PERIODIC LINKS AND AUGMENTED GROUPS

Daniel S. Silver
Susan G. Williams

Received 3 June 1998

ABSTRACT. Given a finitely presented group G and an epimorphism $\chi : G \to \mathbf{Z}$, constraints on the orders of automorphisms $F : G \to G$ such that $\chi \circ F = \chi$ are obtained via symbolic dynamics. The techniques provide new obstructions to periodicity for knots and links.

1. INTRODUCTION. Given a finitely presented group G together with an epimorphism $\chi : G \to \mathbf{Z}$, we study the automorphisms $F : G \to G$ of finite order such that $\chi \circ F = \chi$. Such automorphisms arise frequently in topology. By considering representation shifts introduced in [29] we obtain easily computable constraints on the order of F.

Definition 1.1. An **augmented group** is a pair (G, χ) consisting of a finitely presented group G together with an epimorphism $\chi : G \to \mathbf{Z}$. Augmented groups (G_1, χ_1) and (G_2, χ_2) are regarded as the same if there exists a group isomorphism $h : G_1 \to G_2$ such that $\chi_2 \circ h = \chi_1$.

Augmented groups arise naturally in knot theory. If $l \subset S^3$ is an oriented link, then the associated augmented group consists of the group $G = \pi_1(S^3 - l)$ and the homomorphism $\chi : G \to \mathbf{Z}$ that sends the class of any oriented meridian to 1. It is clear the augmented group associated to a link is well defined.

Definition 1.2. A **symmetry** of an augmented group (G, χ) is an automorphism $F : G \to G$ such that $\chi \circ F = \chi$.

Many links exhibit symmetry. We will see that a rotation leaving an oriented link invariant induces a symmetry of the associated augmented group. Consequently, representation shifts contain information about link symmetries. Sections 4 and 5 are devoted to this topic. In Section 6 we apply our techniques to obtain new results about covering links.

We thank J. Hillman, J. Przytycki, K. Schmidt and R. Strebel for helpful information and suggestions. The authors are grateful to the Erwin Schrödinger Institute for its hospitality.

Both authors partially supported by National Science Foundation Grant DMS-9704399

1991 *Mathematics Subject Classication.* Primary 57Q45; Secondary 54H20, 20E06, 20F05.

2. AUGMENTATION SUBGROUPS.

Let (G,χ) be an augmented group. Throughout this paper the kernel of χ will be called the **augmentation subgroup** of (G,χ), and it will be denoted by K.

Lemma 2.1. Assume that F is a finite-order symmetry of (G,χ). If the center of K is torsion free, then the restriction $F|_K$ has the same order as F.

Proof. Assume that F is a nontrivial symmetry of order q. Choose $x \in G$ such that $\chi(x) = 1$. Suppose that $(F|_K)^j = \mathrm{id}$, for some j, $0 < j < q$. The element $F^j(x)$, which must be different from x, can be written as $F^j(x) = xw$, for some nontrivial $w \in K$. Since $x = (F^j)^{q/j}(x) = xw^{q/j}$, the element w has finite order. Moreover, for any $a \in K$, we have $a = F^j(a) = F^j(x^{-1} \cdot xax^{-1} \cdot x) = w^{-1}x^{-1} \cdot xax^{-1} \cdot xw = w^{-1}aw$. Hence w is a nontrivial element of finite order in the center of K, a contradiction. ∎

Defintion 2.2. If F is a symmetry of (G,χ), then the **quotient augmented group** is the pair $(\bar{G},\bar{\chi})$, where $\bar{G} = G/\langle\langle F(g)g^{-1}, \quad (g \in G)\rangle\rangle$ and $\bar{\chi} : \bar{G} \to \mathbf{Z}$ is the induced homomorphism. (Here $\langle\langle \ \cdot \ \rangle\rangle$ denotes normal closure.) The augmentation subgroup of $(\bar{G},\bar{\chi})$ is denoted by $\bar{K}$.

If F is a symmetry of (G,χ), then it is often an easy matter to find presentations for the augmentation subgroups K and $\bar{K}$. Choose a distinguished element $x \in G$ such that $\chi(x) = 1$. Beginning with any finite presentation of G, we can obtain a new presentation of the form

$$\langle x, a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle,$$

where $a_1, \ldots, a_n$ are elements of K. It follows from the Reidemeister-Schreier method for presenting subgroups [16] that K has presentation

$$\langle a_{1,j}, \ldots, a_{n,j} \mid r_{1,j} \ldots, r_{m,j}, \quad j \in \mathbf{Z} \rangle.$$

Here $a_{i,j}$ denotes the generator $x^{-j}a_ix^j$, while $r_{i,j}$ denotes the relator $x^{-j}r_ix^j$, for each $j \in \mathbf{Z}$.

Lemma 2.3. Assume F is a symmetry of (G,χ).

(i.) The kernel $\bar{K}$ has presentation

$$\bar{K} = \langle a_{1,j}, \ldots, a_{n,j} \mid r_{1,j}, \ldots, r_{m,j}, \ w_j, \ F(a_{1,j})a_{1,j}^{-1}, \ldots, F(a_{n,j})a_{n,j}^{-1}, \quad j \in \mathbf{Z} \rangle,$$

where $w = x^{-1}F(x)$ and w_j denotes the element $x^{-j}wx^j$, for each j;

(ii.) If F has finite order q, then $w_j F(w_j) \cdots F^{q-1}(w_j) = 1$, for each j.

Proof. Since $\bar{G}$ has presentation $\langle x, a_1, \ldots, a_n \mid r_1, \ldots, r_m,\ w,\ F(a_1)a_1^{-1}, \ldots, F(a_n)a_n^{-1}\rangle$, the first assertion follows immediately from the Reidemeister-Schreier method. The second assertion follows directly from the equation $F^q(x) = x$ ∎

Example 2.4. Consider the group $G = \langle x, a \mid x^{-1}a^2x = a^2\rangle$. Let $\chi : G \to \mathbf{Z}$ be the homomorphism determined by $x \mapsto 1$ and $a \mapsto 0$. The kernel K has presentation

$$\langle a_j \mid a_{j+1}^2 = a_j^2, \quad j \in \mathbf{Z}\rangle.$$

The augmented group (G, χ) has an order 2 symmetry F mapping $x \mapsto x^{-1}a^{-1}xax$ and $a \mapsto x^{-1}axa^{-1}x^{-1}ax$. The group $\bar{G}$ of the quotient augmented group $(\bar{G}, \bar{\chi})$ is isomorphic to $\langle x, a \mid xa = ax\rangle$, a free abelian group of rank 2. The kernel $\bar{K}$ is isomorphic to $\langle a_j \mid a_{j+1} = a_j, \quad j \in \mathbf{Z}\rangle$, an infinite cyclic group generated by a_0.

The group G is, in fact, the group of the 2-component link 4_1^2 oriented as in Figure 8a. The automorphism F is induced by the obvious rotation of order 2. We will return to this example in section 4.

3. REPRESENTATION SHIFTS AND MARKOV SUBGROUPS.

Let $\mathcal{G} = (G, \chi, x)$ be an augmented group (G, χ) together with a distinguished element $x \in G$ such that $\chi(x) = 1$. Following [28] we will call such a triple an **augmented group system** (AGS). Two augmented group systems (G_1, χ_1, x_1) and (G_2, χ_2, x_2) are regarded as the same if there exists a group isomorphism $h : G_1 \to G_2$ such that $\chi_2 \circ h = \chi_1$ and $h(x_1) = x_2$.

Endow K with the discrete topology, and let Σ be a topological group. (Whenever Σ is finite we will assume that it has the discrete topology.) The **representation shift** $\Phi_\Sigma(\mathcal{G})$ (also denoted by Φ_Σ) is the set $\mathrm{Hom}(K, \Sigma)$ of representations $\rho : K \to \Sigma$ together with the **shift map** $\sigma_x : \Phi_\Sigma \to \Phi_\Sigma$ defined by

$$\sigma_x\rho(a) = \rho(x^{-1}ax), \quad \forall a \in K.$$

We give Φ_Σ the compact-open topology. When Σ is a metric space, the topology of Φ_Σ is determined by the basic sets

$$\mathcal{N}_{a_1,\ldots,a_s,\epsilon}(\rho) = \{\rho' \mid d(\rho'(a_i), \rho(a_i)) < \epsilon, \quad i = 1, \ldots, s\},$$

where $\rho \in \Phi_\Sigma, a_1, \ldots, a_s \in K$, and $\epsilon > 0$. It is easy to see that σ_x is a homeomorphism. Hence (Φ_Σ, σ_x) is a topological dynamical system; that is, a compact topological space together with a homeomorphism.

Dynamical systems (Φ_i, σ_i) $(i = 1, 2)$ are **topologically conjugate** if there exists a homeomorphism $h : \Phi_1 \to \Phi_2$ such that $h \circ \sigma_1 = \sigma_2 \circ h$. In such a case, h induces a bijection between the fixed point sets Fix $\sigma_1^r = \{\rho \mid \sigma_1^r = \rho\}$ and Fix $\sigma_2^r = \{\rho \mid \sigma_2^r \rho = \rho\}$, for each $r \geq 1$. In other words, topologically conjugate dynamical systems have the same number of periodic points for every period.

It is easy to see that the representation shift $\Phi_\Sigma(\mathcal{G})$ is well defined up to topological conjugacy. Indeed, if $\mathcal{G}_1 = (G_1, \chi_1, x_1)$ and $\mathcal{G}_2 = (G_2, \chi_2, x_2)$ are equivalent AGS's, then any isomorphism $h : G_2 \to G_2$ such that $\chi_2 \circ h = \chi_1$ and $h(x_1) = x_2$ induces a topological conjagacy $h : \Phi_\Sigma(\mathcal{G}_1) \to \Phi_\Sigma(\mathcal{G}_2)$ such that $h \circ \sigma_{x_1} = \sigma_{x_2} \circ h$.

Let Γ be a finite directed graph with edge set E. We give E the discrete topology, and $E^{\mathbf{Z}}$ the product topology. The space of bi-infinite paths (e_j) in Γ is a closed subset X_Γ of $E^{\mathbf{Z}}$ that is invariant under the **shift map** σ given by $\sigma(e_j) = (e'_j)$, where $e'_j = e_{j+1}$. The pair (X_Γ, σ) is the **shift of finite type** described by Γ. If Γ consists of a single vertex and n edges (self-loops), then (X_Γ, σ) is called the **full n-shift**.

If E is a group, then $E^{\mathbf{Z}}$ is a group under coordinate-wise mulitiplication. A closed shift-invariant subgroup is called a **Markov subgroup** of $E^{\mathbf{Z}}$.

In [29] we proved that if Σ is a finite group, then the representation shift (Φ_Σ, σ_x) is a shift of finite type described by a graph that can be constructed algorithmically from a presentation for G. Details of the construction can be read in [29], [30], [31] or [32]. However, we summarize the main points in the examples that follow.

It follows from results of [29] that for a compact *abelian* group Σ, the representation shift Φ_Σ corresponding to any augmented group (G, χ) is a Markov subgroup of $(\Sigma^n)^{\mathbf{Z}}$, for some $n \geq 1$. When the abelianization G_{ab} is infinite cyclic, Φ_Σ is finite.

An **automorphism** of a Markov subgroup Φ is an automorphism $\mathcal{F}$ of the underlying compact group such that $\sigma \circ \mathcal{F} = \mathcal{F} \circ \sigma$. For any positive integer r, we let n_r denote the number of elements of Φ with least period r.

Lemma 3.1. If $\mathcal{F}$ is an automorphism of a finite Markov subgroup (Φ, σ) such that $\mathcal{F}$ has finite order q, then

$$q \leq \max_{r \geq 1} \text{ g.c.d.}(q, r) \cdot n_r / r. \qquad (3.1)$$

Note that (3.1) provides a bound on q. Since g.c.d$(q, r)/r$ never exceeds 1, the value of q is no greater than $\max_{r \geq 1} n_r$.

Proof. Since Φ is finite, the graph Γ consists of finitely many disjoint cycles, and the elements of Φ are in one-to-one correspondance with the vertices of Γ. The automorphism

$\mathcal{F}$ induces a mapping of Γ. If v is any vertex, then the length of the orbit $v, \mathcal{F}v, \ldots$ is a divisor of q.

Let r be any positive integer. Consider a vertex v in any cycle of length r. The quantity n_r/r is the number of cycles in Γ of length r. Let a be the smallest positive integer such that $\mathcal{F}^a v$ is in the same cycle as v. Then $a \leq n_r/r$. We can write $q = ab$, for some positive integer b. The automorphism $\mathcal{F}^a$ has order b, and it takes the cycle of Γ containing v to itself. It follows that b divides r, and hence $b \leq \text{g.c.d.}(q, r)$. Then $q \leq a \cdot \text{g.c.d.}(q, r)$. Thus $q \leq \text{g.c.d.}(q, r) \cdot n_r/r$. ∎

Let $\mathcal{G} = (G, \chi, x)$ be an AGS. If Σ is a finite abelian group, then it is straightforward to see that the representation shift (Φ_Σ, σ_x) is independent of the choice of distinguished generator x. In other words, (Φ_Σ, σ_x) is an invariant of (G, χ).

We will assume throughout the remainder of the section that Σ is abelian. This implies that the induced map $F^* : \Phi_\Sigma \to \Phi_\Sigma$, described by $\rho \mapsto \rho \circ F$, is shift-commuting in the sense that $\sigma_x \circ F^* = F^* \circ \sigma_x$. (To see this, recall that $F(x) = xw$, for some $w \in K$.) We define Φ_Σ^F to be the fixed-point set of F^*; that is,

$$\Phi_\Sigma^F = \{\rho \in \Phi_\Sigma \mid F^*\rho = \rho\}.$$

Since Φ_Σ^F is closed and F^* commutes with σ_x, it follows that $(\Phi_\Sigma^F, \sigma_x)$ is a Markov subgroup.

We will denote the natural projection homomorphism $\pi : G \to \bar{G}$, mapping $g \mapsto \bar{g}$, by π. Then π restricts to a map from K to $\bar{K}$, and it induces a group homomorphism $\pi^* : \bar{\Phi}_\Sigma \to \Phi_\Sigma^F$ defined by $\bar{\rho} \mapsto \bar{\rho} \circ \pi$, where $\bar{\Phi}_\Sigma$ denotes the representation shift of $(\bar{G}, \bar{\chi})$. It is easy to check that π^* is shift commuting; that is, $\sigma_x \circ \pi^* = \pi^* \circ \sigma_{\bar{x}}$.

Proposition 3.2. (Cf. Theorem III.2.4 of [1].) Assume that F is an order-q symmetry of (G, χ). If Σ is any finite abelian group with order $|\Sigma|$ relatively prime to q, then π^* is a Markov isomorphism between $(\bar{\Phi}_\Sigma, \sigma_{\bar{x}})$ and $(\Phi_\Sigma^F, \sigma_x)$.

Proof. Define $\mu^* : \Phi_\Sigma^F \to \bar{\Phi}_\Sigma$ by $\mu^*\rho(\bar{a}) = \sum_{j=0}^{q-1} \rho(F^j(a))$, for all $\bar{a} \in \bar{K}$. The map μ^* is well defined, and using Lemma 2.3 it is not difficult to see that μ^* is a homomorphism. The compositions $\pi^* \circ \mu^*$ and $\mu^* \circ \pi^*$ are both multiplication by q on their respective groups. Since $|\Sigma|$ is relatively prime to q, both μ^* and π^* are group isomorphisms. Also, $\sigma_{\bar{x}} \circ \mu^* = \mu^* \circ \sigma_x$. Hence $\bar{\Phi}_\Sigma$ and Φ_Σ^F are isomorphic as Markov subgroups. ∎

Proposition 3.2 enables us to regard $\bar{\Phi}_\Sigma$ as a subshift of Φ_Σ; that is, a closed shift-invariant subspace. Let $\bar{n}_r$ (respectively, n_r) denote the number of points of least period r in $\bar{\Phi}_\Sigma$ (respectively, Φ_Σ). These numbers are always finite. In fact, they can be computed from the traces of powers of the adjacency matrix of the graph describing the shift.

Corollary 3.3. For every nonnegative integer r,

(i.) $n_r \geq \bar{n}_r$;
(ii.) $n_r \equiv \bar{n}_r$ (mod p), if q is a prime power p^s.

Proof. The first assertion is immediate. In order to prove the second assertion, recall that F^* commutes with σ_x. Consequently, F^* permutes the set Fix σ_x^r. Since $n_r - \bar{n}_r$ is equal to the cardinality of a set on which F^* acts without fixed points, the second statement of Corollary 3.3 follows. ■

The infinite cyclic group C_∞ acts on Φ_Σ via σ_x while the cyclic group C_q of order q acts via F^*. The next two propositions examine the second action.

Proposition 3.4. Assume that F is an order-q symmetry of (G, χ), and assume that Σ is any finite abelian group. If q is a prime, then $\Phi_\Sigma/\bar{\Phi}_\Sigma$ has the structure of a $\mathbf{Z}[\zeta]$-module, where ζ is a primitive qth root of unity..

Proof. We adapt an argument of Naik [22] (cf. proof of Proposition 2.4). Let t be a generator of C_q and let $s = 1 + t + \cdots + t^{q-1} \in \mathbf{Z}[C_q]$. Since q is a prime, $\mathbf{Z}[C_q]/\langle s \rangle \cong \mathbf{Z}[\zeta]$. Since $sa \in \Phi_\Sigma^F$ for all $a \in \Phi_\Sigma$, the result follows. ■

Proposition 3.4 imposes severe restrictions on the structure of $\Phi_\Sigma/\Phi_\Sigma^F$. For example, the following is a consequence of Theorem 2.5 of [3]. It also follows from arguments in [22].

Corollary 3.5. Assume the hypotheses of Proposition 3.4. Let $\Sigma = \mathbf{Z}/p^l$, where p is a prime different from q, and let r be the order of p in the group of units of $\mathbf{Z}/q$. Then

$$\Phi_\Sigma/\Phi_\Sigma^F \cong (\mathbf{Z}/p)^{a_1 r} \oplus (\mathbf{Z}/p^2)^{a_2 r} \oplus \cdots \oplus (\mathbf{Z}/p^{l_1})^{a_{l_1} r},$$

for some nonnegative integers $a_1, \ldots, a_{l-1}$.

A result somewhat stronger than that of Corollary 3.5 is possible in certain situtations. See Theorem 5 of [23].

Example 3.6. Consider the group $G = \langle x, a, b, c \mid x^{-1}ax = b, x^{-1}bx = c, x^{-1}cx = a \rangle$. Let $\chi : G \to \mathbf{Z}$ be the homomorphism determined by $x \mapsto 1$ and $a, b, c \mapsto 0$. The kernel K has presentation

$$K = \langle a_j, b_j, c_j \mid a_{j+1} = b_j,\ b_{j+1} = c_j,\ c_{j+1} = a_j, \quad j \in \mathbf{Z} \rangle.$$

The augmented group (G, χ) has an order-3 symmetry F mapping $x \mapsto x$ and $a \mapsto b \mapsto c \mapsto a$. The group $\bar{G}$ of the quotient augmented group $(\bar{G}, \bar{\chi})$ is isomorphic to $\langle x, a \mid x^{-1}ax = a \rangle$, and the kernel $\bar{K}$ is isomorphic to $\langle a_j \mid a_{j+1} = a_j \rangle$.

Consider the finite abelian group $\Sigma = \mathbf{Z}/2$. We think of the elements of $\Phi_{\mathbf{Z}/2}$ as functions ρ from the generator set $\{a_j, b_j, c_j, \ \ j \in \mathbf{Z}\}$ to $\mathbf{Z}/2$ satisfying the equations

$$\rho(a_{j+1}) = \rho(b_j), \rho(b_{j+1}) = \rho(c_j), \text{and } \rho(c_{j+1}) = \rho(a_j) \text{ for all } j \in \mathbf{Z}. \qquad (*)$$

Such functions are encoded in the finite directed graph Γ in Figure 1. The graph was constucted from vertices $\rho_0 = (\alpha, \beta, \gamma)$, where α, β and γ are arbitrary elements of $\mathbf{Z}/2$. Two vertices $\rho_0 = (\alpha, \beta, \gamma)$ and $\rho_0' = (\alpha', \beta', \gamma')$ are connected by an edge $\rho_0\rho_0'$ if $\alpha' = \beta$, $\beta' = \gamma$ and $\gamma' = \alpha$. A vertex $\rho_0 = (\alpha, \beta, \gamma)$ in Γ should be regarded as a function from the subset $\{a_0, b_0, c_0\}$ of generators to $\mathbf{Z}/2$. We can extend ρ_0 to a function $\rho_0\rho_0' : \{a_0, a_1, b_0, b_1, c_0, c_1\} \to \mathbf{Z}/2$, mapping a_1, b_1 and c_1 to α_1, β_1 and γ_1, respectively, and satisfying the equations (*), if and only if ρ_0 and ρ_0' are connected by an edge in Γ. By similar reasoning, the set Φ_Σ of representations $\rho : K \to \mathbf{Z}/2$ are in one-to-one correspondence with the bi-infinite paths of Γ. Notice too that the shift σ_x corresponds to shifting in Γ: if ρ maps a_j, b_j, c_j to $\alpha_j, \beta_j, \gamma_j$, respectively, then $\sigma_x\rho$ maps a_j, b_j, c_j to $\alpha_{j+1}, \beta_{j+1}, \gamma_{j+1}$. Hence the path in Γ corresponding to $\sigma_x\rho$ is that of ρ shifted by one edge.

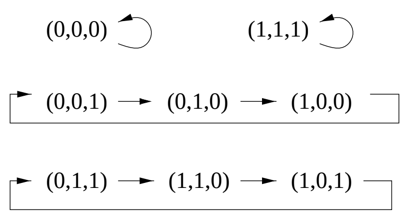


Figure 1: Graph Γ describing $\Phi_{\mathbf{Z}/2}$

The same construction applied to the quotient augmented group reveals that there are exactly two representations from $\bar{K}$ to $\mathbf{Z}/2$. One representation maps each generator a_j to 0, while the other maps each generator to 1. Alternatively, we can apply Proposition 3.2 and think of these representations as the two fixed points of $\Phi_{\mathbf{Z}/2}$. Corollary 3.3 predicts that $n_r - \bar{n}_r$ is divisible by 3, for every nonnegative integer r. This is indeed the situation, for $n_1 - \bar{n}_1 = 0$ and $n_3 - \bar{n}_3 = 6$.

The quotient of $\Phi_{\mathbf{Z}/2}$ by $\bar{\Phi}_{\mathbf{Z}/2}$ is easily seen to be a $\mathbf{Z}/2$-module isomorphic to $(\mathbf{Z}/2)^2$.

Example 3.7. Consider the augmented group in Example 2.4. Clearly, the representation shift $\Phi_{\mathbf{Z}/2}$ is topologically conjugate to the full shift on $\mathbf{Z}/2$. Any function from the set $\{a_j, \ \ j \in \mathbf{Z}\}$ of generators to $\mathbf{Z}/2$ determines a representation $\rho : K \to \mathbf{Z}$. Notice that $\Phi_{\mathbf{Z}/2}$ is uncountable. By contrast, $\bar{\Phi}_{\mathbf{Z}/2}$ consists of two elements. The quotient of $\Phi_{\mathbf{Z}/2}$ by $\bar{\Phi}_{\mathbf{Z}/2}$ is uncountable. This example shows that the hypothesis of Corollary 3.5 that p be different from q is necessary for the conclusion.

In [37] H. Trotter considered conditions on a group K ensuring that if an automorphism of K has some finite period, then the induced automorphism of K_{ab} has the same order. Recall that the lower central series of a group K is inductively defined by $K_1 = K,\ K_{i+1} = [K, K_i]$, for all positive integers i.

Lemma 3.8. [37] Assume that K is a group satisfying:

(T1) K_i/K_{i+1} is torsion-free for all i;

(T2) $\bigcap_{i=1}^{\infty} K_i$ consists of the identity of K.

If F is an automorphism of K with order q, then the induced automorphism of K_{ab} also has order q.

Every knot group satisfies condition T1 (see [35] page 153). Condition T2 (residual nilpotence of K) is harder to detect. Nevertheless, examples are common. C. Gordon observed in [10] that if k is any knot obtained from fibered or 2-bridge knots under the operations of cabling and/or connect summing, then the commutator subgroup K satisfies T2.

The lower central series of K can be extended using transfinite induction. If β is a limit ordinal, then K_β is defined to be the intersection of all K_α, with $\alpha < \beta$. If Trotter's conditions are replaced by the following weaker conditions, then the conclusion of Lemma 3.8 follows from a straightforward adaptation of the arguments in [37]. (See Appendix I.)

(T1*) $K_\alpha/K_{\alpha+1}$ is torsion-free for all ordinals α;

(T2*) $K_\alpha = 1$ for some ordinal α.

It is possible that the commutator subgroup of any alternating knot is transfinitely nilpotent; that is, it satisfies condition T2* (cf. Question 5.2 of [10]).

If (G, χ) is any augmented group, then K_{ab} is a finitely generated module over $\Lambda = \mathbf{Z}[t, t^{-1}]$. (The action of t corresponds to conjugation in G by x, a distinguished element such that $\chi(x) = 1$.) We define the "Alexander polynomial" $\Delta_{K_{\mathrm{ab}}}(t)$ in the usual way. Generally, any finitely generated module A over a Noetherian U.F.D. R can be described by an $m \times n$ presentation matrix M representing a homomorphism $R^m \to R^n \to A \to 0$. Without loss of generality we can assume that $m \geq n$. The ideal $E^k \subset R$ generated by determinants of the $(n-k) \times (n-k)$-submatrices of M is independent of the presentation matrix [38]. The Alexander polynomial $\Delta_A(t)$ is a generator of the minimal principal ideal containing E^0. It is defined up to a unit.

Theorem 3.9. Suppose that (G, χ) is an augmented group such that K satisfies conditions T1*, T2* and $\Delta_{K_{\mathrm{ab}}}(t) = c_d t^d + \cdots + c_1 t + c_0$ has positive degree. If (G, χ) has a symmetry of order q, then for every prime p not dividing c_0, c_d or q, inequality (3.1) is satisfied, where n_r is the number of points of least period r in $\Phi_{\mathbf{Z}/p}$.

Conditions T1* and T2* can possibly be replaced by some weaker condition. However, they cannot be merely eliminated, as the following example shows.

Example 3.10. Consider the Baumslag-Solitar group $G = \langle x, a \mid x^{-1}a^2x = a^4\rangle$ and the homomorphism χ that maps $x \mapsto 1$ and $a \mapsto 0$. According to Lemma 3.7 of [4], the assignment $x \mapsto x^{r+1}ax^{-r}$ and $a \mapsto a$ determines an automorphism γ_r of order 2^r, for each $r \geq 1$. Hence (G, χ) has symmetries of order 2^r, for all ≥ 1. On the other hand, the associated representation shift $\Phi_{\mathbf{Z}/3}$ comprises 3 representations, one of them fixed by the shift map, the remaining two permuted. The bound provided by (3.1) is equal to 3. Theorem 3.8 does not apply in this example because K does not satisfy T1*, the nonzero element $a_0 - 2a_1$, for example, being 2-torsion. The group K also fails to satisfy T2*. In fact, since $H_2(K_{\rm ab}; \mathbf{Z}) = 0$ by a Mayer-Vietoris argument, the conclusion $K_2 = K_3 = \cdots$ follows easily. We are indebted to R. Strebel for this observation.

Our proof of Theorem 3.9 requires the following.

Lemma 3.11. Assume that A is a finitely generated Λ-module, and let p be a prime. If $pA = A$, then $\Delta_A(t) \equiv 1 \pmod{p}$.

Proof. R. Crowell proved Lemma 3.11 in [6] under the additional assumption that A has a square presentation matrix. His proof can be easily modified in the general case. For the reader's convenience we will do this.

Assume that A has an $m \times n$ presentation matrix $M = (m_{i,j}(t))$, with $m \geq n$. By this we mean that there is an exact sequence $\Lambda^m \overset{d_2}{\to} \Lambda^n \overset{d_1}{\to} A \to 0$, where Λ^m and Λ^n are free Λ-modules with bases $\{y_i\}$ and $\{x_j\}$, respectively, and for each $1 \leq i \leq m$,

$$d_2 y_i = \sum_{j=1}^{n} m_{i,j}(t) x_j.$$

By hypothesis there exist $f_{k,j}(t) \in \Lambda$ such that

$$d_1 x_k = p d_1 \sum_{j=1}^{n} f_{k,j}(t) x_j,$$

for each $k = 1, \dots n$. Then

$$x_k - p \sum_{j=1}^{n} f_{k,j}(t) x_j = d_2 \sum_{h=1}^{m} g_{k,h}(t) y_h,$$

for some $g_{k,h}(t) \in \Lambda$. We can write

$$\sum_{j=1}^{n} (\delta_{k,j} - p f_{k,j}(t)) x_j = \sum_{h=1}^{m} \sum_{j=1}^{n} g_{k,h}(t) m_{h,j}(t) x_j.$$

(Here $\delta_{k,j}$ is the Kronecker delta.) In matrix form:

$$I - p(f_{k,j}(t)) = (g_{k,h}(t))M.$$

Since M has a left inverse as a matrix in $(\mathbf{Z}/p)[t, t^{-1}]$, it follows that the mod-p rank of M is equal to n. Thus the determinant of some $(n \times n)$-submatrix of M is a unit in $(\mathbf{Z}/p)[t, t^{-1}]$. Hence $\Delta_A(t) \equiv 1 \pmod{p}$. ∎

Lemma 3.12. Suppose that A is a finitely generated Λ-module such that A is $\mathbf{Z}$-torsion free. Assume that $\Delta_A(t)$ has positive degree, and let p be a prime that divides neither its first nor its last coefficient. If F is an automorphism of order q such that g.c.d.$(p, q) = 1$, then the order of the induced automorphism $F \otimes I$ of $A \otimes \mathbf{Z}/p$ is also q.

Proof. The order of $F \otimes I$ must divide q. By passing to a suitable power of F, it suffices to show that $F \otimes I$ is not trivial. We will assume that $F \otimes I$ is trivial and derive a contradiction.

The hypothesis that A is $\mathbf{Z}$-torsion free implies that the kernel of the natural projection $A \to A \otimes \mathbf{Z}/p,\ a \mapsto a \otimes 1$, is the submodule pA. Therefore, we can write $F \otimes I = I + pG$, for some nonzero homomorphism G. Corresponding to the short exact sequence

$$0 \to G(A) \to A \to A/G(A) \to 0$$

is a factorization $\Delta_A = \Delta' \Delta''$, where Δ' and Δ'' are the Alexander polynomials of $G(A)$ and $A/G(A)$, respectively [15]. Since $G(A)$ is nonzero and torsion free, the finitely generated $\mathbf{Q}[t, t^{-1}]$-module $G(A) \otimes \mathbf{Q}$ is nonzero. The Alexander polynomial of the latter module, chosen to have integer cooefficients, must have postive degree and it must divide Δ'. Hence Δ' has positive degree.

The assumption that F has order q implies that

$$I = (I + pG)^q = I + qpG + \binom{q}{2} p^2 G^2 + \cdots + p^q G^q.$$

Hence

$$p[qG + \binom{q}{2} pG^2 + \cdots + p^{q-1} G^q] = 0.$$

Since A is $\mathbf{Z}$-torsion free,

$$qG + \binom{q}{2} pG^2 + \cdots + p^{q-1} G^q = 0.$$

If α and β are integers such that $\alpha p + \beta q = 1$, then

$$G = \alpha pG + \beta qG = \beta[-\binom{q}{2} pG^2 - \cdots - p^{q-1} G^q] + \alpha pG$$

It follows that $pG(A) = G(A)$. Lemma 3.10 now implies that $\Delta' \equiv 1 \pmod p$, implying that the degree of $\Delta_A \pmod p$ is less than that of Δ_A. However, this contradicts our choice of p. ∎

Proof of Theorem 3.9. Let F be a symmetry of (G, χ) of order q. By Lemmas 3.8 and 3.12 the induced automorphism $F \otimes I$ of the vector space $K_{\text{ab}} \otimes \mathbf{Z}/p$ has order q. Regard $\text{Hom}(K, \mathbf{Z}/p) \cong \text{Hom}(K_{\text{ab}} \otimes \mathbf{Z}/p, \mathbf{Z}/p)$ as the dual space. Since F^* is the adjoint of $F \otimes I$, the desired conclusion follows from Lemma 3.1. ∎

Example 3.13. Consider the augmented group (G, χ) associated to the trefoil knot. The group G has a presentation

$$\langle x, y, z \mid xy = yz, yz = zx, zx = xy \rangle,$$

from which we immediately see an order-3 symmetry of (G, χ), mapping x, y and z cyclically. The group G has another presentation, one with generators $x_1, y_1, x_2, y_2, x_3, y_3$ and relators $x_1 y_1 = y_2 x_1,\ y_1 x_1 = x_2 y_1,\ x_2 y_3 = y_1 x_2,\ y_2 x_3 = x_1 y_2,\ x_2 y_2 = x_3 x_2,\ y_2 x_2 = y_3 y_2$ that reveals an order-2 symmetry of (G, χ), interchanging x_i with y_i. These presentations can be obtained from Wirtinger's method applied to symmetric diagrams for the trefoil.

We will use shift techniques to conclude that (G, χ) admits no symmetry of prime-power order other than 2 and 3. (Of course, this can be shown by other techniques. Splitting field arguments such as those in [37] can be used.)

Application of the Reidemeister-Schreier method shows that the commutator subgroup K has the well-known presentation

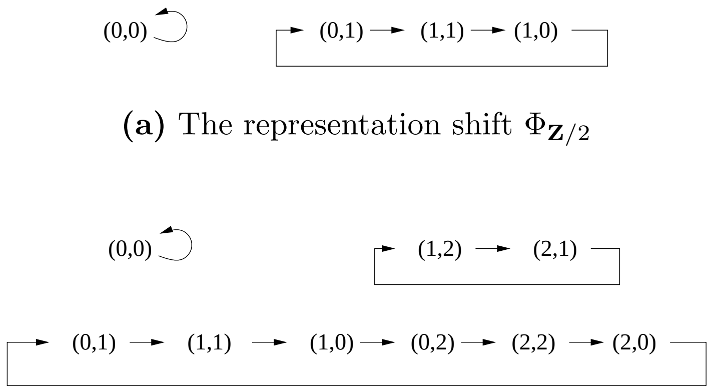
$$\langle a_j \mid a_{j+2} = a_{j+1} a_j^{-1}, \quad j \in \mathbf{Z} \rangle.$$

(See Example 2.3 of [31] for details.) Since K is evidently a free group on generators a_0 and a_1, we can apply Theorem 3.9.

Considerations such as those in the previous examples show that the representation shift $\Phi_{\mathbf{Z}/2}$ consists of a single fixed point (the trivial representation) and three points of least period 3 (see Figure 2a). From this we find that (G, χ) can have symmetries of prime-power orders 3 and 2^l, $l \geq 1$ only.

In order to complete the analysis, we consider the representation shift $\Phi_{\mathbf{Z}/3}$ with graph shown in Figure 2b. Applying Theorem 3.9 once more, (G, χ) has no symmetry of order 2^l, if $l \geq 2$.

By a theorem of S. Meskin [17] K has no automorphism of period 6, and consequently neither does G. Hence (G, χ) has symmetries of order 2 and 3 only. We remark that K_{ab} admits an automorphism of period 6. In fact, the composition of the order 2 and order 3 automorphisms of G described above will induce such a map. Hence our shift techniques of this section will not always decide whether or not an augmented group admits a symmetry of a a specified order.



(a) The representation shift $\Phi_{\mathbf{Z}/2}$

(b) The representation shift $\Phi_{\mathbf{Z}/3}$

Figure 2: Representation shifts for the trefoil

4. PERIODIC KNOTS AND LINKS.

Let l be an *oriented* link. Recall that the associated group system (G, χ) consists of the fundamental group $G = \pi_1(S^3 - l)$ together with the homomorphism χ that maps the class of each oriented meridian to 1. Given a diagram for the link, it is an easy matter to find a presentation for the kernel K of χ using the following procedure that combines the Wirtinger and Reidemeister-Schreier methods. We denote the meridinal generators of G corresponding to the arcs of the diagram as $x, ax, bx, cx, \ldots$, were $a, b, c, \ldots$ are elements of K. We label the first arc with the identity element of K, and we label the remaining arcs by the corresponding elements $a, b, c, \ldots$ The kernel K has generators $a_j, b_j, c_j, \ldots$ with relations corresponding to the crossings of the diagram, as in Figure 3. (Conjugates of the identity are of course trivial and can be ignored.) As is true for Wirtinger presentations of links, any single relator is a consequence of the remaining relators and can be omitted.

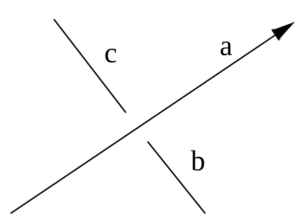


Figure 3: Relation $a_{j+1}c_j = b_{j+1}a_j$

The resulting presentation can usually be simplified greatly. In fact, if we place a "height function" on the diagram, then the generators corresponding to arcs containing the local maxima generate K, and all other generators can easily be written in terms of them.

Definition 4.1. An oriented link l has **period q** if there exists a rotation f of S^3 of order q about an unknotted circle (the axis of rotation) disjoint from l, leaving the link invariant.

The axis of the rotation is equal to the fixed point set $\mathrm{Fix}(f)$. We do not require that f map each component of l to itself. However, we do require that orientations be preserved. Adapting the notation of [21], we refer to (l, f) as a **periodic link of period q**. The quotient space S^3/f is again the 3-sphere, and it contains a **quotient link** denoted $\bar{l}$.

There is a large body of literature about periodic knots, a subject introduced by R. Fox [7]. The interested reader can find an excellent surveys in [12],[13] and [14]. Many of the papers describe obstructions for given periods. One of the most basic results is due to K. Murasugi [18]. It states that the Alexander polynomial $\Delta(t)$ of k factors as

$$\Delta(t) = \bar{\Delta}(t) \prod_{i=1}^{q-1} D(t, \xi^i) \qquad (4.1)$$

where $\bar{\Delta}(t)$ is the Alexander polynomial of the quotient knot $\bar{k}$, $D(t, u)$ is the 2-variable Alexander polynomial of the link $k \cup \mathrm{Fix}(f)$, q is the order of f and ξ is a primitive qth root of unity. Murasugi proved that for any prime power p^r dividing q, the equation (4.1) reduces to the following congruence.

$$\Delta(t) \equiv \bar{\Delta}(t)^{p^r}(1 + t + \cdots + t^{\lambda-1})^{p^r-1} \pmod{p} \qquad (4.2)$$

where λ is the linking number of k and $\mathrm{Fix}(f)$.

A generalization of Equation (4.1) was proven for periodic links by M. Sakuma [26].

Proposition 4.2. If (l, f) is a periodic link of period q, then the rotation f induces an order q symmetry F of the augmented group (G, χ) of the link. The augmented group of $\bar{l}$ is the quotient augmented group $(\bar{G}, \bar{\chi})$.

Proof. Choose a diagram for l that displays its symmetry under f. Consider the Wirtinger presentation for G with basepoint chosen on the axis of rotation, and let F be the induced automorphism. Clearly the order of F must divide q. In fact, the order of F must be q, since otherwise some iterate f^j, $0 < j < q$, would induce the trivial automorphism on fundamental group, thereby contradicting results of P. E. Conner [5] (see Proposition 3.1 (i) of [21]).

The second statement follows easily from the symmetry of the Wirtinger presentation. Details of the presentation can be found in [21] or [2]. ∎

The following observation of [18] (see page 171), explicitly stated in [11], will be very useful to us.

Lemma 4.3. Let (k, f) be a periodic knot of prime-power period $q = p^s$. Assume that the Alexander polynomial Δ of k is not a product of nontrivial Alexander polynomials. If Δ is not congruent to 1 mod p, then the quotient knot $\bar{k}$ has trivial Alexander polynomial.

In [21] Murasugi excluded periods for knots by studying the action of a rotation f on the set of equivalence classes of representations of G into a finite permutation group. Our techniques are similar in spririt to Murasugi's. However, by examining the representations of K instead of the knot group, we can often use smaller target groups.

Example 4.4. In [21] Murasugi considered the knot $k = 10_{137}$, which has Alexander polynomial $t^4 - 6t^3 + 11t^2 - 6t + 1$. If k had period 5, then by Lemma 4.3 the quotient knot would have trivial Alexander polynomial. Since the results of [18] alone do not exclude the possibility, Murasugi studied the representations of the knot group G. By combining the observation that G has a unique representation onto the alternating group A_4, up to conjugation in S_4, with a result of R. Hartley he could prove that k does not have period 5. We show that k does not have period 5 by considering the action of f on representations of K into $\mathbf{Z}/2$.

Working from a Wirtinger presentation for k, one can verify the well-known fact that K_{ab} is a cyclic $\mathbf{Z}[t, t^{-1}]$-module with the following abelian group presentation

$$\langle a_j \mid a_{j+4} - 6a_{j+3} + 11a_{j+2} - 6a_{j+1} + a_j, \quad j \in \mathbf{Z} \rangle.$$

From this a directed graph Γ describing the Markov subgroup $\Phi_{\mathbf{Z}/2}$ can be described. It consists of 4-tuples $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbf{Z}/2)^4$; there is an edge from $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ whenever $a_4 - 6\alpha_3 + 11\alpha_2 - 6\alpha_1 + \alpha_0 \equiv 0 \pmod 2$ or equivalently whenever $a_4 \equiv a_0 + a_2 \pmod 2$. From the graph, which appears in Figure 4, we see at once that $\Phi_{\mathbf{Z}/2}$ has exactly three points of least period 3.

Suppose that (k, f) is a periodic knot of period 5. Since the Alexander polynomial of the quotient knot must be trivial, $\bar{K}_{\text{ab}}$ is trivial. Proposition 3.2 implies that F^* fixes only the trivial representation. In particular, it freely permutes the three points of least period 3. Since that is impossible, $k = 10_{137}$ does not have period 5.

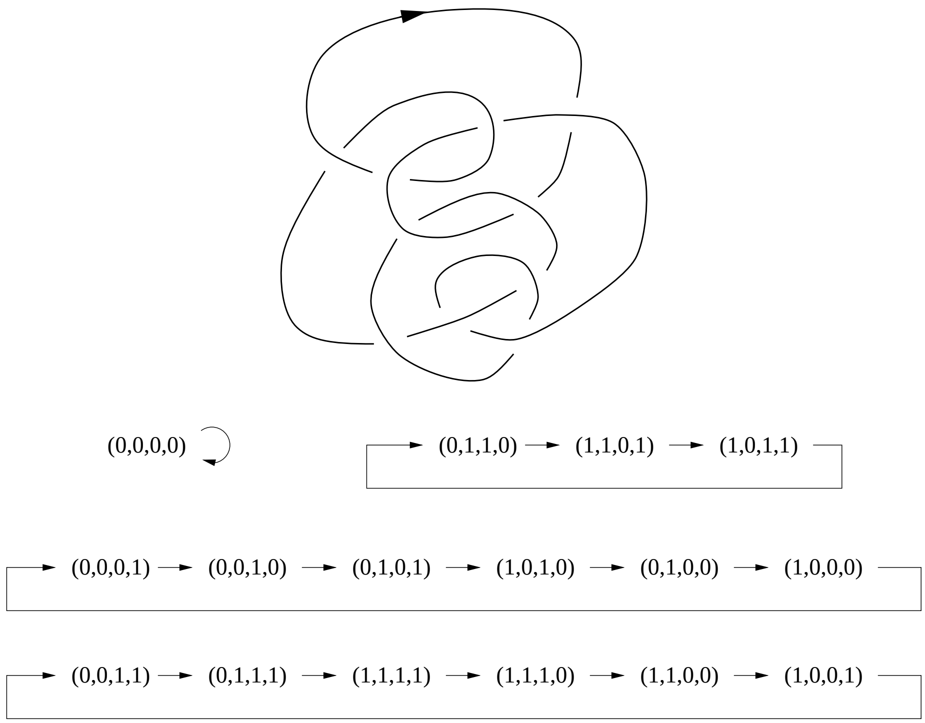


Figure 4: The knot $k = 10_{137}$ and shift $\Phi_{\mathbf{Z}/2}$

Example 4.5. P. Traczyk's paper [36] was motivated by the fact that for many knots the possiblity of period 3 is difficult to exclude using only Murasugi's results in [18]. Traczyk developed techniques based on the FLYPMOTH polynomial to decide the matter for many 10-crossing knots. In particular, he showed that the knots $10_{10}, 10_{69}$ and 10_{164} do not have period 3. Here we use shift techniques to decide the matter.

We consider $k = 10_{10}$, which has Alexander polynomial $3t^4 - 11t^3 + 17t^2 - 11t + 3$. As in the previous example, $K_{\rm ab}$ is a cyclic $\mathbf{Z}[t, t^{-1}]$-module. A directed graph describing the corresponding Markov subgroup $\Phi_{\mathbf{Z}/p}$, for any prime p, consists of vertices which are 4-tuples $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbf{Z}/p)^4$; there is an edge from $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, where $3\alpha_4 - 11\alpha_3 + 17\alpha_2 - 11\alpha_1 + 3\alpha_0 \equiv 0 \pmod p$.

If we choose $p = 5$, then Γ contains precisely two cycles of length 2. Equivalently, $\Phi_{\mathbf{Z}/5}$ contains exactly five elements of period 2 (including the fixed point). This can be seen, of course, by direct calculation. Alternatively, one can make use of the fact that the submodule of $\Phi_{\mathbf{Z}/5}$ consisting of all elements of period 2 is isomorphic to $H_1(\hat{X}_2; \mathbf{Z}/5)$, the homology group with $\mathbf{Z}/5$ coefficients of the 2-fold cyclic cover branched over the knot (see [31]). Since the first (and only) torison number of k is $T = 45$ [2], the homology group is isomorphic to $\mathbf{Z}/45 \otimes \mathbf{Z}/5 \cong \mathbf{Z}/5$.

The Alexander polynomial of k is irreducible over $\mathbf{Z}$ and not congruent to 1 modulo 3. Suppose that (k, f) is a periodic knot of period 3. The quotient knot must have trivial Alexander polynomial by Lemma 4.3. Proposition 3.2 implies that F^* fixes only the trivial element of $\Phi_{\mathbf{Z}/5}$. That is impossible since F^* must leave the set of period 2 elements invariant. Hence k does not have period 3. In fact, our argument shows more: the group G of the knot k admits no automorphism of order 3.

Very similar arguments apply to the knots 10_{69} and 10_{164}. The torsion number of the first knot is $T = 87$, and one uses $p = 29$. In the second case, the torsion number is $T = 45$, and $p = 5$ will again work.

The same techniques used in Example 4.5 can exclude periods other than 3. Consideration of period 2 elements in $\Phi_{\mathbf{Z}/13}$ will show, for example, that the knot 10_{105} does not have period 7. This was proven by S. Naik in [22]. Indeed, the techniques of [22] can also be used in Examples 4.4 and 4.5. However, the following example demonstrates that Markov subgroup techniques have different capabilities.

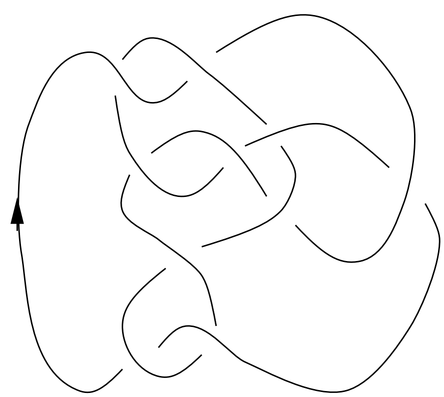


Figure 5: The knot $k = 10_{105}$

Example 4.6. Consider the knot $k = 8_{10}$, which has Alexander polynomial $(t^2 - t + 1)^3$. Using the results of [18] it is not difficult to see that if k has period 3, then the quotient knot must be a trefoil. The techniques of [21] enabled Murasugi to prove that no quotient of k can be a trefoil, and hence k does not have period 3. We show this using Markov subgroups.

Using Wirtinger's algorithm together with the Reidemeister-Schreier method (see top of section) we obtain the following presentation of the abelianization K_{ab}:

$$\langle a_j, b_j \mid a_{j+4} = -a_j + 2a_{j+1} - 3a_{j+2} + 2a_{j+3},\ b_{j+2} = a_j - a_{j+1} - b_j + b_{j+1},\ j \in \mathbf{Z} \rangle$$

A diagram for k with generator labels appears in Figure 6.

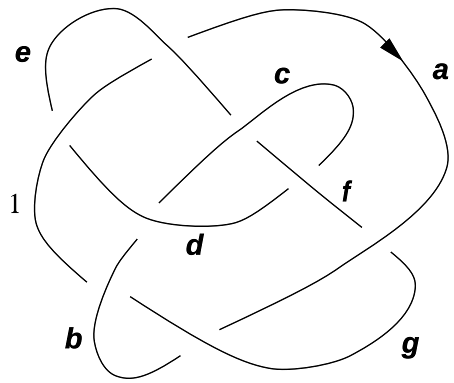


Figure 6: The knot $k = 8_{10}$

Modulo 2 the relations become:

$$a_{j+4} = a_j + a_{j+2}, \quad b_{j+2} = a_j + a_{j+1} + b_j + b_{j+1}$$

The representation shift $\Phi_{\mathbf{Z}/2}$ can be described by a directed graph Γ with vertex set $(\mathbf{Z}/2)^6$. There is an edge from $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1)$ to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2)$ whenever

$$\alpha_4 \equiv \alpha_0 + \alpha_2, \quad \beta_2 \equiv \alpha_0 + \alpha_1 + \beta_0 + \beta_1 \pmod 2$$

A portion of the graph appears in Figure 7.

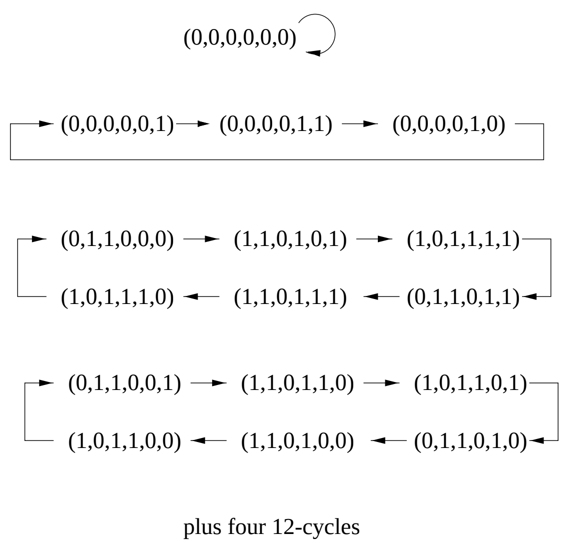


Figure 7: Shift $\Phi_{\mathbf{Z}/2}$ for $k = 8_{10}$

Assume that (k, f) is a periodic knot with period 3. The rotation f induces an order 3 automorphism F of K_{ab} by Lemma 3.11. Hence the induced map F^* of the representation shift $\Phi_{\mathbf{Z}/2}$ has order 3. Recall from Example 3.12 that the representation shift $\bar{\Phi}_{\mathbf{Z}/2}$

associated to the group system of the trefoil consists of a single fixed point and three points of least period 3. Proposition 3.2 implies that these four representations are the only fixed points in $\Phi_{\mathbf{Z}/2}$. Consequently, each of the two 6-cycles in Γ must be left invariant but not fixed. In particular, the vertex $v_1 = (0,1,1,0,0,0)$ in the first 6-cycle must be mapped to either $(1,0,1,1,1,1)$ or $(1,1,0,1,1,1)$, while the vertex $v_2 = (0,1,1,0,0,1)$ must be mapped to $(1,0,1,1,0,1)$ or $(1,1,0,1,0,0)$. Since $v_1 + v_2$ is the fixed vertex $(0,0,0,0,0,1)$ in the 3-cycle of Γ, the sum of the images of v_1 and v_2 must be equal to $(0,0,0,0,0,1)$. Evidently this is impossible. Thus $k = 8_{10}$ does not have period 3.

Example 4.7. Consider the 2-component link 4_1^2. If the components are oriented as in Figure 8a with Wirtinger generators indicated, then the group G of the link is given in Example 2.4. On the other hand, if we orient the components as in Figure 8b, then the link, which we denote by l, has group $G = \langle x, a \mid x^{-3}ax^3 \cdot x^{-1}ax = x^{-2}ax^2 \cdot a\rangle$. The augmentation subgroup $K = \langle a_j \mid a_{j+3} = a_{j+2}a_ja_{j+1}^{-1},\ \ j \in \mathbf{Z}\rangle$ is free on a_0, a_1, a_2.

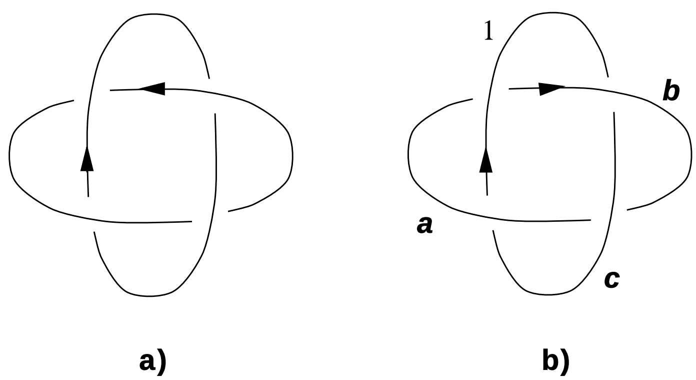


FIGURE 8: The link 4_1^2

Since K is free, it satisfies Trotter's conditions T1 and T2. Hence if (l, f) is a periodic link of period q, then f induces an order 3 automorphism F of K_{ab} by Lemma 3.12. The techniques of [37] can show that l has nontrivial period $q = 4$ (and hence 2) but no other. This approach makes use of the fact that the characteristic polynomial $t^3 - t^2 + t - 1$ of the automorphism of $K_{\text{ab}} \otimes \mathbf{Q}$ mapping $a_j \mapsto a_{j+1}$ has distinct roots, and consequently its splitting field must contain the qth roots of unity. In contrast, our methods do not require the characteristic polynomial to have distinct roots. They work only with Markov subgroups and do not appeal to algebraic field extensions.

A directed graph describing the Markov subgroup $\Phi_{\mathbf{Z}/2}$ associated to the augmented group of l consists of two fixed points, one 2-cycle and one 4-cycle. From Theorem 3.9 we see at once that (G, χ) admits no symmetry of odd order. Consequently, any period of l has the form 2^j.

A directed graph describing $\Phi_{\mathbf{Z}/3}$ appears in Figure 9. Theorem 3.9 implies that (G, χ) admits no symmetry of order 2^j, for $j > 4$. A closer examination of the graph shows that (G, χ) has no order 16 symmetry. Indeed, under such a symmetry F, the orbit of some vertex v must be the union of four 4-cycles, and $F^4 v$ must be $\sigma_x v$ or $\sigma_x^3 v$. Multiplication by 2 transposes the pairs of cycles C, C' and D, D', and leaves each of E and E' invariant. Hence F must leave $E \cup E'$ invariant, and we may assume that $v \in C$. Then Fv is in D or D' while $F^2 v$ is in C', say $F^2 v = \sigma_x^k(2v)$. Now $F^4 v = \sigma_x^{2k} v$, a contradiction. Hence (G, χ) has no symmetry of order 16.

The structure of $\Phi_{\mathbf{Z}/3}$ does not preclude a symmetry of order 8. In fact, the homomorphism $a_0 \mapsto 2a_0 + 2a_1,\ a_1 \mapsto 2a_0 + 2a_1 + a_2,\ a_2 \mapsto 2a_1 + a_2$ of $K_{\rm ab}$ induces an order 8 automorphism of $\Phi_{\mathbf{Z}/3}$.

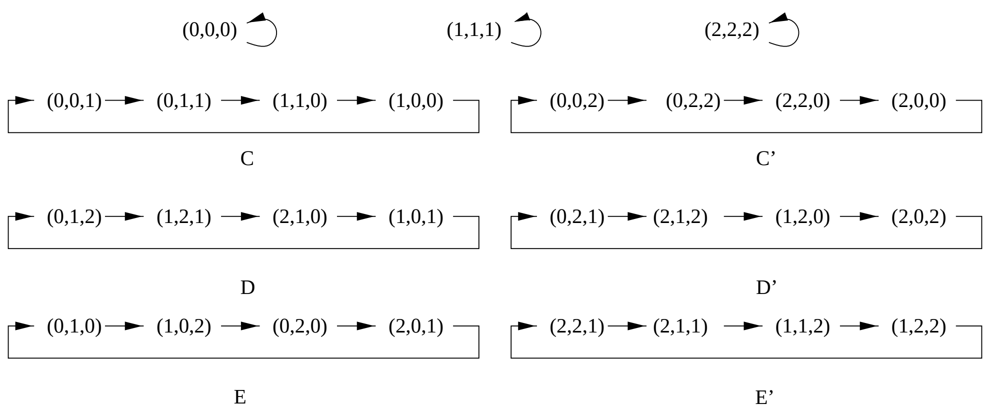


Figure 9: Shift $\Phi_{\mathbf{Z}/3}$ for 4_1^2

5. THE MARKOV SUBGROUP $\Phi_{\mathbf{T}}$.

Every cyclic group $\mathbf{Z}/n$ can be regarded as a subgroup of the compact group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ via the embedding $i \pmod n \mapsto i/n \pmod 1$. Consequently, if (G, χ) is any augmented group, then all of the associated representation shifts $\Phi_{\mathbf{Z}/n}$ embed in $\Phi_{\mathbf{T}}$. The underlying group of $\Phi_{\mathbf{T}}$ is known as the *dual group* of $K_{\rm ab}$.

Proposition 5.1. Let F be a symmetry of an augmented group (G, χ) inducing an order q automorphism of $K_{\rm ab}$. Assume that $K_{\rm ab}$ is $\mathbf{Z}$-torsion free and $\Delta_{K_{\rm ab}}$ has positive degree. Then F induces an order q automorphism F^* of $\Phi_{\mathbf{T}}$. Furthermore, $F^* \circ \sigma_x = \sigma_x \circ F^*$.

Proof. It is easy to see that F induces an automorphism F^* of the compact group $\Phi_{\mathbf{T}}$. The first assertion of Proposition 5.1 follows immediately from Lemma 3.12. It is also a consequence of the Pontryagin duality theorem [27]. The second claim holds more generally for any representation shift Φ_Σ, with Σ abelian (see section 3). ■

When $K_{\rm ab}$ is finitely generated, $(\Phi_{\mathbf{T}}, \sigma_x)$ is topologically conjugate to $(\mathbf{T}^N, \sigma)$, for some nonnegative integer N, where $\mathbf{T}^N$ is the N-torus $S^1 \times \cdots \times S^1$, and σ is the topological group automorphism adjoint to the automorphism $a \mapsto x^{-1}ax$ of $K_{\rm ab}$.

Example 5.1. Consider the augmented group (G, χ) associated to any 2-bridge knot with Alexander polynomial $c_d t^d + \cdots + c_1 t + c_0$ such that $c_0 c_d = \pm 1$. Then $K_{\rm ab}$ has an abelian group presentation

$$\langle a_j \mid c_d a_{j+d} + \cdots + c_1 a_{j+1} + c_0 a_j, \quad j \in \mathbf{Z} \rangle.$$

Since any representation $\rho : K \to \mathbf{T}$ is determined by the values $\rho(a_0), \rho(a_1), \ldots, \rho(a_{d-1})$, the elements of $\Phi_{\mathbf{T}}$ correspond to points $(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbf{T}^d$. The isomorphism σ sends $(\alpha_0, \alpha_1, \ldots, \alpha_{d-1})$ to $(\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $a_d = -(c_0/c_d)\alpha_0 - \cdots - (c_{d-1}/c_d)\alpha_{d-1}$. Such a Markov subgroup is finite since every representation is determined by its values on a fixed finite set of generators of $K_{\rm ab}$. Nevertheless, such a dynamical system can have surprising behavior. In particular, its topological entropy can be positive [27], [33].

When $K_{\rm ab}$ is not finitely generated, $(\Phi_{\mathbf{T}}, \sigma_x)$ can be very complicated, as the next example shows.

Example 5.2. Consider the augmented groups (G, χ) associated to the oriented link in Figure 8a. Here K has a presentation

$$\langle a_j \mid a_{j+1}^2 = a_j^2, \quad j \in \mathbf{Z} \rangle,$$

and the assignment $x \mapsto x^{-1}a^{-1}xax,\ a \mapsto x^{-1}axa^{-1}x^{-1}ax$ determines an order 2 symmetry of (G, χ).

Elements of $\Phi_{\mathbf{T}}$ correspond to bi-infinite sequences $(\alpha_j) \in \mathbf{T}^{\mathbf{Z}}$ such that $2\alpha_{j+1} \equiv 2\alpha_j \pmod 1$. Given any $\alpha_i, \ldots, \alpha_j \in \mathbf{T}$ satisfying the relation, there are two possibile values for α_{j+1} such that the relation remains satisfied, and likewise there are two possible values for α_{i-1}. The term "solenoid" is used to describe the topological structure of $\Phi_{\mathbf{T}}$ (see [27]). The symmetry F described above induces an order 2 automorphism F^* of $\Phi_{\mathbf{T}}$ mapping (α_j) to $(-\alpha_j)$.

In [33] elements of $\Phi_{\mathbf{T}}$ are used to define colorings of knot and link diagrams. Results about braid entropy are obtained using symbolic dynamics.

6. COVERING LINKS AND BRANCHED CYCLIC COVERS.

Let T be an n-tangle oriented in such a way that composition $T^s = T \cdot T \cdots T$ (s factors) is defined. The closure of T^s is an oriented link denoted here by l_s. For any positive integer r, we consider the r-fold cyclic cover $\hat{X}_r(l_s)$ of S^3 branched over l_s.

Theorem 6.1. For any finite group Σ, the cardinality $N_{r,s} = |\mathrm{Hom}(\pi_1 X_r(l_s), \Sigma)|$ satisfies linear recurrence relations in both r and s.

In [19] Murasugi proved that for any n-braid B and prime power p^k, the Jones polynomial $V_{B^{p^k}}(t)$ is congruent to $[V_B(t)]^{p^k}$ modulo a certain ideal of $\mathbf{Z}[t, t^{-1}]$. It seems possible that Theorem 6.1 and Murasugi's result are somehow theorems of the same type. Pat Gilmer's recent discovery of connections between topological quantum field theory and representations shifts [8] supports the view.

Proof of Theorem 6.1. Linear recurrence in r (s fixed) was shown in [31]. (A special case was also proven by W. Stevens [34].) We will show that $N_{r,s}$ also satisfies a linear recurrence relation in s.

Consider the oriented proper link l_∞ obtained by composing countably many copies T_k. Let S denote the translation that carries T_k to T_{k+1}, for each $k \in \mathbf{Z}$. We label the arcs of l_∞ by $a^{(k)}, b^{(k)} \ldots$ in such a way that S maps each to $a^{(k+1)}, b^{(k+1)} \ldots$, respectively. An example is given in Figure 10.

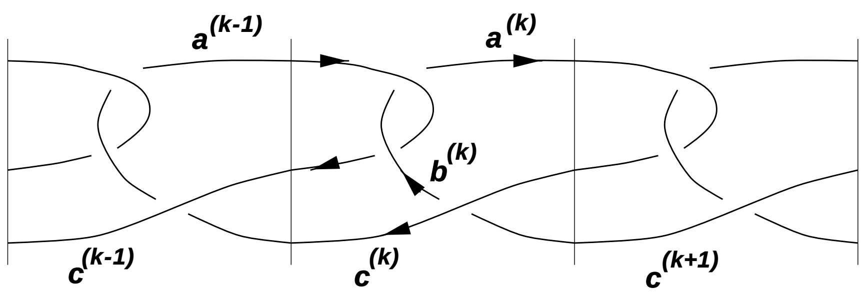


Figure 10: Labeled diagram for proper link l_∞

Define

$$K_\infty = \langle a_j^{(k)}, b_j^{(k)}, \ldots \mid r_j^{(k)}, s_j^{(k)}, \ldots,\ j, k \in \mathbf{Z} \rangle,$$

where $r_j^{(k)}, s_j^{(k)}, \ldots$ are determined as in Figure 3. For example, the group determined by the diagram in Figure 10 has presentation with generators $a_j^{(k)}, b_j^{(k)}$ and relations

$$a_{j+1}^{(k-1)} a_j^{(k)} = b_{j+1}^{(k)} a_j^{(k-1)}, \quad b_{j+1}^{(k)} c_j^{(k-1)} = a_{j+1}^{(k-1)} b_j^{(k)}, \quad c_{j+1}^{(k)} c_j^{(k+1)} = b_{j+1}^{(k)} c_j^{(k)}.$$

Let Σ be any group. A homomorphism $\rho : K_\infty \to \Sigma$ is a function $a_j^{(k)} \mapsto \alpha_j^{(k)}$, $b_j^{(k)} \mapsto \beta_j^{(k)} \ldots$ from the set of generators of K_∞ to Σ such that whenever the corresponding elements of Σ are substituted for generators in a relation of K_∞, the resulting equation holds in Σ. We can therefore regard ρ as an "admissible" labeling of the integer lattice $\{(j,k) \mid j,k \in \mathbf{Z}\}$ by n-tuples of Σ, where n is the number of distinct letters $a, b, \ldots$

appearing the presentation of K_∞. A labeling is admissible if a finite number of local translation-invariant conditions, determined by the presentation for K_∞, hold everywhere. For example, the relation $a_{j+1}^{(k-1)} a_j^{(k)} = b_{j+1}^{(k)} a_j^{(k-1)}$ determines a condition that must be satisfied at every parallelogram with vertices $(j-1, k+1), (j, k), (j, k+1), (j-1, k)$.

Let $X_r(l_s)$ denote the unbranched r-fold cyclic cover of $S^3 - l_s$. The fundamental group of $X_r(l_s)$ is a quotient of the augmentation subgroup of $\pi_1(S^3 - l_s)$, and a presentation for it has generators $y, a_j^{(k)}, b_j^{(k)}, \ldots$ and relations

$$a_j^{(0)} = 1,\ r_j^{(k)} = 1, s_j^{(k)} = 1, \ldots, y^{-1} a_j^{(k)} y = a_{j+r}^{(k)},\ y^{-1} b_j^{(k)} y = b_{j+r}^{(k)}, \ldots,$$

$$a_j^{(k+s)} = a_j^{(k)},\ b_j^{(k+s)} = b_j^{(k)}, \ldots,$$

where j, k range over $\mathbf{Z}$. The first relation $a_j^{(0)} = 1$ results from relabeling with the identity element an arc of the diagram for l_s, in this case, the arc originally labeled by $a^{(0)}$, and applying the combined Wirtinger and Reidemeister-Schreier methods of section 4; in other words, we must choose a distinguished meridian generator x for the group of l_s. The generator y corresponds to x^r. If l_s is a knot, then we obtain a presentation for $\pi_1(\hat{X}_r(l_s))$ by adding the relation $y = 1$. However, if l_s has more than one component, then we must also add $z^r = 1$, for every meridianal generator z of the group of l_s. In general, we add the relations $(x a_0^{(k)})^r = 1,\ (x b_0^{(k)})^r = 1, \ldots$ or equivalently, $y^{-1}(x a_0^{(k)})^r = 1,\ y^{-1}(x b_0^{(k)})^r = 1, \ldots$. These last relations can be rewritten as $a_{r-1}^{(k)} a_{r-2}^{(k)} \cdots a_0^{(k)} = 1,\ b_{r-1}^{(k)} b_{r-2}^{(k)} \cdots b_0^{(k)} = 1, \ldots$. Summarizing, the representations $\rho : \pi_1(\hat{X}_r(l_s)) \to \Sigma$ are labelings of the lattice by admissible n-tuples $(a_j^{(k)}, b_j^{(k)}, \ldots)$ of Σ such that

$$(1) \quad \alpha_j^{(0)} = 1 \quad \forall\ j \in \mathbf{Z};\ \text{and}$$

$$(2) \quad \alpha_{r-1}^{(k)} \alpha_{r-2}^{(k)} \cdots \alpha_0^{(k)} = 1,\ \beta_{r-1}^{(k)} \beta_{r-2}^{(k)} \cdots \beta_0^{(k)} = 1, \ldots\ \forall\ k \in \mathbf{Z}.$$

A trick introduced in [31] enables us to remove the second condition. Replace labels of the lattice points (j, k) by $2n$-tuples $(\alpha_j^{(k)}, \tilde{\alpha}_j^{(k)},\ \beta_j^{(k)},\ \tilde{\beta}_j^{(k)}, \ldots)$, where $\tilde{\alpha}_j^{(k)},\ \tilde{\beta}_j^{(k)}, \ldots$ are defined in the following manner:

$$\tilde{\alpha}_0^{(k)} = \tilde{\beta}_0^{(k)} = \cdots = 1$$

$$\tilde{\alpha}_{j+1}^{(k)} = \alpha_j^{(k+1)} \tilde{\alpha}_j^{(k)}, \quad \tilde{\beta}_{j+1}^{(k)} = \beta_j^{(k+1)} \tilde{\beta}_j^{(k)} \ldots$$

The elements $\tilde{\alpha}_j^{(k)},\ \tilde{\beta}_j^{(k)}, \ldots$ should be regarded as "counters," for it is not difficult to see that the condition $\tilde{\alpha}_{j+r}^{(k)} = \tilde{\alpha}_j^{(k)},\ \tilde{\beta}_{j+r}^{(k)} = \tilde{\beta}_j^{(k)}, \ldots$ is equivalent to condition (2) above.

Recall that each relation in the presentation of K_∞ determines a shape, usually a parallelogram, such that wherever that shape is translated in the lattice, the vertex labels must satisfy a condition dictated by the relation. The width of any such shape is at most 2. Let M denote the maximum height of the shapes (there are only finitely many different shapes). Fix a positive integer r, and let V denote the set of all labelings $R = \{(j,k) \mid 0 \le j \le r,\ 0 \le k \le M\}$ by $2n$-tuples $v = (\alpha_j^{(k)}, \tilde{\alpha}_j^{(k)},\ \beta_j^{(k)},\ \tilde{\beta}_j^{(k)}, \ldots) \in \Sigma^{2n}$ such that whenever a shape fits insider R, the corresponding equation is satisfied. Also, we require that $\alpha_r^{(k)} = \alpha_0^{(k)},\ \beta_r^{(k)} = \beta_0^{(k)}, \ldots$

Assume that Σ is a finite group. Then the set V is also finite. We construct a directed graph Γ with vertex set V. We place an edge from $v = (\alpha_j^{(k)}, \tilde{\alpha}_j^{(k)},\ \beta_j^{(k)},\ \tilde{\beta}_j^{(k)}, \ldots)$ to $v' = (\alpha_j^{(k)'}, \tilde{\alpha}_j^{(k)'},\ \beta_j^{(k)'},\ \tilde{\beta}_j^{(k)'}, \ldots)$ whenever

$$\alpha_j^{(k+1)} = \alpha_j^{(k)'}, \quad \bar{\alpha}_j^{(k+1)} = \bar{\alpha}_j^{(k)'},$$

$$\beta_j^{(k+1)} = \beta_j^{(k)'}, \quad \bar{\beta}_j^{(k+1)} = \bar{\beta}_j^{(k)'},$$

$$\vdots$$

for $k = 1, \ldots, M-1$. Admissible labelings of the lattice that have period r in the j-coordinate direction are in one-to-one correspondance with bi-infinite paths in Γ.

We call a vertex $v = (\alpha_j^{(k)}, \tilde{\alpha}_j^{(k)},\ \beta_j^{(k)},\ \tilde{\beta}_j^{(k)}, \ldots)$ of Γ "based" if $\alpha_0^{(k)} = \beta_0^{(k)} = \cdots = 1$. From what has been said it follows that the number $N_{r,s}$ of representations $\rho : \pi_1(\hat{X}_r(l_s)) \to \Sigma$ is equal to the number in paths in Γ of length r beginning and ending at a based vertex. This quantity can be computed from powers of the adjacency matrix for Γ, and it satisfies a linear recurrence relation. ■

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Dept. of Mathematics and Statistics, Univ. of South Alabama, Mobile, AL 36688
email: silver@mathstat.usouthal.edu, williams@mathstat.usouthal.edu