**Determinants of Commuting-Block Matrices**

by

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Let $R$ be a commutative ring, and $\mathrm{Mat}_n(R)$ the ring of $n \times n$ matrices over $R$. We can regard a $k \times k$ matrix $M = (A^{(i,j)})$ over $\mathrm{Mat}_n(R)$ as a *block matrix*, a matrix that has been partitioned into $k^2$ submatrices (*blocks*) over $R$, each of size $n \times n$. When $M$ is regarded in this way, we denote its determinant by $|M|$. We will use the symbol $D(M)$ for the determinant of $M$ viewed as a $k \times k$ matrix over $\mathrm{Mat}_n(R)$. It is important to realize that $D(M)$ is an $n \times n$ matrix.

**Theorem 1**. *Let $R$ be a commutative ring. Assume that $M$ is a $k \times k$ block matrix of blocks $A^{(i,j)} \in Mat_n(R)$ that commute pairwise. Then*

$$|M| = |D(M)| = \left| \sum_{\pi \in S_k} (\mathrm{sgn}\ \pi) A^{(1,\pi(1))} A^{(2,\pi(2))} \cdots A^{(k,\pi(k))} \right|. \tag{1}$$

Here $S_k$ is the symmetric group on $k$ symbols; the summation is the usual one that appears in the definition of determinant. Theorem 1 is well known in the case $k = 2$; the proof is often left as an exercise in linear algebra texts (see [4, page 164], for example). The general result is implicit in [3], but it is not widely known. We present a short, elementary proof using mathematical induction on $k$. We sketch a second proof when the ring $R$ has no zero divisors, a proof that is based on [3] and avoids induction by using the fact that commuting matrices over an algebraically closed field can be simultaneously triangularized.

**Proof**. We use induction on $k$. The case $k = 1$ is evident. We suppose that (1) is true for $k - 1$ and then prove it for $k$. Observe that the following matrix equation holds.

$$\begin{pmatrix} I & O & \cdots & O \\ -A^{(2,1)} & I & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ -A^{(k,1)} & O & \cdots & I \end{pmatrix} \begin{pmatrix} I & O & \cdots & O \\ O & A^{(1,1)} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A^{(1,1)} \end{pmatrix} M = \begin{pmatrix} A^{(1,1)} & * & * & * \\ O & & & \\ \vdots & & N & \\ O & & & \end{pmatrix},$$

where $N$ is a $(k-1) \times (k-1)$ matrix. For the sake of notation, we write this as

$$PQM = R, \tag{2}$$

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where the symbols are defined appropriately. By the multiplicative property of determinants we have $D(PQM) = D(P)D(Q)D(M) = (A^{(1,1)})^{k-1}D(M)$ and $D(R) = A^{(1,1)}D(N)$. Hence we have $(A^{(1,1)})^{k-1}D(M) = A^{(1,1)}D(N)$. Take the determinant of both sides of the last equation. Using $|D(N)| = |N|$, a consequence of the induction hypothesis, together with (2) we find

$$|A^{(1,1)}|^{k-1}|D(M)| = |A^{(1,1)}||D(N)| = |A^{(1,1)}||N| = |R| = |P||Q||M| = |A^{(1,1)}|^{k-1}|M|.$$

If $|A^{(1,1)}|$ is not zero nor a zero divisor, then we can divide the sides by $|A^{(1,1)}|^{k-1}$ to get (1). For the general case, we embed $R$ in the polynomial ring $R[z]$, where $z$ is an indeterminant, and replace $A^{(1,1)}$ by the matrix $zI + A^{(1,1)}$. Since the determinant of $zI + A^{(1,1)}$ is a monic polynomial of degree $n$, and hence is nonzero and not a zero divisor, equation (1) again holds. Substituting $z = 0$ (equivalently, equating constant terms of both sides) yields the desired result. ■

We sketch an alternative proof of Theorem 1 when $R$ has no zero divisors, a proof suggested to us by the referee. It is based on ideas of [3] (see also [1]). If $R$ is a commutative ring with no zero divisors, then we can embed it in its quotient field and then pass to the algebraic closure $F$. We now regard the blocks $A^{(i,j)}$ as operators on the vector space $F^n$, and $M$ as an operator on the tensor product $V = F^n \otimes F^k$. Since the blocks $A^{(i,j)}$ commute pairwise, there exists a basis for $F^n$ with respect to which each $A^{(i,j)}$ is upper triangular (see [2], for example). We form the tensor product of such a basis with the standard one for $F^k$, thereby constructing a new basis for $V$. The change of basis has the effect on $M$ of simultaneously triangularizing each block. Thus it suffices to assume that each block $A^{(i,j)}$ is upper triangular.

The matrix $M$ is permutation-similar to a $k \times k$ block matrix $\tilde{M} = (\tilde{A}_{p,q})$ such that $\tilde{A}_{p,q} = (A^{(i,j)}_{p,q})$ is an $n \times n$ matrix consisting of the $p, q$-entries of the $A^{(i,j)}$. Since each $A^{(i,j)}$ is upper triangular, $\tilde{A}_{p,q} = 0$ whenever $p > q$. Hence $|\tilde{M}| = |\tilde{A}_{1,1}| \cdots |\tilde{A}_{n,n}| = \prod_{r=1}^{n} \sum_{\pi \in S_k} (\text{sgn } \pi) A^{(1,\pi(1))}_{r,r} \cdots A^{(k,\pi(k))}_{r,r}$. Since each $A^{(i,j)}$ is upper triangular, the last product is equal to $\prod_{r=1}^{n} \sum_{\pi \in S_k} (\text{sgn } \pi)(A^{(1,\pi(1))} \cdots A^{(k,\pi(k))})_{r,r}$. But this is equal to $|\sum_{\pi \in S_k} (\text{sgn } \pi) A^{(1,\pi(1))} \ldots A^{(k,\pi(k))}|$. Hence equation (1) holds.

The second proof shows that the commutativity hypotheses of Theorem 1 can be replaced by the weaker condition that the blocks can be simultaneously triangularized. However, *some* hypothesis about the blocks is certainly needed for the conclusion of the theorem to hold, as the reader can see by considering the matrix $M$ below.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We conclude by describing a class of block matrices that satisfy the commutativity hypothesis of Theorem 1. Matrices of this type arose in [5], and were the original motivation for this investigation. Let $p^{(i,j)}(t)$ be polynomials, $1 \le i, j \le k$, and let $N$ be an $n \times n$

matrix. All coefficients are in $R$, which can be taken to be the field of complex numbers, if the reader desires. Since the matrices $p^{(i,j)}(N)$ commute pairwise, the block matrix

$$M = \begin{pmatrix} p^{(1,1)}(N) & \cdots & p^{(1,k)}(N) \\ \vdots & \ddots & \vdots \\ p^{(k,1)}(N) & \cdots & p^{(k,k)}(N) \end{pmatrix}$$

satisfies the hypothesis of Theorem 1. In fact, using the theorem we can say something about the determinant of $M$. Let $p(t)$ be the determinant of

$$\begin{pmatrix} p^{(1,1)}(t) & \cdots & p^{(1,k)}(t) \\ \vdots & \ddots & \vdots \\ p^{(k,1)}(t) & \cdots & p^{(k,k)}(t) \end{pmatrix},$$

and let $\zeta_1, \ldots, \zeta_n$ be the (not necessarily distinct) eigenvalues of $N$. We leave the proof of the following assertion as an exercise.

$$|M| = \prod_{r=1}^{n} p(\zeta_r).$$

## Bibliography


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