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POLYNOMIAL INVARIANTS OF VIRTUAL LINKS

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ABSTRACT
Properties of polynomial invariants Δ_i for oriented virtual links are established. The effects of taking mirror images and reversing orientation of the link diagram are described. The relationship between $\Delta_0(u, v)$ and an invariant of F. Jaeger, L. Kauffman, H. Saleur and J. Sawollek is discussed.

Keywords: virtual knot, Alexander group, Alexander polynomial.

1. Introduction.

A classical knot or link l can be profitably regarded as an equivalence class of diagrams under combinatorial moves. By a **diagram** we mean as usual a 4-valent plane graph D resulting from a regular projection of l with a *trompe l'oeil* device at each vertex conveying "over" and "under" information. The neighborhood of a vertex is called a **crossing** (Figure 1 (a)). It is well known that two links are isotopic if and only if one can be transformed into the other by a finite sequence of Reidemeister moves.

In 1996 L. Kauffman introduced virtual links [K97] by allowing another type of crossing which he called a **virtual crossing** (Figure 1 (b)). (Henceforth a classical crossing is a crossing that is not virtual.) A virtual link is an equivalence class of virtual link diagrams under **generalized Reidemeister moves** (Figure 2).

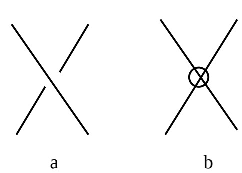

Figure 1. Classical and virtual crossings

An oriented virtual link is defined in the usual way, by specifying a direction for each component of a diagram. Such a link is an equivalence class of diagrams under **oriented generalized Reidemeister moves**. These are the same moves as those in Figure 2 except that all possible assigned directions for arcs must be considered. A result ot V. Turaev [T88] (see also [K91, p.81]) reduces the number of necessary moves: If we consider one Reidemeister type II move in which both arcs are oriented in the same direction and another in which the arcs are oriented differently, then we need only consider a single move of type III.

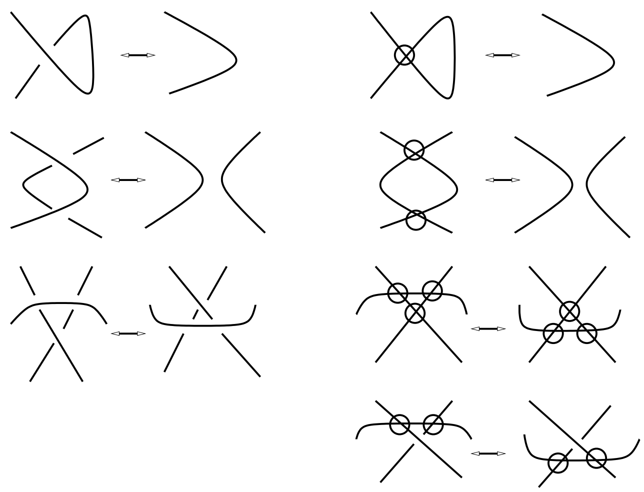

Figure 2. Generalized Reidemeister moves

A theorem of Goussarov, Polyak and Viro [GPV00] states that if two classical knot diagrams are equivalent under generalized Reidemeister moves, then they are equivalent under the classical moves. In this sense virtual knot theory is a nontrivial extension of the classical theory. Further motivation for the subject comes from consideration of Gauss codes. The reader is encouraged to consult any of the surveys [K99], [K00] or [K01] for details.

It is clear from the above discussion that any invariant of virtual links must be a generalization of a classical invariant, possibly the trivial invariant. In a seminal paper [K00] on this subject Kauffman showed that the group of an oriented link, the bracket polynomial and hence the Jones polynomial generalize naturally in the new category.

Motivated by a study [SW01] of the Burau representation, the authors introduced in [SW01'] an extension $\tilde{A}_l$ of the group of an oriented virtual link l. We review the definition below. From $\tilde{A}_l$ one immediately obtains a sequence of invariants $\Delta_i,\ i \geq 0$, in the Laurent polynomial ring $\Lambda = \mathbf{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}, v]$. When l is classical, Δ_i is seen to be the well-known ith Alexander polynomial of l in the variables $u_1 v, \ldots, u_d v$. In particular, Δ_0 vanishes whenever l is classical. This can be used to show that many virtual links are not classical. (See [K02], [SW01'] for examples.)

Shortly after completing [SW01'] we learned of a polynomial invariant $Z(x, y)$ for oriented virtual links described by J. Sawollek in [S99]. It first appeared as an invariant of F. Jaeger, L. Kauffman and H. Saleur [JKS94] for links in thickened surfaces. We will refer to $Z(x, y)$ as the **JKSS invariant**. Close inspection showed $Z(x, y)$ is essentially the same as the invariant Δ_0. One purpose of this paper is to provide details of the relationship.

The invariant $Z(x, y)$ has an advantage over our original definition of Δ_0 in that it is defined up to multiples of x rather than arbitrary units in $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$. Consequently, it can often detect chirality and noninvertibility of virtual links. On the other hand, the algebraic approach used in [SW01'] yields useful "higher invariants" $\Delta_i,\ i > 0$, via elementary divisor theory. (See Example 6.2.)

Motivated by [S99] we sharpen the algebraic methods of [SW01']. The polynomial invariant Δ_0 becomes well defined up to multiples of $u_i v,\ 1 \le i \le d$, as one would hope in view of the above discussion. (The higher invariants $\Delta_i,\ i > 0$, stubbornly remain well defined only up to units in Λ.) Setting $u_1 = \cdots = u_d = u$ reduces Δ_0 to a 2-variable polynomial $\Delta_0(u, v)$ which we identify with $Z(x, y)$. We prove a version of a skein relation in [S99], using an algebraic argument that is very short and self-contained. We also establish some basic properties of $\Delta_0(u, v)$. Finally, we use both algebraic and combinatorial methods to examine the effect on all of the polynomials Δ_i of an oriented virtual link l when reflections and orientation changes are performed on a diagram. A key argument was provided by Sawollek [S02'].

2. Normalizing the Alexander polynomial.

Let $l = l_1 \cup \cdots \cup l_d$ be an oriented virtual link with diagram D. Regard each classical crossing arc as the union of two edges joined at the point of overcrossing. By an **edge** of Δ we mean a segment of l going from one classical crossing to another. Every edge is an input, left or right, for a unique classical crossing (see Figure 3). Assign letters $a, b, c, d, \ldots$ to edges so that a, b correspond to left and right inputs, respectively, for some crossing, while c, d correspond similarly to another crossing, and so forth. The reader can find amusing solutions for the problem of labeling a diagram with more than 13 classical crossings.

We associate to D ordered pairs of generator families $a_{\mathbf{n}},\ b_{\mathbf{n}}$ and $c_{\mathbf{n}},\ d_{\mathbf{n}}$, etc., each generator indexed by elements $\mathbf{n}$ of the free abelian group Π generated by $u_1, \ldots, u_d, v$. It is convenient to abbreviate the families of generators by $a, b, c, d, \ldots$. To each classical crossing (equivalently, to each pair of generator families) we associate an ordered pair of indexed relator families. If the crossing is positive, we associate $ab^{u_i}(b_+^{u_j})^{-1}a_+^{-1},\ a_+^v b^{-1}$; if the crossing is negative, we associate $ab^{u_i}(b_+^{u_j})^{-1}a_+^{-1},\ a^v(b_+)^{-1}$ and, as in Figure 3. Here a_+ (respectively b_+) is the letter directly below a (respectively b) when the crossing is viewed with both arrows pointing down. The index i is that of the component corresponding to a, while j is the index corresponding to b. The symbol $ab^{u_i}(b_+^{u_j})^{-1}a_+^{-1}$ represents the family of relators $a_{\mathbf{n}} b_{\mathbf{n}+u_i}(b_+)^{-1}_{\mathbf{n}+u_j}(a_+)^{-1}_{\mathbf{n}}$. The other exponential symbols are similarly defined. We regard Π as an additive group when elements appear as subscripts, and a multiplicative group when they appear as exponents.

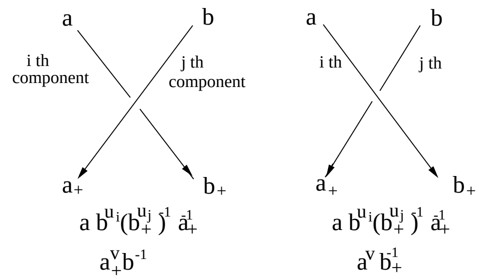

Figure 3. Relators for $\tilde{\mathcal{A}}_l$

Let $\langle X \mid R\rangle_D$ denote the presentation with ordered generators $a, b, c, d, \ldots$ and corresponding ordered relations. We regard relators as cyclic words, allowing them to be cyclically permuted. Such a presentation is unique up to the numbering of the classical crossings in D and hence up to even permutations of generators and relators.

From the point of view of combinatorics, a group is an equivalence class of presentations, the equivalence relation being generated by Tietze transformations. The group $\tilde{\mathcal{A}}_l$ presented by $\langle X|R\rangle_D$ was studied in [SW01'], where it is called the **extended Alexander group** of the oriented virtual link l. (It is an extension of an "Alexander group" $\mathcal{A}_l$ defined in [SW01'] for classical links, a group that will not appear here.) In [SW01'] it was seen that if a diagram D' is gotten from D by applying a generalized Reidemeister move, then $\langle X \mid R\rangle'_D$ differs from $\langle X \mid R\rangle_D$ by a sequence of Tietze transformations. In this way the group $\tilde{\mathcal{A}}_l$ was shown to be an invariant of l.

In [SW01'] we observed that $\tilde{\mathcal{A}}_l$ has the structure of a finitely generated Π-**group** (see [R96]). This means there is a homomorphism $\sigma : \Pi \to \mathrm{Aut}(\tilde{\mathcal{A}}_l),\ \mathbf{n} \mapsto \sigma_{\mathbf{n}}$. Two Π-groups are isomorphic if there exists an isomorphism of the underlying groups that respects the Π-actions. Isomorphisms in this paper are of this sort.

The automorphism $\sigma_{\mathbf{n}}$ acting on $g \in \tilde{\mathcal{A}}_l$ translates by $\mathbf{n}$ all subscripts of generators in a word representing g. We abbreviate $\sigma_{\mathbf{n}}(g)$ by $g^{\mathbf{n}}$. The Tietze transformations used to show that $\tilde{\mathcal{A}}_l$ is well defined can be chosen to respect the action, and hence the Π-group $\tilde{\mathcal{A}}_l$ is an invariant of l. We remark that in [K01] Kauffman sets $u_1 = \ldots = u_d = t$ and uses the resulting action to go in a different direction, associating to l an invariant called a *biquandle.* This approaoch is developed further in [KR01].

The abelianization $(\tilde{\mathcal{A}}_l)_{\mathrm{ab}}$ is a finitely generated Λ-module. When l is classical, it is essentially the Alexander module of l (see [SW01']). The well known theory of elementary divisors produces a sequence $\Delta_i(l) = \Delta_i(l)(u_1, \ldots, u_d, v) \in \Lambda,\ i \geq 0$, of invariants for l such that $\Delta_{i+1}(l)$ divides $\Delta_i(l)$, for each i. (When no confusion results, we will write Δ_i instead of $\Delta_i(l)$.) We recall the definition: Without any loss of generality assume that $(\tilde{\mathcal{A}}_l)_{\rm ab}$ has an $m \times m$ relation matrix. Then Δ_i is the greatest common divisor of the $(m-i)\times(m-i)$-minors of the relation matrix. It is well defined up to multiplication by units in Λ. We call Δ_i the ith **virtual Alexander polynomial** of l. When l is classical, Δ_i agrees with the well-known d-variable ith Alexander polynomial of l, evaluated at $u_1v, \ldots, u_dv$. Details can be found in [SW01'].

We sharpen our algebraic methods, keeping track of the effects on a presentation $\langle X|R\rangle_D$ when Tietze transformations that correspond to generalized Reidemeister moves on D are performed.

Definition 2.1. The **presentation module** $\mathcal{P}$ is the Λ-module generated by Π-group presentations $\langle X|R\rangle = \langle x_1,\ldots,x_m \mid r_1,\ldots,r_n\rangle$ modulo the submodule generated by the relators:

$$\begin{aligned}
&\langle x_1,\ldots,x_m \mid R\rangle - \mathrm{sgn}(\pi)\ \langle x_{\pi(1)},\ldots,x_{\pi(m)} \mid R\rangle\\
&\langle X \mid r_1,\ldots,r_n\rangle - \mathrm{sgn}(\pi)\ \langle X \mid r_{\pi(1)}\ \ldots, r_{\pi(n)}\rangle\\
&\langle X \mid r_1^{-1}, r_2,\ldots,r_n\rangle + \langle X \mid r_1,\ldots,r_n\rangle\\
&\langle X \mid r_1^g, r_2,\ldots,r_n\rangle - g\langle X \mid r_1,\ldots,r_n\rangle,\ g\in\Pi\\
&\langle X \mid r_1,\ldots,r_n\rangle - \langle X \mid r_1(r_j^g)^{\pm1}, r_2,\ldots,r_n\rangle,\ j\neq 1,\ g\in\Pi\\
&\langle y, X \mid yw, R\rangle - \langle X \mid R\rangle,\ (y \text{ does not occur in } w)\\
&\langle X \mid 1, r_2,\ldots,r_n\rangle - \langle X \mid r_2,\ldots,r_n\rangle
\end{aligned}$$

It is immediate that if $\langle X|R\rangle$ has a redundant relator (that is, a relator that is a consequence of the other relators) then $\langle X|R\rangle$ is trivial in $\mathcal{P}$. In particular, $\langle X|R\rangle_D$ vanishes whenever l is classical. However, the module $\mathcal{P}$ is nontrivial. We see this as follows.

Let D be a diagram for an oriented virtual d-component link. Assume that D' is obtained from D by applying a single generalized Reidemeister move. We distinguish four cases of Type I moves (see Figure 4). The proof of the following lemma is routine, and we leave it to the reader.

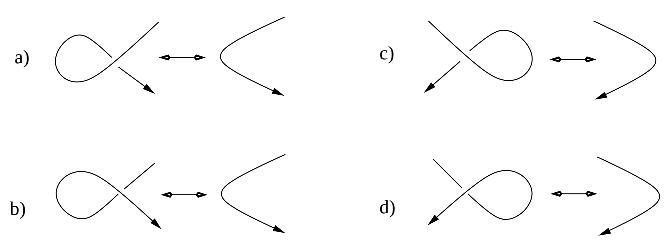

Figure 4. Reidemeister type I moves

Lemma 2.2. We have

$$\begin{aligned}
\langle X \mid R\rangle_{D'} &= -u_iv\langle X \mid R\rangle_D,\ \text{for type Ia, b moves;}\\
\langle X \mid R\rangle_{D'} &= -\langle X \mid R\rangle_D,\ \text{for type Ic, d moves;}\\
\langle X \mid R\rangle_{D'} &= \langle X \mid R\rangle_D,\ \text{for type II or III moves.}
\end{aligned}$$

Let $\langle X \mid R\rangle = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_m\rangle$ be a finite presentation for a Π-group. We mildly abuse notation, letting R also denote the $m \times m$ matrix representing the Λ-module obtained from the group by abelianization. The relator r_j becomes the jth column of R with entries in Λ.

Proposition 2.3. The map $\delta : \mathcal{P} \to \Lambda$ given by $\langle X \mid R\rangle \mapsto \det(R)$ is a homomorphism.

Proof. One checks easily that the relators in Definition 2.1 are mapped to zero. □

Let D and D' be two diagrams for the same virtual oriented link l. Assume that D (resp. D') has N (resp. N') classical crossings. It follows immediately from Lemma 1.2 that $(-1)^N\delta(\langle X \mid R\rangle_D)$ and $(-1)^{N'}\delta(\langle X \mid R\rangle_{D'})$ differ only by multiplicative factors of the form $(u_iv)^{\pm 1}$. We normalize to eliminate this indeterminancy, defining the **normalized** 0th **virtual Alexander polynomial** $\Delta_0(l)(u_1, \ldots, u_d, v)$ to be

$$(-1)^N(u_1v)^{-n_1}\cdots(u_dv)^{-n_d}\delta(\langle X \mid R\rangle_D),$$

where n_i is the lowest exponent of u_i in $\delta(\langle X \mid R\rangle_D)$. This definition is consistent with the original definition given in [SW01'], where the invariant was given only up to multiplication by units in Λ.

When l has just one component, we write u instead of u_1. When l has more than one component, we can set all the u_i equal to u and carry out the above procedures. In this way we obtain a normalized 2-variable polynomial that we will denote by $\Delta_0(l)(u, v)$.

3. Relationship with the JKSS invariant.

In [S99] Sawollek adapted ideas of [JKS94] to define a polynomial invariant $\tilde{Z}(x, y)$ for any oriented virtual link l. Let D be a diagram for l with classical crossings $C_1, \ldots, C_N$. Let

$$M_+ = \begin{pmatrix} 1-x & -y \\ -xy^{-1} & 0 \end{pmatrix} \quad \text{and} \quad M_- = \begin{pmatrix} 0 & -x^{-1}y \\ -y^{-1} & 1-x^{-1} \end{pmatrix}.$$

For $1 \le i \le N$, let M_i be M_+ (resp. M_-) if C_i is positive (resp. negative). Let M be the $2N \times 2N$ diagonal block matrix $\mathrm{diag}(M_1, \ldots, M_N)$. At each classical crossing, label left and right input arcs a, b and $c, d, \ldots$ as we have done previously. The diagram D determines a permutation $a \mapsto a_+, b \mapsto b_+, \ldots$, where $a_+,\ b_+, \ldots$ are as shown in Figure 3. Regard the letters as column vectors, and let P be the associated permutation matrix. By considering the effects of generalized Reidemeister moves, one verifies that $Z_l(x, y) = (-1)^N\det(M - P)$ is an invariant of l well defined up to multiplication by powers of x. Normalization as above yields a completely well defined polynomial invariant $\tilde{Z}_l(x, y) = x^{-e}Z_l(x, y)$, where e is the lowest exponent of x in $Z_l(x, y)$.

Proposition 3.1. Let l be an oriented virtual link. Let $l^\sharp$ be the link obtained by changing each classical crossing in a diagram for l. Then

$$\Delta_0(l)(u,v) = \tilde{Z}_{l^\sharp}(uv, -v).$$

Proof. Consider a positive classical crossing of a diagram D for l. Assume without loss of generality that the left and right input arcs are labeled a and b, respectively. The corresponding pair of columns in the relation matrix R for $\Delta_0(l)(u,v)$ describe a pair of consecutive relators:

$$a + ub - ub_+ - a_+, \quad va_+ - b \tag{2.1}$$

Now change the crossing, and consider the corresponding pair of columns in the matrix $M - P$ above. The pair of relations that they describe is

$$v^{-1}b - a_+, \quad u^{-1}a + (1 - u^{-1}v^{-1})b - b_+ \tag{2.2}$$

Multiply the first relator of (2.2) by $-v$, multiply the second by u, and then interchange the two relators to obtain

$$a + ub - v^{-1}b - ub_+, \quad va_+ - b \tag{2.3}$$

Finally, replacing the second occurrence of b in (2.3) by va_+ changes (2.3) into (2.1). The corresponding modifications on $M - P$ change the determinant by a multiplicative factor of uv, a factor that is removed by normalization.

A similar but easier argument applies for any negative crossing of D. We leave the details to the reader. □

4. Factors of $\Delta_0(u,v)$.

In the last section we showed that for any oriented virtual link l, the polynomial $\Delta_0(l)(u,v)$ is equal to $\tilde{Z}_{l^\sharp}(uv,-v)$.

Proposition 4.1. Let l be an oriented virtual link.

(i) $(u-1)(v-1)$ divides $\Delta_0(u,v)$.

(ii) If l is a knot, then $uv - 1$ divides $\Delta_0(u,v)$.

Proof. The result that $v-1$ divides $\Delta_0(u,v)$ is proved in [S99]. For the reader's convenience we prove it here.

Let D be a diagram for l with N classical crossings. Recall the matrices M and P of the previous section. Replace x, y with $uv, -v$, respectively. Then $\Delta_0(u,v)$ is equal to $(-1)^N \det(M-P)$.

If we let $u = 1$, then each row of $M - P$ sums to zero, and hence the columns are dependent and $\det(M-P)$ vanishes. Consequently, $u - 1$ divides $\Delta_0(u,v)$. Similarly, if we let $v = 1$, then each column sums to zero, and so $v - 1$ divides $\Delta_0(u,v)$.

Now assume that l has a single component. As in [S99] we set

$$T = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

and observe that PT is the matrix of the permutation π of edge labels that we read by following the diagram D for l in the preferred direction. We have

$$\Delta_0(u,v) = (-1)^N \det(MT - PT).$$

MT is a matrix of diagonal blocks

$$\begin{pmatrix} v & 1-uv \\ 0 & u \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u^{-1} & 0 \\ 1-u^{-1}v^{-1} & v^{-1} \end{pmatrix}.$$

If we set $uv = 1$, then $MT - PT$ becomes a matrix $(a_{i,j})$ with $a_{i,i} = u^{\pm 1}$, $a_{i,\pi(i)} = -1$ and all other entries equal to zero. Write the determinant as

$$\sum_{\sigma \in S_{2N}} \mathrm{sgn}\ \sigma \cdot a_{1,\sigma(1)} \cdots a_{2N,\sigma(2N)}.$$

We claim that the only nonzero terms are given by $\sigma = \pi$ and σ equal to the identity. For if the term corrsponding to σ is nonzero and $\sigma(i) = \pi(i)$ for some i, then $\sigma^2(i)$ must be $\pi(\sigma(i))$, and so on; since π is a single cycle, $\sigma = \pi$. Consequently the determinant is $u^N u^{-N} + \mathrm{sgn}\ \pi(-1)^{2N}$, which is zero, since π is a cycle of even length. Hence $1 - uv$ divides $\Delta_0(u,v)$. □

Remark 4.2. When l has more than one component, $uv - 1$ need not divide $\Delta_0(u,v)$. The simplest example is provided by a "virtual Hopf link" with one classical and one virtual crossing, for which $\Delta_0(u,v) = (u-1)(v-1)$.

In [S99] Sawollek describes a skein relation for $Z(x,y)$. The proof relies on a connection between partition functions and Pfaffians, and the reader is referred to [JKS94] for details of proof. We offer a short, elementary proof of the skein relation using the algebraic approach of Section 2.

Let l_+, l_-, l_0 be a skein triple of oriented virtual links, that is, links with corresponding diagrams D_+, D_-, D_0 that differ only in the neighborhood of a point, as in Figure 5. In the neighborhood we have a postive crossing, a negative crossing and a smoothing, respectively.

Theorem 4.3. [S99] There exist integers i, j such that

$$(uv)^i \Delta_0(l_+) - (uv)^j \Delta_0(l_-) = (uv - 1)\ \Delta_0(l_0).$$

Proof. Consider the presentations $\langle X \mid R \rangle_{D_+}, \langle X \mid R \rangle_{D_-}$ and $\langle X \mid R \rangle_{D_0}$ with $u_1, \ldots, u_d$ set equal to u (see Section 2). Each has $2N$ generators and $2N$

relators. Note that we introduced two vertices in D_0 so that labels would remain unchanged outside a neighborhood of the smoothing. The abelianized relator pairs are respectively

$$a + ub - a_+ - ub, \qquad -b + va_+;$$

$$a + ub - a_+ - ub_+, \qquad va - b_+;$$

$$a - a_+, \qquad b - b_+.$$

In the second pair we add $-v$ times the first relator to the second. In the third pair we add u times the second relator to the first. The result is $a+ub-a_+-ub, -b+va_+$ $a+ub-a_+-ub_+, -uvb+va_+ + (uv-1)b_+$ and $a+ub-a_+-ub_+, b-b_+$. In matrix form we have

$$\begin{pmatrix} 1 & 0 & & \\ u & -1 & & \\ \vdots & \vdots & & \\ -1 & v & * & * \\ \vdots & \vdots & & \\ -u & 0 & & \\ \vdots & \vdots & & \end{pmatrix}, \begin{pmatrix} 1 & 0 & & \\ u & -uv & & \\ \vdots & \vdots & & \\ -1 & v & * & * \\ \vdots & \vdots & & \\ -u & uv-1 & & \\ \vdots & \vdots & & \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & & \\ u & 1 & & \\ \vdots & \vdots & & \\ -1 & 0 & * & * \\ \vdots & \vdots & & \\ -u & -1 & & \\ \vdots & \vdots & & \end{pmatrix}.$$

Note that the three matrices differ only in the second column. Computing the determinant of each matrix by expanding along the second column gives the desired result. □

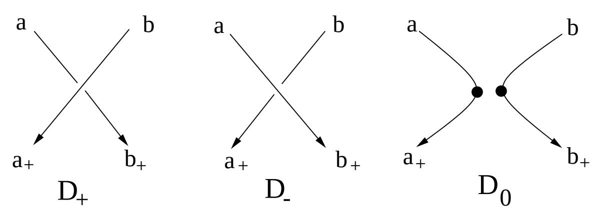

Figure 5. Skein triple of oriented virtual links

5. Effect on $\Delta_i(u,v)$ of three link involutions.

Given a diagram D of an oriented virtual link l, we denote by $D^\sharp$ the diagram obtained by changing the sign of each classical crossing; by D^* the reflection in a line on the paper; and by $-D$ the diagram with all orientations reversed. We let $l^\sharp, l^*$ and $-l$ denote the corresponding virtual links. Note that $l^* = l^\sharp$ for any classical link l. Sawollek [S02'] has shown that $\tilde{Z}_{-l^\sharp}(x,y) = \tilde{Z}_l(x^{-1}, y^{-1})$, or equivalently $\Delta_0(-l^\sharp)(u,v) = \Delta_0(l)(v^{-1}, u^{-1})$. Examining his proof we see that it also gives $\Delta_i(-l^\sharp)(u,v) = \Delta_i(l)(v^{-1}, u^{-1})$, for $i > 0$. We derive analagous results for the effects of reflection and reversal, not only on the polynomial but on the group $\tilde{\mathcal{A}}_l$ and the module of presentations. We will see, however, that the effect of crossing change on $\tilde{\mathcal{A}}_l$ is not so simple, and reveals a fundamental asymmetry of $\tilde{\mathcal{A}}_l$.

Setting $u_1, \ldots, u_d$ equal to u in a presentation $\langle X \mid R\rangle_D$ for $\tilde{\mathcal{A}}_l$ yields a presentation $\langle X \mid R(u,v)\rangle_D$ of a quotient group that we denote by $\tilde{\mathcal{A}}_l(u,v)$. It has an action by Π', the free abelian group on u and v. Replacing u and u^{-1} by v and v^{-1} in $\langle X \mid R(u,v)\rangle_D$ gives a presentation $\langle X \mid R(u^{-1},v^{-1}\rangle_D$ of the Π'-group $\tilde{\mathcal{A}}_l(u^{-1},v^{-1})$.

Theorem 5.1. Let l be an oriented virtual d-component link with diagram D having N classical crossings.

(i) $\langle X \mid R(u,v)\rangle_{D^*} = (uv)^N \langle X \mid R(u^{-1},v^{-1})\rangle_D$ in the presentation module $\mathcal{P}$, and $\tilde{\mathcal{A}}_{l^*}(u,v) \cong \tilde{\mathcal{A}}_l(u^{-1},v^{-1})$.

(ii) $\langle X \mid R(u,v)\rangle_{-D} = (-1)^d (uv)^N \langle X \mid R(u^{-1},v^{-1})\rangle_D$ in $\mathcal{P}$, and $\tilde{\mathcal{A}}_{-l}(u,v) \cong \tilde{\mathcal{A}}_l(u^{-1},v^{-1})$.

Corollary 5.2. For each $i \geq 0$,
(i) $\Delta_i(l^*)(u,v) = \Delta_i(l)(u^{-1},v^{-1})$
(ii) $\Delta_i(-l)(u,v) = (-1)^d \Delta_i(l)(u^{-1},v^{-1})$
(iii) $\Delta_i(l^\sharp)(u,v) = (-1)^d \Delta_i(l)(v,u)$
(Here equality is understood to hold up to powers of uv.)

Proof of Corollary. Parts (i) and (ii) follow from the corresponding parts of Theorem 5.1. Part (iii) comes from combining (ii) with Sawollek's result [S02'] for $-l^\sharp$.

Remark 5.3. The group theoretic analogue of part (iii) is not valid. We will see Example 6.3 that $\tilde{\mathcal{A}}_{l^\sharp}(u,v)$ need not be isomorphic to $\tilde{\mathcal{A}}_l(v,u)$.

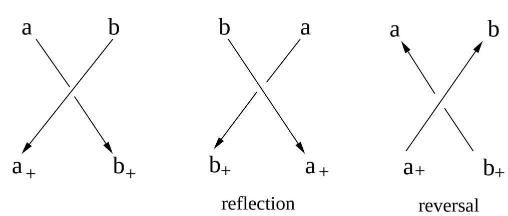

Figure 6. Positive crossing and reflection

Proof of Theorem 5.1. (i) We label each edge in the reflected diagram D^* by the same letter as its mirror image. The local effect of reflection on a positive crossing is shown in Figure 6. The ordered pair of relators corresponding to the crossing in D is

$$ab^u(b_+^u)^{-1}a_+^{-1},\ a_+^v b^{-1}. \tag{5.1}$$

Replacing u by u^{-1} and v by v^{-1}, and then acting by u on the first relator and by v on the second, gives

$$a^u b(b_+)^{-1}(a_+^u)^{-1},\ a_+(b^v)^{-1}. \tag{5.2}$$

Inverting the first relator and cyclically permuting both relators gives

$$b^{-1}(a^u)^{-1}a_+^u b_+^{-1}, (b^v)^{-1}a_+,$$

which is the pair corresponding to the negative crossing in D^*. The reader can check that the same sequence of steps transforms the relator pair for a negative crossing in D to the pair for the corresponding positive crossing in D^*.

The net effect of these transformations in the module $\mathcal{P}$ of presentations is to introduce a factor of $-uv$ for every classical crossing. In order to obtain the correct order of generators in the the presentation $\langle X \mid R\rangle_{D^*}$ we must interchange each input pair a, b, since in D^*, the label b is on the left. This permutation introduces another factor of $(-1)^N$, and hence $\langle X \mid R(u,v)\rangle_{D^*} = (uv)^N \langle X \mid R(u^{-1}, v^{-1})\rangle$ in $\mathcal{P}$. Comparing the Π'-groups given by the two presentations we find that $\tilde{\mathcal{A}}_{l^*}(u,v) = \tilde{\mathcal{A}}_l(u^{-1}, v^{-1})$.

(ii) The proof of (ii) is similar to that of (i). We keep the same labels on the edges of $-D$, although now a_+ is the predecessor rather than the successor of a. As before, we take relators (4.1) corresponding to a positive crossing of D and transform them to (5.2). Now replacing each generator by its inverse and cyclically permuting yields

$$b_+a_+^u(a^u)^{-1}b^{-1},\ b^v a_+^{-1},$$

which is the relator pair for $-D$. Again, negative crossings admit the same transformation as positive ones. In $\mathcal{P}$ this transformation introduces a factor of uv for each crossing.

Finally, to get the presentation $\langle X \mid R\rangle_{-D}$ we must put the generators in the order $b_+, a_+, \ldots$. The permutation $\pi(a) = b_+,\ \pi(b) = a_+, \ldots$ is a product of cycles that are encountered as we traverse each component of the link l in the preferred direction. Since each cycle has even length, $\text{sgn}(\pi) = (-1)^d$. □

6. Examples.

As part of his Master's Thesis under the direction of A. Kawauchi at Osaka City University, T. Kishino produced a census of virtual knots with small numbers of virtual and classical crossings [K02]. Computations of $\tilde{Z}(x,y)$ (as well as Jones polynomial and group presentations) are included. We are indebted to S. Kamada for informing us about Kishino's work and providing us with an advance copy. It contains a wealth of fascinating diagrams and computations.

For each virtual knot k that he treats, Kishino also considers $k^*, k^\sharp$ and $k^{*\sharp}$. As Corollary 5.2 predicts, $\Delta_0(k)(u,v)$ determines the polyomials for all four.

Example 6.1. Consider the virtual knot k in Figure 7. It is $(3,2)_5$ in Kishino's table; that is, the fifth knot with 3 classical crossings and 2 virtual crossings. The eight knots gotten from k by applying the operations $*, -, \sharp$ have pairwise distinct polyomials $\Delta_0(u,v)$. Using $\Delta_0(k)(u,v) = -(u-1)(v-1)(uv-1)(u+1)$ and Corollary 5.2 all the polynomials can be determined.

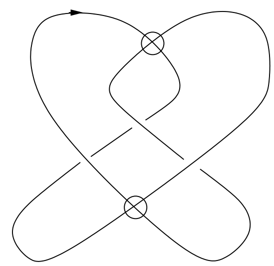

Figure 7. The virtual knot $(3, 2)_5$

Example 6.2. In [K02] Kauffman observed that the apparently nontrivial virtual knot k in Figure 8 has unit Jones polynomial. He asked if k is classical. For this knot $\Delta_0(u, v) = 0$. However, $\Delta_1(u, v)$ is equal to $uv + v - 1$. Recall from [SW01'] that if k is classical then $\Delta_i(u, v)$ must be a polynomial in uv. Since that is not the case here, k is nonclassical; in particular, it is nontrivial. Corresponding polynomials for k^* and $k^\sharp$ can be found using Corollary 5.2. We see immediately that the three virtual knots are distinct.

There is a second way to see that k is not classical, by computing its knot group. By the usual Wirtinger method we have $G \cong \langle x, y \mid xyx^{-1} = yxy^{-1}\rangle$. If k were classical, then its first Alexander polynomial would coincide with that of G. The polynomial is easily computed (see Section 3.4 of [MKS76]) and is equal to $2t - 1$. But any Alexander polynomial of a classical knot is reciprocal. Hence k is not classical.

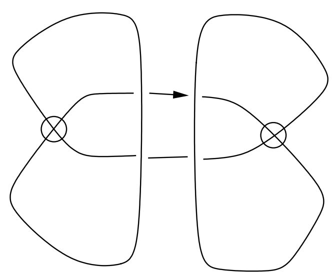

Figure 8. Knot k with trivial Jones polynomial

Example 6.3. Consider the pair $k, k^\sharp$ where k is the virtual knot in Figure 9. For both of these, $\Delta_0(u, v) = (u-1)(v-1)(uv-1)(u^2v^2 - uv^2 + u^2v - u + v + 1)$ while $\Delta_i(u, v) = 1$ for $i > 0$. One verifies that with the given input labels,

$$\tilde{\mathcal{A}}_k(u, v) \cong \langle a, c \mid (a^{uv^2})^{-1}a^{-1}a^v c^u c^{u^2v^2}(c^{u^2v})^{-1},\ (c^{uv^2})^{-1}c^{-1}c^v a^u a^{u^2v^2}(a^{u^2v})^{-1}\rangle$$
$$\tilde{\mathcal{A}}_{k^\sharp}(u, v) \cong \langle a, c \mid c^v a^u c^{u^2v^2}(a^{u^2v})^{-1}(c^{uv^2})^{-1}a^{-1}\ a^v c^u a^{u^2v^2}(c^{u^2v})^{-1}(a^{uv^2})^{-1}c^{-1}\rangle.$$

Note that $\tilde{\mathcal{A}}_{k^\sharp}(1, 1) \cong \langle a, c \mid aca^{-1}c^{-1}a^{-1}c, cac^{-1}a^{-1}c^{-1}a\rangle \cong \langle a, c \mid aca = cac\rangle$, which is the group of the trefoil, while $\tilde{\mathcal{A}}_k(1, 1)$ is trivial. Consequently, the groups

$\tilde{\mathcal{A}}_{k^\sharp}(u,v)$ and $\tilde{\mathcal{A}}_k(v,u)$ are not isomorphic as one might expect from Theorem 5.1 and Corollary 5.2. The operation $\sharp$ appears to be more subtle than $-$ or $*$.

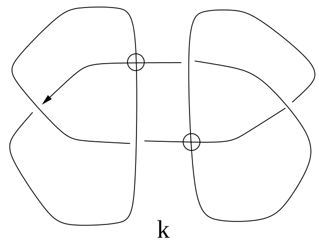

Figure 9. The knot k

Example 6.4. Consider the oriented virtual 2-component link l in Figure 10. We have $\Delta_0(u_1,u_2,v) = (u_2-1)(v-1)(u_1v-1)(u_2v-1)$. From this we see that there is no sequence of generalized Reidemeister moves that interchanges the two components.

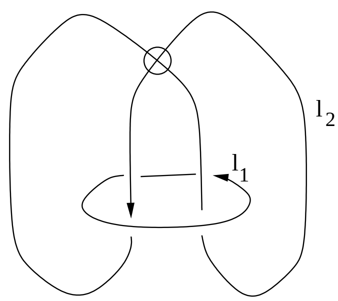

Figure 10. 2-component link l

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