COLORING SPATIAL GRAPHS

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ABSTRACT

Coloring invariants for spatial graphs are defined, inspired by Fox colorings of knots and links. A new proof of a the nontriviality of Suzuki's n-theta curves is given. Necessary and sufficient conditions for colorings of θ_n-curves are described in terms of an Alexander polynomial defined by Litherland.

Keywords: spatial graph, Fox coloring, theta curve.

1. Introduction.

Definition 1.1. A *(treed) spatial graph* is the image of an embedding $f : \Gamma \to S^3$ of a finite graph $\Gamma = (V_\Gamma, E_\Gamma)$ with a specified maximal tree $\mathcal{T}$. The spatial graph is *oriented* if the edges of $\Gamma - \mathcal{T}$, and hence their images, are directed.

A spatial graph is a pair $\mathcal{G} = (f(\Gamma), f(\mathcal{T}))$. Spatial graphs $\mathcal{G}_1, \mathcal{G}_2$ are regarded as the same if there is an isotopy of S^3 taking $(f_1(\Gamma_1), f_1(\mathcal{T}_1))$ to $(f_2(\Gamma_2), f_2(\mathcal{T}_2))$. For oriented spatial graphs we require that the isotopy preserve directions. A spatial graph $\mathcal{G}$ is *trivial* if $f(\Gamma)$ is contained in a standard 2-sphere in S^3; otherwise it is *nontrivial*. An obvious necessary condition for $\mathcal{G}$ to be trivial is that Γ be planar.

The usual definition of spatial graph does not require that a maximal tree be specified. We include the additional data in our definition for convenience. However, all of the results of this paper can be reformulated without the additional data, if desired, by taking into account all possible maximal trees $\mathcal{T}$ and directions for edges in $\Gamma - \mathcal{T}$. In Section 5 we will further assume that the edges of $\Gamma - \mathcal{T}$ are ordered.

An embedded graph $f(\Gamma)$ can be described by a *diagram*, a regular projection in the plane with crossing information at each double point, just as for a knot. It is well known that two diagrams represent ambiently isotopic embedded graphs if and only if one can be obtained from the other by a finite sequence of *graphical Reidemeister moves*. (See [**7**] or [**21**], for example.) Many of the results of this paper can be proved combinatorially by using such moves. However, since such arguments can be laborious and obscure topological significance, we appeal instead to algebraic topology.

The technique of tricoloring (or more generally n-coloring) a diagram due to R. Fox [**3**], [**4**], [**5**] is an elementary method that distinguishes many pairs of knots and links. The technique is in fact a computation of the first homology group with coefficients in $\mathbf{Z}_n$ of the 2-fold branched cyclic covers of a knot or link. In [**17**] we defined (n, r)-colorings for a an oriented link diagram, for any positive integer r. We showed that $(n, 2)$-colorings are the same as Fox n-colorings (and independent of orientation). Our general technique is a computation of the first homology group with coefficients in $\mathbf{Z}_n$ of the r-fold branched cyclic cover.

Fox n-colorings were introduced in [**6**] for a special class of spatial graphs. We present an extremely general coloring theory for oriented spatial graphs. We provide a homology interpretation for it in terms of suitable branched covering spaces.

2. Coloring spaces for oriented spatial graphs.

Let $\mathcal{G} = (f(\Gamma), f(\mathcal{T}))$ be an oriented spatial graph. We find orientations and nonnegative integer weights w for the edges of Γ in the following manner. Assign each edge in $\Gamma - \mathcal{T}$ weight 1; assign nonnegative integers and directions to the edges of $\mathcal{T}$ so that at any vertex the sum of weights of edges incident and directed in minus the the sum of weights of edges incident and directed out vanishes. If an edge receives weight 0 then its direction can be chosen arbitrarily. With this exception all directions and weights are uniquely determined.

Let D be a diagram for $f(\Gamma)$. Each arc of D represents part of the image of an edge. We decorate the arc with an arrow and an integer indicating the direction and weight of the edge.

Consider a vertex v of D with incident arcs $x_1, \ldots, x_k$, listed in clockwise order. We associate two quantities to each arc in a small neighborhood of v. The *local sign* ϵ_j is $+1$ if the arc is directed in toward v, and it is -1 if the arc is directed out. Assume that w_j is the weight of the arc x_j. The *cumulative weight* is $m_j = \epsilon_1 w_1 + \cdots + \epsilon_{j-1} w_{j-1} + \min\,\{\epsilon_j, 0\}\; w_j$.

Let Σ be a topological group. Although here we consider only finite or countable groups with the discrete topology, arbitrary compact Lie groups may be used as well (cf. [**18**]). Let r be a positive integer. An element $\alpha = (\alpha_1, \ldots, \alpha_r) \in \Sigma^r$ is a *color*. The identity element is the *trivial color*.

Definition 2.1. A (Σ, r)-*coloring* of D is an assignment of colors $\alpha, \beta, \ldots$ to the arcs of D such that:

(C1) any color $\alpha = (\alpha_1, \ldots, \alpha_r)$ assigned to an arc of $\overline{f(\Gamma - \mathcal{T})}$ satisfies

$$\alpha_r \cdots \alpha_1 = 1;$$

(C2) at any crossing as in Figure 1a we have

$$\alpha_j \beta_{j-w_1} = \gamma_j \alpha_{j-w_2}, \quad (0 \le j < r);$$

(C3) at any vertex as in Figure 1b we have

$$\alpha^{\epsilon_1}_{j+m_1} \beta^{\epsilon_2}_{j+m_2} \cdots \delta^{\epsilon_k}_{k+m_k} = 1, \quad (0 \le j < r).$$

All subscripts are taken modulo r.

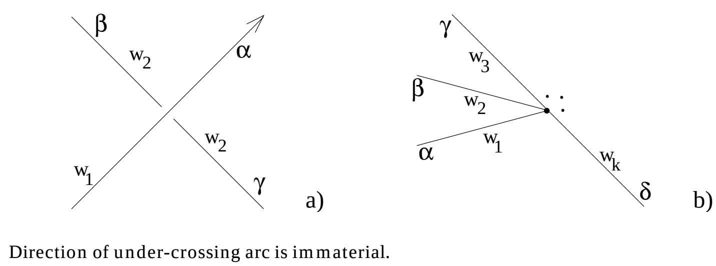


Figure 1. a) crossing; b) vertex

The (Σ, r)-coloring that assigns the trivial color to each arc of D is the *trivial coloring*. The collection of (Σ, r)-colorings of D is a subspace of Σ^{rN}, where N is the number of arcs in D; the linearity of conditions C1-C3 ensures that it is a subgroup when Σ is abelian. We call it the (Σ, r)-*color space* of D, and denote it by $\mathrm{Col}_{\Sigma,r}(D)$.

Remarks. 1. When Σ is abelian, the condition C1 will automatically hold for arcs of $f(\mathcal{T})$. The reader can check this easily by induction, first considering arcs of the pendant edges of $f(\mathcal{T})$ and using C2-C3.

2. When considering (Σ, r)-colorings, we can regard the weights of edges as elements of $\mathbf{Z}/r\mathbf{Z}$.

3. If a diagram D' is obtained from D by simultaneously changing the direction of an edge and the sign of its weight, then the (Σ, r)-color spaces of D and D' are homeomorphic by a map that sends the color $(\alpha_1, \ldots, \alpha_r)$ of the affected arc to $(\alpha^{-1}_{1+w}, \ldots, \alpha^{-1}_{r+w})$, where w is the weight of the arc, leaving the remaining colors unchanged. In particular, when $r = 2$ and each edge weight is 1 mod 2, the coloring rules are independent of orientation.

4. An oriented spatial graph $\mathcal{G}$ can be obtained from an oriented link $l = l_1 \cup \cdots \cup l_d \subset S^3$ by placing a vertex on each component. The maximal tree $\mathcal{T}$ is

0-dimensional, and so every color $\alpha = (\alpha_1, \ldots, \alpha_r)$ must satisfy $\alpha_r \cdots \alpha_1 = 1$. Also, since each arc has weight $+1$, rule C2 can be written

$$\alpha_{j+1}\beta_j = \gamma_{j+1}\alpha_j, \ (0 \leq j < r).$$

Rule C3 implies that colors of arcs separated by a vertex are the same. In this case (Σ, r)-colorings correspond to those already defined in [**17**], [**18**]. In particular, when $r = 2$ and $\Sigma = \mathbf{Z}_n$ we recover the classical Fox n-colorings of the link diagram.

It is well known that Wirtinger's method for presenting the fundamental group of a link complement from a diagram generalizes for embedded graphs. To find such a presentation for $G = \pi_1(S^3 - f(\Gamma))$, one first orients the arcs of a diagram (the choice affects the presentation but not the isomorphism class of G). To each arc corresponds a Wirtinger generator; each crossing then determines a relation, as does each vertex. Any one relation is a consequence of the others, just as for links. Details can be found in [**19**].

For any oriented spatial graph $\mathcal{G}$ there is a well-defined *augmentation homomorphism* $\chi : G \to \mathbf{Z} = \langle t \mid \rangle$, mapping each Wirtinger generator to t raised to the weight of its associated arc. We denote by E the *exterior* of $\mathcal{G}$; that is, the closure of S^3 minus a regular neighborhood of $\mathcal{G}$. By standard algebraic topology E has an r-fold cyclic covering space $\tilde{E}_r$ corresponding to the composition $G \xrightarrow{\chi} \mathbf{Z} \xrightarrow{t \mapsto t} C_r = \langle t \mid t^r = 1 \rangle$. The covering map can be extended to a branched covering map $M_r \to S^3 - f(\mathcal{T})$ much like the "cyclic branched covering without singularity" that is defined in [**14**].

Theorem 2.2. Assume that $\mathcal{G}$ is an oriented spatial graph. Let Σ be a topological group. The color space $\mathrm{Col}_{\Sigma,r}(D)$ is homeomorphic to $\mathrm{Hom}(\pi_1 M_r, \Sigma) \times \Sigma^{r-1}$. If Σ is abelian, $\mathrm{Col}_{\Sigma,r}(D)$ and $\mathrm{Hom}(\pi_1 M_r, \Sigma) \times \Sigma^{r-1}$ are isomorphic topological groups.

Here we assume that $\pi_1 M_r$ has the discrete topology. It follows from the theorem that the color space $\mathrm{Col}_{\Sigma,r}(D)$ is independent of the particular diagram D for $\mathcal{G}$ that we choose. Hence we can refer to it as the (Σ, r)-color space of $\mathcal{G}$.

Proof. (cf. [**16**]) The diagram D for $f(\Gamma)$ yields a Wirtinger presentation

$$G \cong \langle x_1, x_2, \ldots, x_n, \ldots x_{n+m} \mid r_1, \ldots, r_N, r_{N+1}, \ldots, r_{N+V} \rangle,$$

where $x_1, \ldots, x_n$ correspond to arcs of $\overline{f(\Gamma - \mathcal{T})}$. The relators $r_1, \ldots, r_N$ correspond to crossings while the remaining relators arise from the vertices of the graph.

Construct a canonical 2-complex P with $\pi_1 P \cong G$ having a single vertex v, directed 1-cells labeled $x_1, \ldots, x_{n+m}$ and 2-cells $c_1, \ldots, c_{N+V}$ such that each ∂c_j is attached to the 1-skeleton of P according to r_j (see [**9**, Chapter 11]).

We denote the covering space corresponding to the homomorphism $G \to C_r$ by $\tilde{P}$. Each of v, x_i, c_j in P is covered by cells $t^\mu \tilde{v}, t^\mu \tilde{x}_i, t^\mu \tilde{c}_j$, respectively, where

t^μ ranges over elements of C_r. We adopt the convention that the lift $\tilde{x}_i$ of a 1-cell beginning at $t^\mu\tilde{v}$ terminates at $t^{\mu-w_i}\tilde{v}$. For each $i = 1, \ldots, n$, the 1-cells $t^{r-1}\tilde{x}_i$, $t^{r-2}\tilde{x}_i$, $\ldots$, $\tilde{x}_i$ form a closed loop, and we attach a 2-cell $\tilde{d}_i$ along it; these additional cells correspond to the 2-handles one might attach to $\tilde{E}$ in order to construct M_r. We obtain a 2-complex Q such that $\pi_1 Q \cong \pi_1 M_r$.

The action of C_r on $\tilde{P}$ extends to Q. Consider the quotient complex Q/Q^0, where $Q^0 = \{\tilde{v}, t\tilde{v}, \ldots, t^{r-1}\tilde{v}\}$ is the 0-skeleton of Q. Generators of $\pi_1 Q/Q^0$ correspond to the 1-cells of Q; each orbit of a generator under C_r corresponds to an arc of D. A homomorphism $\rho : \pi_1 Q/Q^0 \to \Sigma$ is an assignment of elements of Σ to the generators such that when the values are read around the boundary of any 2-cell, the resulting element of Σ is trivial. Clearly the requirement imposed by the 2-cells $\tilde{d}_i$ is equivalent to the rule C1.

The conditions imposed by the 2-cells coming from Wirtinger crossing relations are equivalent to C2. For example, a closed path in P corresponding to the crossing relation $x_1x_2x_1^{-1}x_3^{-1}$ lifts to closed paths in Q given by $(t^j\tilde{x}_1)(t^{j-w_1}\tilde{x}_2)(t^{j-w_2}\tilde{x}_1)^{-1}$ $(t^j\tilde{x}_3)^{-1}$, where the product is the usual concatenation of paths. Hence colors α, β, γ assigned to the corresponding arcs of D must satisfy $\alpha_j\beta_{j-w_1}\alpha_{j-w_2}^{-1}\gamma_j^{-1}$ or equivalently C2. In a similar way the reader can verify that the conditions that come from Wirtinger vertex relations are equivalent to C3. Thus $\mathrm{Col}_{\Sigma,r}(D)$ is homeomorphic to $\mathrm{Hom}(\pi_1 Q/Q^0, \Sigma)$.

The space Q/Q^0 is homotopy equivalent to $X = Q \cup_{Q^0} CQ^0$, where CQ^0 is a cone on Q^0. We can regard X as the union of Q and $T \cup_{Q^0} CQ^0$, where T is any maximal tree in the 1-skeleton of Q. By the Seifert-van Kampen theorem, $\pi_1 X$ is isomorphic to the free product $\pi_1 Q * F_{r-1}$, where F_{r-1} is a free group of rank $r-1$. Also, $\pi_1 Q \cong \pi_1 M_r$, by the previous paragraph. In any representation $\rho : \pi_1 M_r * F_{r-1} \to \Sigma$ the generators of F_{r-1} can be mapped arbitrarily. Hence $\mathrm{Col}_{\Sigma,r}(D)$ is homeomorphic to $\mathrm{Hom}(\pi_1 M_r, \Sigma) \times \Sigma^{r-1}$.

When Σ is abelian $\mathrm{Col}_{\Sigma,r}(D)$ and $\mathrm{Hom}(\pi_1 M_r, \Sigma) \times \Sigma^{r-1}$ are topological groups. It is clear that in this case the homeomorphism described is an isomorphism. □

Remarks. 1. In [**6**] the authors extend Fox n-colorings for embedded (unoriented) graphs $f(\Gamma)$ such that every vertex has even degree. We observe that regardless of the maximal tree chosen for Γ each weight must be 1 mod 2. Their $\mathbf{Z}_n$-colorings now are seen to be the same as $(\mathbf{Z}_n, 2)$-colorings defined here. (Note that by Remark 3 following Definition 2.1 the colorings are independent of orientation in this case.) Theorem 2.2 implies that they comprise an abelian group isomorphic to $\mathrm{Hom}(H_1 M_2, \mathbf{Z}_n) \oplus \mathbf{Z}_n$. This provides a topological interpretation of the results of [**6**].

2. An examination of the proof of Theorem 2.2 reveals more. Generators of the free factor F_{r-1} correspond to a maximal tree of the 1-skeleton of Q. Let x_i be a Wirtinger generator corresponding to an arc of weight 1. If we assign any values in Σ to $\tilde{x}_i, \ldots, t^{r-2}\tilde{x}_i$, the edges of a particular maximal tree, then the value of $t^{r-1}\tilde{x}_i$ is uniquely determined. The (Σ, r)-colorings of D with these values

comprise a subspace of $\mathrm{Col}_{\Sigma,r}(D)$ that is homeomorphic to the representation space $\mathrm{Hom}(\pi_1 M_r, \Sigma)$. Motivated by this observation we define a *based* (Σ, r)*-coloring* of D to be a (Σ, r)-coloring that assigns the trivial color to some fixed arc of the diagram. We will denote the space of all based (Σ, r)-colorings by $\mathrm{Col}^0_{\Sigma,r}(D)$. This terminology extends that for links in [**17**], [**18**].

Corollary 2.3. (cf. [**18**]) Assume the hypotheses of Theorem 2.2. The space $\mathrm{Col}^0_{\Sigma,r}(D)$ of based (Σ, r)-colorings of D is homeomorphic to $\mathrm{Hom}(\pi_1 M_r, \Sigma)$. When Σ is abelian, they are isomorphic as topological groups.

Important examples of embedded graphs are provided by θ_n-curves. A θ_n*-curve* is an embedded graph $f(\Gamma)$ consisting of two vertices v_-, v_+, joined by n edges. We regard θ_n as an oriented spatial graph by selecting one of the edges as the maximal tree $\mathcal{T}$, and orienting the remaining edges from v_- to v_+. However, when we consider (Σ, n)-colorings, the edge of $\mathcal{T}$ will also be oriented from v_- to v_+, with weight 1. In that case, $\mathrm{Col}_{\Sigma,n}(D)$ is independent of $\mathcal{T}$.

Corollary 2.4. (cf. [**12**, Theorem 5.1]) Let D be a diagram of an n-theta curve. Assume that for some topological group Σ and positive integer r there exists a nontrivial (Σ, r)-coloring of D such that every arc incident to some vertex receives the trivial color. Then the θ_n-curve is nontrivial.

Proof. If $\mathcal{G}$ is a trivial θ_n-curve then by Theorem 2.2 the arcs in a neighborhood of either vertex correspond to a set of generators for $\pi_1(Q/Q^0)$. Any homomorphism from this group to Σ is uniquely determined by its values on the generators. □

3. Examples.

Example 3.1. A diagram for the Borromean links appears in Figure 2. Consider the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, and let $n = 2$. It is not difficult to show that any assignment of colors $\alpha, \beta, \gamma \in Q$ to arcs as indicated in the diagram can be uniquely extended to a coloring of the diagram. Hence there are 8^3 distinct Q-colorings.

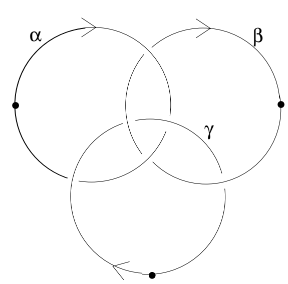


Figure 2. Borromean rings

Example 3.2. A $(\mathbf{Z}_2, 3)$-coloring of S. Kinoshita's θ_3-curve Θ_3 appears below. The spatial graph is known to be nontrivial [**8**]. Figure 3 together with Corollary 2.4 provide a quick proof.

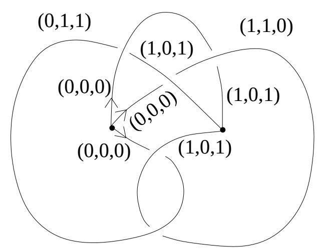
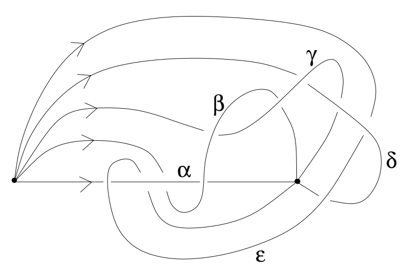


Figure 3. Nontrivial $(\mathbf{Z}_2, 3)$-Coloring of Kinoshita's θ_3-curve

Example 3.3. Kinoshita's θ_3-curve is *Brunnian* in the sense that every proper sub-spatial graph is trivial. In [**20**] S. Suzuki extended Kinoshita's example to a family Θ_n of Brunnian θ_n-curves ($n \geq 3$). A diagram for Θ_5 appears below together with a nontrivial $(\mathrm{A}_5, 5)$-coloring that appeared in [**12**], where A_5 is the alternating group of degree 5. By Corollary 2.4, Θ_5 is nontrivial.



$$\alpha = ((12345), (13542), (14523), (13425), (14352))$$
$$\beta = ((13542), (14523), (13425), (14352), (12345))$$
$$\gamma = ((14523), (13425), (14352), (12345), (13542))$$
$$\delta = ((13425), (14352), (12345), (13542), (14523))$$
$$\epsilon = ((14352), (12345), (13542), (14523), (13425))$$

unlabeled arcs receive trivial color

Figure 4. Nontrivial A_5-Coloring of Suzuki's θ_5-curve

The arguments in [**20**] imply that there is a nontrivial $(\mathbf{Z}_n, 5)$-coloring of Θ_5 for some finite cyclic group $\mathbf{Z}_n$, and initially we tried to find such a coloring. Unfortunately, the proof in [**20**] that Θ_n is nontrivial has a gap when n is congruent to either 1 or 5 mod 6. (The gap occurs on page 21 where it is implicitly assumed that $1-t+t^2$ is noninvertible in the integral group ring $\mathbf{Z}[C_n]$.) We take the opportunity to remark that M. Scharlemann's argument in [**15**] is the first complete proof that Θ_n is nontrivial for all n. (See also Section 4.)

Example 3.4. We display a nontrivial $(\mathbf{Z}_2, 3)$-coloring for Θ_5. We indicate our choice of maximal tree $\mathcal{T}$ by thickening the corresponding arcs on the diagram. (Symmetry implies that the coloring space is independent of $\mathcal{T}$. See Section 4.) Again by Corollary 2.4 we see that Θ_5 is nontrivial.

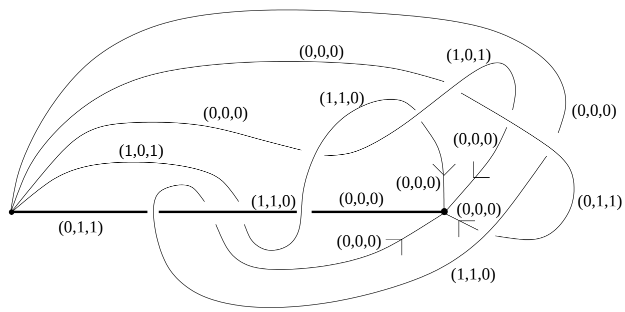


Figure 5. Nontrivial $(\mathbf{Z}_2, 3)$-Coloring of Suzuki's θ_5-curve

4. Symmetric θ_n-curves and a theorem of Livingston.

C. Livingston gave another proof of the nontriviality of Suzuki's θ_n-curves by making explicit use of symmetrical diagrams that represent them. We give an alternate, combinatorial proof of the main result of [**11**], using graph colorings.

Figure 6 displays a symmetrical diagram D for Suzuki's Θ_3 with one vertex at infinity. Also shown (Figure 6b) is a fundamental domain that generates D by an action of C_3. Identifying the two edges of the fundamental region results in a curve (Figure 6c) that we refer to as a *doohicky*.

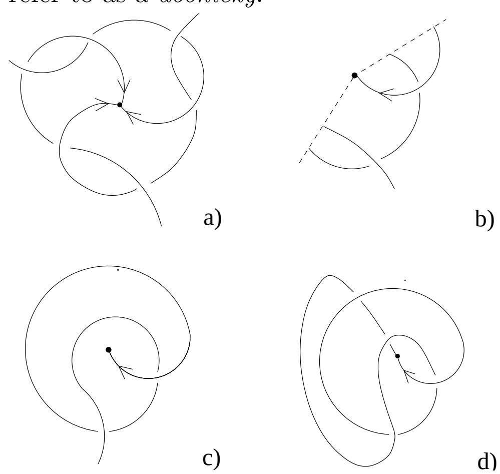


Figure 6. a) symmetric diagram; b) fundamental domain;
c) doohicky; d) quotient knot

Closing a doohicky, connecting its two endpoints by arcs that travel under (Figure 6d) produces a diagram for the *quotient knot* k of the θ_n-curve. A (Σ, n)-coloring of the diagram for k that assigns the trivial color to the initial arc that travels underneath restricts to a coloring of the doohicky and then lifts to a (Σ, n)-coloring of D. The solution to the Smith Conjecture [**1**] ensures that the n-fold branched cyclic cover of k has nontrivial fundamental group, and hence by Corollary 2.3 there exists a nontrivial (Σ, n)-coloring of D assigning the trivial color to each arc meeting the vertex. (We may simply choose Σ to be $\pi_1 M_n$.) Corollary 2.4 implies that the θ_n-curve is nontrivial.

The argument that we have given applies equally well to any θ_n-curve that is *symmetric* in the sense that $f(\Gamma)$ is invariant under a period n homeomorphism of S^3. We have then an alternate proof of

Theorem 4.1. [**11**] Assume that $\mathcal{G}$ is a symmetric θ_n-curve. If the quotient knot k is nontrivial, then $\mathcal{G}$ is nontrivial.

5. Colorings and Litherland's Alexander polynomial for θ_n-curves.

It is well known that a diagram for a knot k has a nonzero based $(\mathbf{Z}_m, 2)$-coloring if and only if the first homology of its 2-fold cyclic branched cover has nontrivial m-torsion. This occurs if and only if m divides $|\Delta_k(-1)|$, the *determinant* of k. Here $\Delta_k(t)$ denotes the Alexander polynomial of the knot. More generally, the diagram has a nonzero based $(\mathbf{Z}_m, r)$-coloring if and only if m divides $|\prod_{\nu=1}^{r-1} \Delta_k(e^{\nu \cdot 2\pi i/r})|$ (see Theorem 8.21 of [**2**], for example).

Let H_{n-1} denote the multiplicative free abelian group on generators $t_1, \ldots, t_{n-1}$. In [**10**] Litherland defined an Alexander module and a polynomial $\Delta_{\mathcal{G}}(t_1, \ldots, t_{n-1})$ in $\mathbf{Z}H_{n-1}$ for an oriented θ_n-curve $\mathcal{G}$, assuming that the edges of its underlying graph are ordered. (He implicitly treated the nth edge as a maximal tree.)

Litherland's definitions are based on Dehn presentations. We give an alternative description using Wirtinger presentations, one that is more convenient for our purpose.

Let $\mathcal{G} = (f(\Gamma), f(\mathcal{T}))$ be an oriented θ_n-curve. Assume that the edges of $\Gamma - \mathcal{T}$ are ordered, say $e_1, \ldots, e_{n-1}$. Let D be a diagram for $f(\Gamma)$. Give new, multiplicative weights to the arcs of D, assigning weight t_i to every arc corresponding to $f(e_i), 1 \leq i < n$, and then extending so that at any vertex the product of weights is 1. Such an extension exists and is unique. Define a $\mathbf{Z}H_{n-1}$-module with generators $a, b, c, \ldots$ corresponding to the arcs of D and relations arising from crossings and vertices. At a crossing as in Figure 1a we require

$$a + w_1 b = c + w_2 a, \tag{C2$'$}$$

where w_1 and w_2 are the (new) weights of the overcrossing and undercrossing arcs, respectively. The *cumulative weight* m_j is defined to be $w_1^{\epsilon_1} w_2^{\epsilon_2} \cdots w_{j-1}^{\epsilon_{j-1}} w_j^{\min\{\epsilon_j, 0\}}$. At a vertex as in Figure 1b we require

$$\epsilon_1 m_1 a + \epsilon_2 m_2 b + \cdots \epsilon_k m_k d = 0. \tag{C3$'$}$$

The *Alexander module* $A_{\mathcal{G}}$ is the quotient module obtained by annihilating all generators corresponding to arcs incident to v_-. Since the spatial graph is connected, it follows that this module has a square relation matrix R. The matrix can be obtained from the relation matrix arising from C2′, C3′ by deleting any row and then deleting columns corresponding to the annihilated generators. The determinant of R is the *Alexander polynomial* $\Delta_{\mathcal{G}}$. As usual it is defined only up to units in $\mathbf{Z}H_{n-1}$.

Clearly $A_{\mathcal{G}}$ and $\Delta_{\mathcal{G}}$ are the usual Alexander module and polynomial, respectively, when D is a knot diagram with a single vertex. One checks that $A_{\mathcal{G}}$ is isomorphic to the module $H_1(\tilde{E}, \partial_- \tilde{E})$ defined in [**10**], where E is the exterior of the theta-curve, $\tilde{E}$ denotes the universal abelian cover of E, and $\partial_- \tilde{E}$ is the preimage of the component of ∂E divided by the meridians and containing v_-.

Previously we defined a (Σ, r)-coloring of a diagram to be based if some specified arc received the trivial color. We say that a (Σ, r)-coloring of a diagram for an oriented θ_n-curve is *totally based* if each arc incident to v_- receives the trivial color. By Corollary 2.4 the existence of a nontrivial totally based (Σ, r)-coloring, for some Σ, r, implies that the spatial graph is nontrivial.

Theorem 5.1. Assume that $\mathcal{G} = (f(\Gamma), f(\mathcal{T}))$ is an oriented θ_n-curve with diagram D and Alexander polynomial $\Delta_{\mathcal{G}}(t_1, \ldots, t_{n-1})$. There exists a nonzero totally based $(\mathbf{Z}_m, 2)$-coloring of D, for some positive integer m, if and only if m divides $|\Delta_{\mathcal{G}}(-1, \ldots, -1)|$. More generally, there exists a nonzero totally based $(\mathbf{Z}_m, r)$-coloring if and only if m divides $|\prod_{\nu=1}^{r-1} \Delta_{\mathcal{G}}(e^{\nu \cdot 2\pi i/r}, \ldots, e^{\nu \cdot 2\pi i/r})|$.

Proof. The following proof, presented for the reader's convenience, is a standard one. It is similar to an argument in [**10**].

The integer matrix obtained from R by specializing $t_1 = \cdots = t_{n-1} = 1$ is a presentation matrix for $H_1(E, \partial_- E)$ (integer coefficients here and throughout). Since the module is trivial, the specialized matrix is nonsingular, and so $|\Delta_{\mathcal{G}}(1, \ldots, 1)| = 1$.

The edge labels of Γ determine a homomorphism from the group $G = \pi_1(S^3 - f(\Gamma))$ to H_{n-1}, each meridian generator mapping to the weight of its corresponding arc in D. Consider the r-fold cyclic covering $\tilde{E}_r$, corresponding to the composition $G \to H_{n-1} \xrightarrow{\pi} C_r$, where π sends each t_i to $t \in C_r = \langle t \mid t^r = 1 \rangle$. Denote the preimage of $\partial_- E_r$ in $\tilde{E}_r$ by $\partial_- \tilde{E}_r$. As in [**13**] we have an exact sequence

$$H_2(E, \partial_- E_r) \to H_1(\tilde{E}_r, \partial_- \tilde{E}_r) \xrightarrow{t-1} H_1(\tilde{E}_r, \partial_- \tilde{E}_r) \to 0,$$

where the last map is multiplication by $t - 1$. Since $\Delta_{\mathcal{G}}(1, \ldots, 1) \neq 0$, the module $H_2(E, \partial_0 E)$ is zero, and hence $t - 1$ is injective.

Generators for $H_1(\tilde{E}_r, \partial_- \tilde{E}_r)$ correspond to those arcs of D not incident to v_-. Let a be a generator, and note that $a + ta + \cdots + t^{r-1}a$ is annihilated by $t - 1$. Since multiplication by $t - 1$ is injective, $a + ta + \cdots + t^{r-1}a = 0$. It is clear that totally based $(\mathbf{Z}_m, r)$-colorings of D correspond to homomorphisms from $H_1(\tilde{E}_r, \partial_- \tilde{E}_r)$ to $\mathbf{Z}_m$. A relation matrix for $H_1(\tilde{E}_r, \partial_- \tilde{E}_r)$ can be obtained from R by replacing each occurrence of t_i with the companion matrix for $t^r - 1$. A matrix calculation (for example, as in [**18**]) completes the argument. □

Example 5.2. A straightforward calculation shows that Kinoshita's θ_3-curve in Figure 3 has Alexander polynomial $\Delta_{\mathcal{G}}(t_1, t_2) = 1 - t_2 - t_1 t_2 - t_1 t_2^2 + t_1 t_2^3 - t_1^2 t_2^4 + t_1^3 t_2^4$. Since $|\prod_{\nu=1}^{2} \Delta_{\mathcal{G}}(e^{\nu \cdot 2\pi i/3}, e^{\nu \cdot 2\pi i/3})| = 4$, Theorem 5.1 implies that the θ_3-curve has nonzero totally based $(\mathbf{Z}_m, 3)$-colorings only for $m = 2, 4$. A nonzero totally based $(\mathbf{Z}_2, 3)$-coloring appears in Figure 3 above. There are exactly 3 such colorings: the assignment of an arbitrary color to some "middle" arc, an arc not incident to a vertex, extends uniquely. Each totally based $(\mathbf{Z}_2, 3)$-coloring is, in fact, the lift of a based $(\mathbf{Z}_2, 3)$-coloring of the quotient trefoil knot (see Section 4).

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References

[**1**] H. Bass and J. Morgan, *The Smith Conjecture*, Academic Press, Orlando, FL, 1984.

[**2**] G. Burde and H. Zieschang, *Knots*, de Gruyter Stud. in Math. **5**, de Gruyter, Berlin (1985).

[**3**] R. H. Crowell and R. H. Fox, *An Introduction to Knot Theory*, Ginn and Co., 1963.

[**4**] R. H. Fox, *A quick trip through knot theory*, in Topology of 3-Manifolds and Related Topics (edited by M. K. Fort), Prentice-Hall, NJ (1961), 120–167.

[**5**] R. H. Fox, *Metacyclic invariants of knots and links*, Canadian J. Math. **22** (1970), 193–201.

[**6**] Y. Ishii and A. Yasuhara, *Color invariant for spatial graphs*, J. Knot Theory and its Ramif. **6** (1997), 319 – 325.

[**7**] L. H. Kauffman, *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. **311** (1989), 697 – 710.

[**8**] S. Kinoshita, *On elementary ideals of polyhedra in the 3-sphere*, Pac. J. Math. **42** (1972), 89 – 98.

[**9**] W. B. R. Lickorish, *An Introduction to Knot Theory*, Springer-Verlag, New York, 1997.

[**10**] R. Litherland, *The Alexander module of a knotted theta-curve*, Math. Proc. Camb. Phil. Soc. **106** (1989), 95 – 106.

[**11**] C. Livingston, *Knotted symmetric graphs*, Proc. Amer. Math. Soc. **123** (1995), 963 – 967.

[**12**] J. M. McAtee, *Spatial graphs and colorings*, Master's thesis, University of South Alabama, 1999 (Advisor: D. S. Silver)

[**13**] J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds (Michigan State University, 1967), Prindle, Weber and Schmidt, Boston, 1968, 115 – 133.

[**14**] H. Naka, *On the n-fold cyclic branched coverings of a spatial θ_n-curve*, Kobe J. Math. **11** (1994), 69 – 87.

[**15**] M. Scharlemann, *Some pictorial remarks on Suzuki's Brunnian graph*, in Topology '90: Proceedings of Ohio State Topology Year (B. Apanasov, W. D. Neumann, A. W. Reid and L. Siebenmann, eds.), Walter de Gruyter, Berlin, 1992, 351 – 354.

[**16**] D. S. Silver and S. G. Williams, *Mahler measure, links and homology growth*, preprint. http://www.mathstat.usouthal.edu/~williams

[**17**] D. S. Silver and S. G. Williams, *Generalized n-colorings of links*, in Knot Theory, Banach Center Publications **42**, Warsaw 1998, 381–394.

[**18**] D. S. Silver and S. G. Williams, *Coloring link diagrams with a continuous palette*, Topology **39** (2000), 1225–1237.

[**19**] J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer-Verlag, New York, 1980.

[**20**] S. Suzuki, *Almost unknotted θ_n-curves in the 3-sphere*, Kobe J. Math. **1** (1984), 19–22.

[**21**] D. N. Yetter, *Category theoretic representations of knotted graphs in S^3*, Advances in Math. **77** (1989), 137–155.