). We adopt the operator notation $ab^{u_i} = cd^{u_j}, a^v = d$ and $ab^{u_i} = cd^{u_j}, c^v = b$, similar to that used in the previous section.

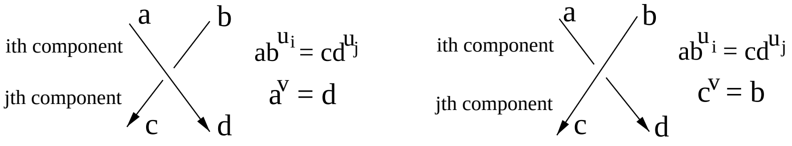

Figure 7. Extended Alexander group relations

Once again we leave it to the reader to check that the extended Alexander groups associated to equivalent diagrams are isomorphic. We speak of the **extended Alexander group** $\tilde{\mathcal{A}}$ associated to an oriented virtual link l. The extended Alexander group of a trivial link of d components is a free $\mathbf{Z}^{d+1}$-group of rank d.

The group $\tilde{\mathcal{A}}$ is an extension by $\mathcal{A}$. For if we adjoin the relations $a^v = a, b^v = b, \ldots$ to the presentation for $\tilde{\mathcal{A}}$, then consequently half of the defining relations for $\tilde{\mathcal{A}}$ identify the members of each generator pair associated to an overcrossing arc.

The remaining relations reduce to the defining relations of $\mathcal{A}$. Hence the quotient $\mathbf{Z}^d$-group obtained is $\mathcal{A}$.

Theorem 4.1. Let l be a classical link of d components. A presentation for $\mathcal{A}$ obtained from a diagram of l can be converted into a presentation for $\tilde{\mathcal{A}}$ by replacing each occurrence of u_i by $u_i v,\ 1 \leq i \leq d$.

Proof. (cf. [16]) Consider a neighborhood of a classical crossing as in Figure 7. If we replace b (resp. d) with b^v (resp. d^v) in the presentation for $\tilde{\mathcal{A}}$, then the families of relations associated to the crossing become $ab^{u_i v} = cd^{u_j v}, a^v = d^v$, if the crossing is negative, and $ab^{u_i v} = cd^{u_j v}, c^v = b^v$, if it is positive. In either case, the two families together are equivalent to the single family that arises in the presentation of $\mathcal{A}$, with u_i, u_j replaced by $u_i v, u_j v$, respectively. It suffices to show that if D has only classical crossings, then such a local change of basis can be extended globally.

Assume that D has only classical crossings. Consider the collection of planar "Seifert circles" that result by smoothing D in the usual way, replacing a small neighborhood of each crossing with two nonintersecting, compatibly oriented arcs (see [12, p. 16]). Each generator for the presentation of $\mathcal{A}$ associated to the diagram corresponds to some circle.

The Seifert circles separate the plane into disjoint regions. We assign the value 0 to the unbounded region; we assign to any bounded region the index ν, defined to be the algebraic intersection number of any arc traveling from that region to the unbounded one with the set of Seifert circles. We replace any generators $a, b, c, \ldots$ with $a^{v^{\nu_1}}, b^{v^{\nu_2}}, c^{v^{\nu_3}}, \ldots$, where ν_i is the index of the region seen to our left as we travel along the Seifert circle corresponding to the ith generator. (cf. [11, p. 247]).

We can easily see that this change of generators accomplishes the desired change in neighborhood of each crossing. □

The abelianization $\tilde{A}_{\rm ab}$ is a finitely generated module over the Noetherian ring $\mathcal{R}_{d+1} = \mathbf{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}, v^{\pm 1}]$. Moreover, since the number of generators is at least the number of relators, we can give an $m \times m$ relation matirx R by adding zero rows, if necessary, so that $\tilde{A}_{\rm ab}$ is isomorphic to $\mathcal{R}_{d+1}^m / R\mathcal{R}_{d+1}^m$. (In practise it is easy to eliminate many of the generators and relators, thereby replacing R with a much smaller square matrix.) The standard theory of elementary divisors yields a sequence $\Delta_0, \Delta_1, \ldots$ of polynomial invariants for l. Each Δ_i is a polynomial in $u_1, \ldots, u_d, v$, defined up to a unit factor. In general, Δ_i is the greatest common divisor of the $(m-i) \times (m-i)$-minors of R. We call Δ_i the ith **virtual Alexander polynomial** of l. Our choice of name is justified by the following immediate corollary of Theorem 4.1.

Remark 4.2. In [15] J. Sawollek also derived the polynomial Δ_0, using an invariant of links in thickened surfaces introduced by F. Jaeger, L. Kauffman and H. Saleur [4]. Sawollek proved that Δ_0 is well defined up to a factor of $\pm v^k$, a fact that can be seen from our approach by considering the effect of generalized

Reidemeister moves on the relation matrix R above. Sawollek showed in [15] that consequently Δ_0 can detect chirality an non-invertiblility for certain virtual links.

Corollary 4.3. If l is an oriented classical link, then Δ_i is equal to the ith Alexander polynomial in the d variables $u_1v, \dots, u_dv$. In particular, Δ_0 vanishes.

Example 4.4. Consider the virtual knot k in Figure 4. As Kauffman remarked, its group is infinite cyclic and its Jones polynomial is equal to 1. Nevertheless, k is nontrivial. We see this by computing the 0th virtual Alexander polynomial, which is

$$\Delta_0(u,v) = u^2v^3 - uv^3 - uv^2 + v^2 - u^2v + uv + u - 1.$$

Since Δ_0 is nonzero, Corollary 4.2 implies that k is nontrivial.

Example 4.5. Figure 7 exhibits a 3-component virtual link l with 0th virtual Alexander polynomial $\Delta_0(u_1, u_2, u_3, v) = 0$ but $\Delta_1(u_1, u_2, u_3, v) = u_2v - u_2 - v + 1$. Since Δ_1 is not a polynomial in u_1v, u_2v, u_3v, the link is not classical. In particular, l is nontrivial.

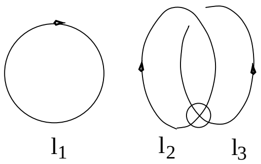

Figure 8. Virtual link l

Example 4.6. In general, a *double* of a virtual knot is constructed from two parallel copies of its diagram by replacing two short parallel segments with a clasp. An *untwisted double* is then obtained by inserting w full twists, where w is sum of the signs of the classical crossings in the original diagram.

Figure 8 displays an untwisted double $D(k)$ of a nontrivial virtual knot k. One can see that $D(k)$ is nontrivial by computing its Jones polynomial, which is

$$V_{D(k)} = t^{-8} - t^{-4} + 1 - 2t^4 + t^8 + 2t^{12} - 2t^{24} + t^{28}.$$

The extended Alexander group of k, which has a presentation

$$\langle a \mid a^v a^u a^{u^2v^2} = a a^{uv^2} a^{u^2v} \rangle,$$

is not the Alexander group of a trivial knot; that is, it is not a free **Z**-group. (We see this, for example, by considering $\Delta_0(u,v)$, which is nonzero). Surprisingly, the extended Alexander group of the untwisted double $D(k)$ is a free **Z**-group of rank 1. This example shows $\tilde{A}$ will not always decide whether a given virtual knot is nontrivial.

That the extended Alexander group of $D(k)$ is a free $\mathbf{Z}$-group of rank 1 can be seen by manipulating presentations. Alternatively, one can verify that a change of generators as in the proof of Theorem 4.1 can be accomplished, and hence the conclusion of the theorem holds for $D(k)$, and indeed for any double of a virtual knot. The idea is that each virtual crossing of k gives rise to 4 virtual crossings of $D(k)$, crossings that can be safely ignored during Seifert smoothing. Consequently, a presentation for the Alexander group of a double $D(k)$ can be converted into a presentation for the extended Alexander group by replacing each occurrence of u by u_1v. Recall now that virtual knot diagrams which differ by a forbidden Reidemeister move (Figure 6) determine isomorphic Alexander groups. From what we have said, if two virtual knot diagrams are equivalent under generalized Reidemeister moves and the forbidden move (such diagrams are said to be *w-equivalent* in [14]) then corresponding untwisted doubled knots have isomorphic extended Alexander groups. The virtual knot k in Figure 9 is in fact w-equivalent to the trivial knot; details can be found in [14].

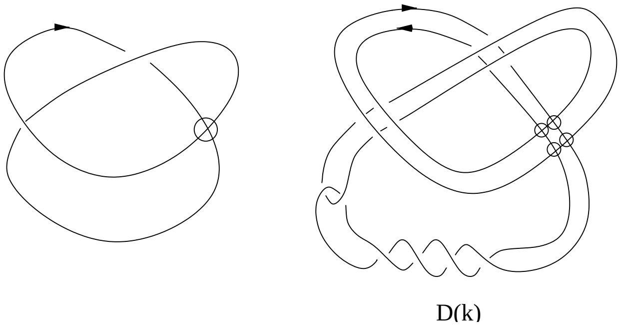

Figure 9. Virtual knot k and untwisted double *D(k)*

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