

PERSISTENT INVARIANTS OF TANGLES

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ABSTRACT. Invariants of virtual $2n$ -tangles t are defined using an analog of the Temperley-Lieb algebra. The invariants yield information about the bracket polynomial of any link in which t embeds.

1. Introduction. Assume that t is a 4-tangle that embeds in a link l . The first author proved in [Kr] that if n is an integer that divides both the determinant of the numerator closure $n(t)$ and the determinant of the denominator closure $d(t)$, then n also divides the determinant of l . One might say that the greatest common divisor of $\det n(t)$ and $\det d(t)$ persists in the determinant of l .

The second and third authors have given a short, alternative proof of the result when n is a prime, a proof based on Fox coloring and valid in the larger category of virtual links and tangles [SiWi2]. The determinant of a classical link is $|H_1(M(l); \mathbf{Z})|$, the order of the first homology of the 2-fold cyclic cover of S^3 branched over l , with the convention that this number is zero when the homology group is infinite. Quite recently, D. Ruberman, motivated by the connection with colorings, gave a third argument in the classical category, showing even more: when the determinant of l is nonzero, the torsion subgroups of both $H_1(M(n(t)); \mathbf{Z})$ and $H_1(M(d(t)); \mathbf{Z})$ embed in $H_1(M(l); \mathbf{Z})$.

It is well known that the determinant of a classical link l can be computed using the Kauffman bracket; it is equal to the absolute value of the bracket $\langle l \rangle$ with variable set equal to a primitive eighth root of unity (see section 11 of [Kr], for example). We generalize the main theorem of [Kr] in a new direction, showing that many other bracket-derived invariants of a tangle persist in the invariant of any link in which the link embeds. Our arguments apply in the virtual category as well as the classical.

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2. Virtual tangles and Temperley-Lieb algebra. A $2n$ -tangle, for n a positive integer, consists of n disjoint arcs and any finite number of simple closed curves properly embedded in the 3-ball. We will refer to a 4-tangle simply as a *tangle*. Two $2n$ -tangles are regarded as the same if one can be transformed into the other by an isotopy of the ball that fixes

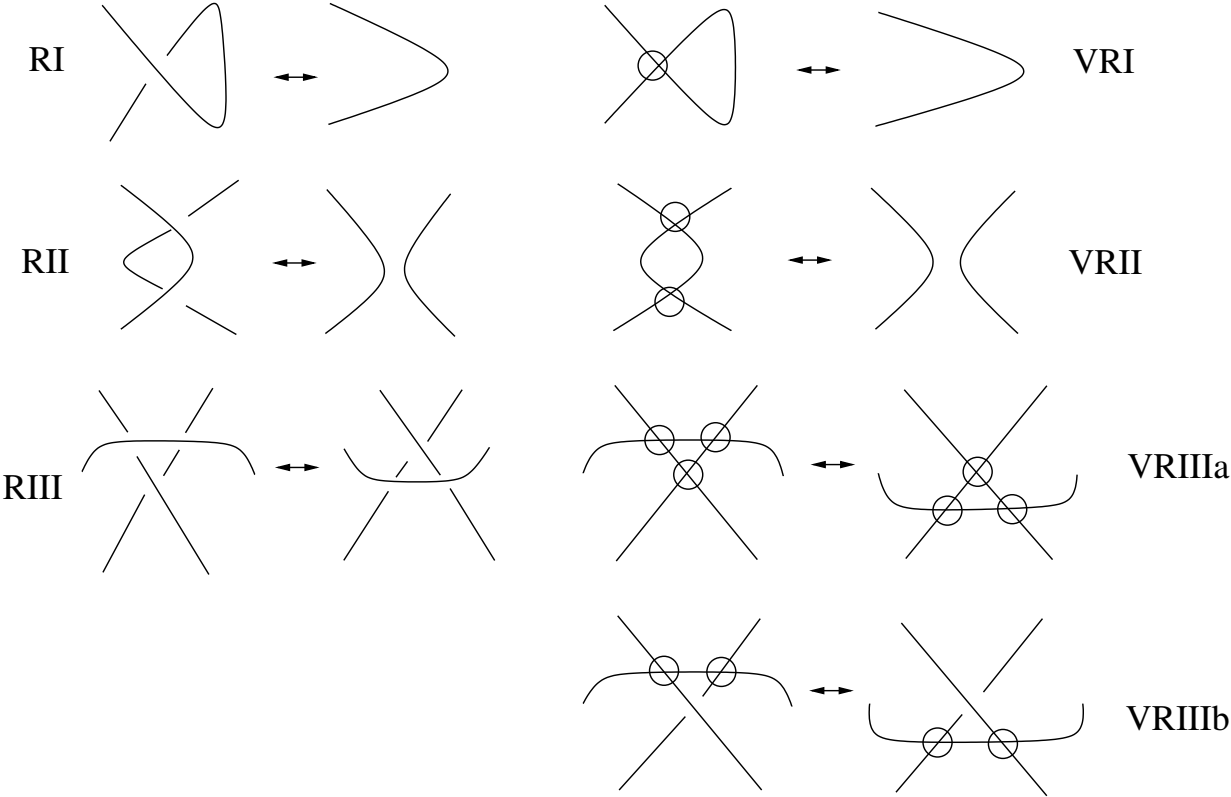
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each point of its boundary. Often we represent $2n$ -tangles by diagrams, as we do for knots and links. Two diagrams represent the same $2n$ -tangle if and only if they are related by a finite number of Reidemeister moves.

In [Ka1] L. Kauffman introduced the concept of a virtual link. Details can be read in [Ka2]. Briefly, a *virtual link* is represented by a diagram that might contain “virtual crossings” as well as the usual, classical kind. Virtual crossings are indicated by a small circle, as in Figure 1.

By definition two diagrams represent the same virtual link if one can be transformed into the other by a finite sequence of generalized Reidemeister moves, shown in Figure 1. Virtual knots generalize classical ones, in the sense that classical diagrams are equivalent under classical Reidemeister moves whenever they are equivalent under generalized ones (see [GuPoVi] or [Ka3]).



Figures 1: Generalized Reidemeister moves

We define a *virtual $2n$ -tangle* as an equivalence class of virtual diagrams, two diagrams being equivalent if one can be transformed into the other by a finite sequence of generalized Reidemeister moves. In our diagrams the $2n$ points where the tangle meets the 3-ball are represented by n free ends on the left side and n free ends on the right side of the diagram. Concatenation of virtual $2n$ -tangle diagrams (see Figure 2) induces the structure of a monoid \mathcal{VT}_n on the set of virtual $2n$ -tangles.

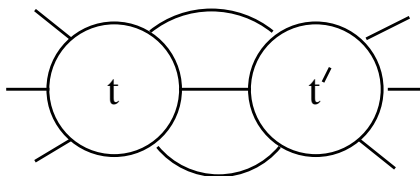


Figure 2: The product of 6-tangle diagrams

A *closure* of a virtual $2n$ -tangle t is a virtual link obtained from a diagram for t by connecting its endpoints with a collection of strands that run outside the diagram and have no classical crossings. When $n = 2$ there are three possible closures: the usual numerator and denominator closures and also a third closure, one that introduces a new virtual crossing. The closure of a virtual $2n$ -tangle that connects each left-hand point to the corresponding point on the right will be called *standard*. Notice that the standard closure of a 4-tangle is just its numerator closure.

As in [SiWi2] we will say that a virtual $2n$ -tangle *embeds* in a virtual link l if some diagram of t extends to a diagram of l .

Lemma 2.1. Assume that a virtual $2n$ -tangle t embeds in a virtual link l . Then l is the standard closure of $t \cdot t'$ for some $t' \in \mathcal{VT}_n$.

Proof. If the diagram for l contains only classical crossings, then a diagram for l of the desired type is easily obtained by dragging arcs, as in the proof of Alexander's Theorem for braids (see page 91 of [Ka4]). The dragging of arcs is accomplished using Reidemeister moves RII and RIII. In the general case, since we are not allowed to drag an arc over a virtual crossing, we use the virtual Reidemeister moves VRII and VRIIIa-b instead. ■

Let Λ denote the ring $\mathbf{Z}[A, A^{-1}]$ of Laurent polynomials in variable A . We denote by \mathcal{VTL}_n the free algebra over Λ generated by all virtual $2n$ -tangle diagrams D modulo:

- (i) $\diagdown = A \diagup + A^{-1} \diagup$;
- (ii) $D \cup \bigcirc = -(A^2 + A^{-2})D$;
- (iii) generalized Reidemeister moves VRI, VRII and VRIIIa-b.

The figures in (i) are skein diagrams in the usual sense: the $2n$ -tangle diagrams differ only in a neighborhood of the indicated crossing. Also, \bigcirc is a simple closed curve in the 3-ball that bounds a disk in the complement of D .

We call \mathcal{VTL}_n the *virtual Temperley-Lieb algebra* (cf. [Ka4], page 99). As a Λ -module it is freely generated by $(2n)!/2^n n!$ virtual $2n$ -tangle diagrams corresponding to the various ways that the $2n$ points on the boundary of the 3-ball can be connected by n arcs without classical crossings. We fix a basis $\{U_i\}$ for \mathcal{VTL}_n .

A virtual $2n$ -tangle diagram determines an element $\sum_i \alpha_i U_i \in \mathcal{VTL}_n$. The coefficients can be found by repeated applications of relations (i) and (ii). Diagrams that differ by finitely many generalized Reidemeister moves other than the first yield the same value. The effect of Reidemeister move RI is multiplication by $-A^{\pm 3}$, a unit in the ring Λ .

It is shown in [Ka2] that the definition of the bracket polynomial $\langle l \rangle$ of a link l can be extended in the virtual category. As usual closed loops are assigned the value $\delta = -A^2 - A^{-2}$. The reader should be aware that the bracket is invariant under all but the first Reidemeister move. Consequently, the function that sends any virtual $2n$ -tangle to the bracket of its standard closure is well defined only up to factors of $-A^{\pm 3}$.

Theorem 2.2. Let $\bar{\Lambda}$ be a quotient ring of Λ with canonical projection $\alpha \mapsto \bar{\alpha}$, and let t be a virtual $2n$ -tangle embedded in a virtual link l . Assume that $\sum_i \alpha_i U_i \in \mathcal{VTL}_n$ is associated to some diagram for t . If $\beta \in \bar{\Lambda}$ divides each coefficient $\bar{\alpha}_i$, then β divides $\overline{\langle l \rangle}$.

Proof. Using Lemma 2.1 we can find a second virtual $2n$ -tangle t' such that l is the standard closure of $t \cdot t'$. Let $\sum_i \alpha'_i U_i$ be associated to a diagram for t' . Then the product $\sum_i \alpha_i U_i \cdot \sum_i \alpha'_i U_i$ can be written uniquely as $\sum_i \alpha''_i U_i$, for some $\alpha''_i \in \Lambda$. Clearly β divides each coefficient $\bar{\alpha}''_i$. The quantity $\overline{\langle l \rangle}$ can be computed from $\sum_i \alpha''_i U_i$ by replacing each U_i with the value $\bar{\delta}^{n_i}$, where n_i is one less than the number of loops in the standard closure of U_i . Since β divides each $\bar{\alpha}''_i$, it also divides $\overline{\langle l \rangle}$. ■

Corollary 2.3. [Kr] Let t be a classical tangle that embeds in a classical link l . If an integer n divides both the determinant of $n(t)$ and the determinant of $d(t)$, then n divides the determinant of l .

Proof. Consider the quotient ring $\bar{\Lambda}$ obtained from Λ by setting the variable A equal to a primitive eighth root of unity. The element of \mathcal{VTL}_2 associated to a classical diagram for t has the form $\alpha_1 U_1 + \alpha_2 U_2$. The basis $\{U_1, U_2, U_3\}$ that we use appears in Figure 3. Since $\bar{\delta} = 0 \in \bar{\Lambda}$, the determinant of $n(t)$ is equal to the absolute value of $\bar{\alpha}_1 + \bar{\alpha}_2 \bar{\delta} = \bar{\alpha}_1$, while the that of $d(t)$ is absolute value of $\bar{\alpha}_1 \bar{\delta} + \bar{\alpha}_2 = \bar{\alpha}_2$. By the previous theorem, n divides the determinant of l . ■

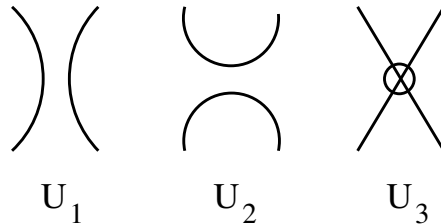


Figure 3: Basis for \mathcal{VTL}_2

3. Examples. We will say that a virtual $2n$ -tangle is *gordian* if it does not embed in a trivial knot. In this section we apply Theorem 2.3 to several tangles, showing in particular that they are gordian.

Example 3.1. Figure 4 displays a classical tangle t_2 . It appears in Figure 13 of [Kr] (top middle picture). The determinants of numerator and denominator closures are relatively prime, and so techniques of that paper cannot dismiss the possibility that t_1 is gordian. Indeed this example was the initial motivation for our efforts.

A routine calculation shows that the element of \mathcal{VTL}_2 associated to this diagram is $\alpha_1 U_1 + \alpha_2 U_2$, where $\alpha_1 = (-1 + 3A^4 - 5A^8 + 6A^{12} - 5A^{16} + 3A^{20} - 2A^{24})A^{-18}$ and $\alpha_2 = ((1 - A^4 + A^8)(1 - 2A^4 + A^8 - 2A^{12} + A^{16}))A^{-12}$. (A technical lemma below expedites the calculation.) Consider the ring homomorphism $\Lambda \rightarrow \bar{\Lambda} = \mathbf{Z}[\sqrt[4]{5}, 1/\sqrt[4]{5}]$ that sends A to $\sqrt[4]{5}$. We find that $\bar{\alpha}_1 = -17 \cdot 1433 \cdot 5^{-9/2}$ while $\bar{\alpha}_2 = 17 \cdot 483 \cdot 5^{-3}$. By Theorem 2.2 the integer 17 must divide $\langle l \rangle$ whenever t_1 embeds in l .

If t_1 embeds in the trivial knot, then $17 \cdot \gamma = (-\sqrt[4]{5})^{3w}$ for some γ in $\bar{\Lambda}$, where w is the writhe of the diagram for the trivial knot in which the tangle embeds (the sum of signs of the classical crossings). Applying the norm map $N : \mathbf{Q}[\sqrt[4]{5}] \rightarrow \mathbf{Q}$, we find $N(17)N(\gamma) = N[(-\sqrt[4]{5})^{3w}]$ which implies that $17^4 g = 5^{3w}$, for some rational number $g \in \mathbf{Z}[1/5]$, an impossibility. Hence t_1 is not gordian. (Background about norms and field extensions can be found in [Is].)

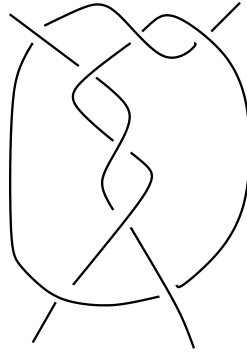


Figure 4: The tangle t_1 .

Lemma 3.2. Consider a virtual diagram that contains a pair of strands with n half-twists. (The integer n is positive if the crossings are positive when the strands are oriented in the same direction; otherwise negative.) Then

$$\boxed{n} = A^n \smile + [|n|A^{\epsilon(n-2)} + g_n \delta] (,$$

for some $g_n \in \Lambda$. Here ϵ is the algebraic sign of n .

Proof. The tangle $\llbracket n \rrbracket$ is equal to $[A^\epsilon \smile + A^{-\epsilon}]^{|n|}$, in the Temperley-Lieb algebra \mathcal{VTL}_2 . The proof is completed by expanding the last expression. ■

Example 3.3. An example of a virtual tangle appears in Figure 5. A short calculation using Lemma 3.2 shows that the element of \mathcal{VTL}_2 associated to the diagram is $\alpha_2 U_2 + \alpha_3 U_3$, where $\alpha_2 = -A^{-4} + 3 - A^4$ and $\alpha_3 = -A^{-6} + A^{-2} + A^2 - A^6$. Under the mapping $\Lambda \rightarrow \mathbf{Z}[e^{\pi i/4}]$ that sends A to $e^{\pi i/4}$, we find that $\bar{\alpha}_2 = 5$ while $\bar{\alpha}_3 (= \bar{\alpha}_1) = 0$. A norm argument as in Example 3.1 establishes that t_2 is gordian.

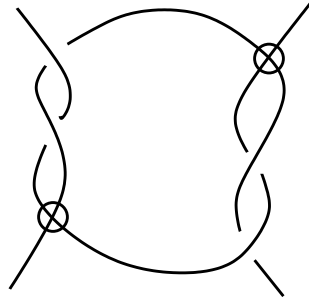


Figure 5: The tangle t_2 .

Remark. The determinant of the numerator tangle $n(t_2)$ (see section 1) is 4, while $\langle n(t_2) \rangle$ evaluated at $A = e^{\pi i/4}$ is 0. Example 3.3 shows that the determinant of a virtual link need not be equal to the absolute value of its bracket evaluated at a primitive eighth root of unity, as is the case for classical links. This anomaly might have an interesting underlying cause, but we do not know what it might be.

Example 3.4. Figure 6 shows a classical $2n$ -tangle $t(m_1, \dots, m_n)$, where m_1, \dots, m_n are arbitrary integers, numbers of half-twists. (The twist number m_j is positive if the crossings of the twist are positive when the strands are coherent oriented.) The tangle $t(3, -3)$, sometimes referred to as the granny tangle, appears in [Kr], where it is shown that 3 divides the determinant of any classical link in which it embeds. We generalize this result. Note that the order of the m_j does not affect the conclusion of Proposition 3.5.

Proposition 3.5. Assume that $t(m_1, \dots, m_n)$ embeds in a virtual link l . Then the greatest common divisor of m_1, \dots, m_n divides $|\langle l \rangle|$ evaluated at $A = e^{\pi i/4}$.

Proof. Assume that $\sum_i \alpha_i U_i$ is the element of \mathcal{VTL}_{2n} associated to the diagram of $t(m_1, \dots, m_n)$. Using Lemma 3.3 we resolve the twists in the diagram. Each coefficient α_i is divisible by some m_j or else by δ . Consider the mapping $\Lambda \rightarrow \mathbf{Z}[e^{\pi i/4}]$ which sends A to the primitive eighth root of unity $e^{\pi i/4}$. Since the image of δ vanishes, each reduced

coefficient $\bar{\alpha}_i$ is divisible by the greatest common divisor of m_1, \dots, m_n . Theorem 2.2 completes the proof. ■

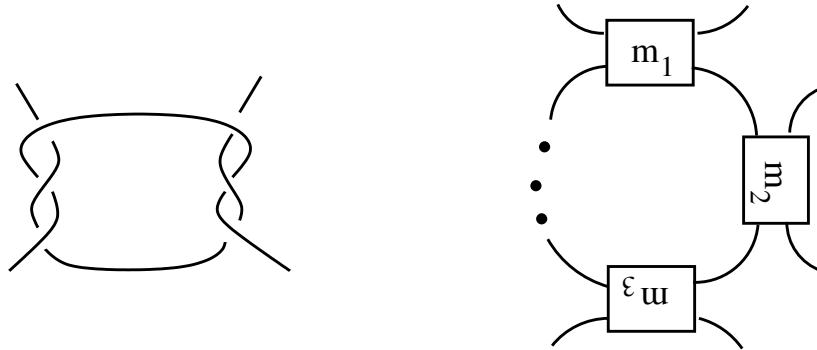


Figure 6: The tangles $t(3, -3)$ and $t(m_1, \dots, m_n)$

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