ON GROUPS WITH UNCOUNTABLY MANY SUBGROUPS OF FINITE INDEX

Daniel S. Silver
Susan G. Williams

ABSTRACT. Let K be the kernel of an epimorphism $\chi : G \to \mathbf{Z}$, for G a finitely presented group. If K has uncountably many normal subgroups of finite index r, then K has uncountably many subgroups (not necessarily normal) of any finite index greater than r. In particular, this is the case whenever G is subgroup separable and K is nonfinitely generated. Assume that G has an abelian HNN base contained in K. If K has infinitely many subgroups of a given finite index, then it has uncountably many.

1. Introduction. By a well-known theorem of M. Hall [**Ha**], a finitely generated group contains at most finitely many subgroups of a given finite index. In contrast, a free group of infinite rank contains uncountably many subgroups of *every* finite index greater than 1. An arbitrary group that contains uncountably many subgroups of a fixed finite index need not contain subgroups of every sufficiently large finite index, as the following easy example shows.

Example 1.1. Let K be the direct sum of countably many copies of the cyclic group $\mathbf{Z}/2$. Since K is abelian, every subgroup H is normal in K and has index equal to the order of the quotient K/H. Any finite quotient of K has order 2^k, for some $k \geq 0$. By constructing homomorphisms from K onto $\mathbf{Z}/2 \oplus \ldots \oplus \mathbf{Z}/2$ (k copies) we see that K contains uncountably many subgroups of index 2^k, for any $k \geq 1$, but that it contains no proper subgroups of any other finite index.

We consider a special class of countable groups, those groups K that are kernels of epimorphisms $\chi : G \to \mathbf{Z}$, where G is finitely presented. Such kernels play important roles in geometric topology, for example as commutator subgroups of knot groups, yet many questions about their structure remain unanswered. Our first result is the following.

Theorem 1.2. Let K be the kernel of an epimorphism $\chi : G \to \mathbf{Z}$, where G is a finitely presented group. If K contains uncountably many normal subgroups of finite index r, then K contains uncountably many subgroups (not necessarily normal) of any finite index greater than r.

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Corollary 1.3. Under the hypotheses of Theorem 1.2, if K contains uncountably many subgroups of finite index r then K contains uncountably many subgroups of any index greater than or equal to $r!$

The hypothesis of Theorem 1.2 that G is finitely presented cannot be replaced by the weaker assumption that G is finitely generated. To see this, consider the wreath product $G = \mathbf{Z} \wr \mathbf{Z}/2$, a group that is finitely generated but not finitely presented according to Theorem 1 of [**Ba1**]. The group G has presentation

$$\langle x,\ a \mid [a, x^{-k}ax^k] = a^2 = e, \quad \forall k \in \mathbf{Z}\rangle.$$

Let $\chi : G \to \mathbf{Z}$ be the epimorphism determined by $x \mapsto 1$ and $a \mapsto 0$. By the Reidemeister-Schreier Theorem (see [**LySc**], for example) the kernel of χ has presentation

$$\langle a_j \mid [a_j, a_{j+k}] = a_j^2 = e, \quad \forall j, k \in \mathbf{Z}\rangle,$$

and it is isomorphic to the group K in Example 1.1, a group that has uncountably many subgroups of finite index r only for $r = 2^k,\ k > 0$.

The proof of Theorem 1.2 is based on new graph techniques introduced in [**SiWi1**] and motivated by symbolic dynamics.

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2. Subgroups of finite index and permutation representations. It is well known that for any group K, there is a finite-to-one correspondence between permutation representations $\rho : K \to S_r$, where S_r is the symmetric group on $\{1, \ldots, r\}$, and subgroups H of K having index no greater than r. The correspondence is described by $\rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}$. The preimage of a subgroup H of index r consists of $(r-1)!$ transitive representations. By *transitive* representation we mean a representation $\rho : K \to S_r$ such that $\rho(K)$ operates transitively on $\{1, \ldots, r\}$. It follows that K contains finitely (resp., countably, uncountably) many subgroups of index r if and only if $\mathrm{Hom}(K, S_r)$ contains finitely (resp., countably, uncountably) many transitive representations.

If G is a finitely presented group with epimorphism $\chi : G \to \mathbf{Z}$, then G can be described as an HNN extension $\langle x, B \mid x^{-1}ax = \phi(a), \quad \forall a \in U\rangle$, such that $\chi(x) = 1,\ B$ is a finitely generated subgroup of $K = \ker\, \chi$, and $U,\ V$ are isomorphic finitely generated subgroups of B with isomorphism $\phi : U \xrightarrow{\sim} V$ (see [**BiSt**]). The subgroup B is a finitely generated HNN *base*. It is possible to choose B so that it contains any prescribed finite subset of K (see [**Si**], for example).

Conjugation by x induces an automorphism of K. Letting $B_j = x^{-j}Bx^j$, $U_j = x^{-j}Ux^j$ and $V_j = x^{-j}Vx^j$, $j \in \mathbf{Z}$, the kernel K can be described as an infinite amalgamated free product

$$K = \langle B_j \mid V_j = U_{j+1}, \quad \forall j \in \mathbf{Z} \rangle.$$

Assume that Σ is a finite group. The set $\mathrm{Hom}(K, \Sigma)$ can be described by a finite directed graph Γ. The vertex set consists of all representations $\rho_0 : U \to \Sigma$, a set which is finite since U is finitely generated. If $\overline{\rho}_0$ is a representation from B to Σ, then we draw a directed edge labeled by $\overline{\rho}_0$ from the vertex $\rho_0 = \overline{\rho}_0|_U$ to the vertex $\rho_0' = \overline{\rho}_0|_V \circ \phi$. Consider a bi-infinite path in Γ given by the edge sequence

$$\cdots \overline{\rho}_{-2}\ \overline{\rho}_{-1}\ \overline{\rho}_0\ \overline{\rho}_1\ \overline{\rho}_2 \cdots$$

The representations from $B_j \to \Sigma$ defined by $y \mapsto \overline{\rho}_j(x^j y x^{-j})$ have a unique common extension $\rho : K \to \Sigma$. Thus the bi-infinite paths of Γ correspond to the elements of $\mathrm{Hom}(K, \Sigma)$.

We will always "prune" the graph Γ, removing any vertex that has no edges leading from it. We discard the vertex togther with any edges that end at the vertex. Similarly, we remove any vertex with no edges leading to it, together with any edges that begin at the vertex. After iterating this process sufficiently many times, we obtain a smaller graph Γ with the property that every edge is part of some bi-infinite path. Such a graph has finitely many bi-infinite paths if and only if it is a collection of disjoint cycles.

In [**SiWi1**] we described an algorithm, based on the Reidemeister-Schreier Theorem, for constructing Γ. Of course, Γ is not uniquely determined; it depends on the presentation for G as well as the steps we use to present K. Nevertheless, there is a surprising amount of information in Γ that depends only on G, χ and x. We used Γ in [**SiWi1**] to prove that $\mathrm{Hom}(K, \Sigma)$ together with the bijection σ_x defined by

$$(\sigma_x \rho)(a) = \rho(x^{-1}ax), \quad \forall a \in K$$

has the structure of a *shift of finite type*, a type of dynamical system with attractive properties (see [**LiMa**]). Following [**SiWi1**], [**SiWi2**] we will call $(\mathrm{Hom}(K, \Sigma), \sigma_x)$ a *representation shift.* The representation shift is uniquely determined up to topological conjugacy. Readers interested in the general theory and applications of representation shifts should consult these papers or [**SiWi3**], which provides an overview.

Example 2.2. The group $G = \langle x,\ a_0,\ a_1 \mid [a_0, a_1] = e,\ x^{-1}a_0x = a_1 \rangle$ is an HNN extension with free abelian base $B = \langle a_0,\ a_1 \mid [a_0, a_1] = e \rangle$ and infinite cyclic subgroups $U = \mathrm{gp}(a_0)$ and $V = \mathrm{gp}(a_1)$. Consider the epimorphism $\chi : G \to \mathbf{Z}$ given by $\chi(x) = 1,\ \chi(a_0) = \chi(a_1) = 0$. The kernel K has presentation

$$K = \langle a_j \mid [a_j, a_{j+1}] = e, \quad \forall j \in \mathbf{Z} \rangle,$$

where $a_j = x^{-j} a_0 x^j$. If we wish to find the subgroups of K having index 2, then we let Σ be the symmetric group S_2. The graph Γ has 2 vertices, corresponding to the trivial representation ρ_0 and the nontrivial representation ρ_0' from U to S_2. There are four edges, which we can label by ordered pairs of elements of $S_2 = \{e, \pi\}$; the label (e, π), for example, describes the representation $\overline{\rho}_0 : B \to S_2$ that maps a_0 to the identity permutation e, while mapping a_1 to the nontrivial permutation π. Every bi-infinite path in Γ that includes an edge labeled by a nontrivial representation of B corresponds to a transitive representation of K. For example, the path

$$\ldots \to (e,\pi) \to (\pi,e) \to \underline{(e,\pi)} \to (\pi,e) \to (e,\pi) \to \ldots,$$

with initial vertex underscored, corresponds to the representation $\rho : K \to S_2$ that maps each a_{2j} to the identity element and each a_{2j+1} to π. There are uncountably many bi-infinite paths in Γ. From this and the remarks in Section 1, we see that K contains uncountably many subgroups of index 2. A similar analysis, replacing S_2 by S_3, would show that K contains uncountably many subgroups of index 3. However, the following lemma, used informally in [**SiWi1**], quickly shows that K contains uncountably many subgroups of every index $r \geq 2$.

The representation ρ in Example 2.2 is periodic. In general, a representation $\rho : K \to \Sigma$ is *periodic* if $\sigma_x^d \rho = \rho$, for some positive integer d. We say that ρ has *period d*.

Lemma 2.3. (**Substitution Lemma.**) *For $R > 1$, assume that $(\mathrm{Hom}(K, S_R), \sigma_x)$ contains a periodic transitive representation ρ such that every permutation in $\rho(U_0)$ fixes the symbol R. Then $\mathrm{Hom}(K, S_r)$ contains uncountably many transitive representations for every $r > R$.*

Proof. Let C be the cycle in Γ that describes the periodic representation ρ. The hypothesis implies that each permutation of $\rho(U_0)$ fixes the symbol R. However, a permutation in some group $\rho(B_j)$ must move R, since ρ is transitive. Adjoin $R+1$ to the set of permuted symbols. The representation ρ' such that $\rho'(g)$ is $\rho(g)$ conjugated by the transposition $(R, R+1)$, for all $g \in K$, corresponds to a cycle C' distinct from C but sharing its initial vertex $\rho|_{U_0}$ with it. The permutations in the groups $\rho'(B_j)$ are those of $\rho(B_j)$ with the symbol $R+1$ substituted everywhere for R. Any bi-infinite path that travels around both C and C' at least once describes a transitive representation in $\mathrm{Hom}(K, S_{R+1})$. There are uncountably many such paths. Repeating the procedure with new cycles $C'', C''', \ldots$, and new symbols $R+2, R+3, \ldots$, completes the proof. ∎

Proof of Theorem 1.2. Assume that K contains uncountably many normal subgroups of some fixed finite index r. Uncountably many of the corresponding quotient groups must be isomorphic to the same finite permutation group Σ. Since two representations of K onto Σ have the same kernel if and only if they differ by an automorphism of Σ, there exist uncountably many representations of K onto Σ.

We begin by proving that K admits a periodic representation ρ into $\Sigma \times \Sigma$ such that the sequence of subgroups $\rho(B_j)$ is nonconstant. Let Γ be a graph as above that describes $\mathrm{Hom}(K, \Sigma)$. Since Γ contains uncountably many bi-infinite paths, it must contain two cycles branching from a common vertex. By replacing one of the cycles by the concatenation of the two and then rechoosing initial vertices, we obtain cycles $C = \{e_1, \ldots, e_m\}$ and $C' = \{e'_1, \ldots, e'_n\}$ such that edges e_1 and e'_1 are equal but e_2 and e'_2 are not. The cycle C describes some periodic representation τ, while C' describes a periodic representation τ'. Consider the periodic representation $\rho : K \to \Sigma \times \Sigma$ defined by $a \mapsto (\tau(a), \tau'(a))$. Since $\rho(B_0)$ lies on the diagonal $\Delta \subset \Sigma \times \Sigma$ while $\rho(B_1)$ does not, the sequence of subgroups $\rho(B_j)$ is nonconstant.

Regard $\Sigma \times \Sigma$ as a permutation group acting by (right) multiplication on the r cosets of Δ. The subgroup $\rho(K)$ acts transitively on some subset containing Δ. List the elements of the subset as $\{1, \ldots, R\}$, where R corresponds to Δ. Note that $R > 1$ since $\rho(B_1)$ is not contained in Δ. However, since $\rho(B_0)$ is contained in Δ, every permutation of $\rho(B_0)$ and hence every permutation of $\rho(U_0)$ fixes the symbol R. Lemma 2.3 completes the proof. ∎

Proof of Corollary 1.3. Assume that K contains uncountably many subgroups of some finite index r. Then K admits uncountably many transitive representations into S_r. Among the kernels of the representations there exist uncountably many subgroups of some index no greater than $r!$. Theorem 1.2 finishes the argument. ∎

Corollary 2.4. *(i)* K has infinitely many subgroups of some finite index if and only if K admits a representation ρ onto a finite group such that the sequence of subgroups $\rho(B_j)$ is nonconstant.

(ii) K has uncountably many subgroups of finite index if and only if K admits a periodic representation ρ onto a finite group such that the sequence of subgroups $\rho(B_j)$ is nonconstant.

Proof. The proof of Theorem 1.2 shows that K has uncountably many subgroups of some finite index if and only if K admits a periodic representation ρ onto a finite group such that the sequence of subgroups $\rho(B_j)$ is nonconstant. This proves statement *(ii)*.

If K admits a representation ρ, not necessarily periodic, onto a finite group such that the sequence of subgroups $\rho(B_j)$ is not constant, then we claim that K contains infinitely many subgroups of some finite index. By statement *(ii)* this is so if ρ is periodic. If ρ is not periodic, then the representation shift is infinite and the claim follows.

Conversely, if K contains infinitely many subgroups of finite index r, then a graph describing the representation shift $(\mathrm{Hom}(K, S_r), \sigma_x)$ branches. As in the proof of Theorem 1.2, we can find a (possibly nonperiodic) representation ρ of K in $S_r \times S_r$ such that the sequence of subgroups $\rho(B_j)$ is nonconstant. ∎

For any positive integer r, define $\hat{K}_r$ to be the quotient of K modulo the relations $x^{-r}ax^r = a$, for all $a \in K$. Then the representations of $\hat{K}_r$ into any group Σ correspond naturally to the periodic representations $\rho : K \to \Sigma$ with period r (see Section 4 of [**SiWi2**] for details). Corollary 2.4 implies that knowledge about the representations of the *finitely presented* groups $\hat{K}_r$ is sufficient to decide whether K contains uncountably many subgroups of finite index. In the case that K is the commutator subgroup of the group of a knot $k \subset S^3$, the group $\hat{K}_r$ is isomorphic to the fundamental group of the r-fold cyclic cover of S^3 branched over k.

3. Abelian HNN bases. In an earlier paper we described an example, due to K.H. Kim and F. Roush, of a nonabelian group K that has infinitely many subgroups of index 5 but not uncountably many (see Example 3.2 of [**SiWi2**]). In contrast, S. Ivanov has observed the following proposition [**Iv**].

Proposition 3.1. Let J be an abelian group. If J contains infinitely many subgroups of a fixed finite index, then J contains uncountably many.

Proof. Suppose that J contains infinitely many subgroups of index r. Then J admits infinitely many representations onto some abelian group A of order r. We can find infinitely many representations of J onto a factor $\mathbf{Z}/p^n$ of A, where p is a prime. An easy induction argument shows that $\mathrm{Hom}(J, \mathbf{Z}/p^n)$ is finite (resp., countable, uncountable) if and only if $\mathrm{Hom}(J, \mathbf{Z}/p)$ is. Hence there are infinitely many representations $\rho_j : J \to \mathbf{Z}/p$. It follows that $N = \cap_j \ker \rho_j$ has infinite index in J; the quotient J/N is an infinite-dimensional vector space over $\mathbf{Z}/p$, a group that admits uncountably many homomorphisms onto $\mathbf{Z}/p$. Hence $\mathrm{Hom}(J, \mathbf{Z}/p)$ is uncountable, and therefore so is $\mathrm{Hom}(J, \mathbf{Z}/p^n)$. It follows easily that $\mathrm{Hom}(J, A)$ is uncountable. Thus J contains uncountably many subgroups of index r. ∎

Corollary 3.2. Let G be a finitely presented group, and $\chi : G \to \mathbf{Z}$ an epimorphism. If the kernel K of χ admits infinitely many representations onto a finite abelian group A, then it admits uncountably many. This occurs if and only if the representation shift $(\mathrm{Hom}(K, A), \sigma_x)$ contains a periodic element ρ such that the sequence of groups $\rho(B_j)$ is nonconstant.

Proof. Let J be the abelianization of K. Since every abelian representation of K factors through J, the hypothesis of Corollary 3.2 implies that J admits infinitely many representations onto A; the proof of Proposition 3.1 shows that J admits uncountably many. Composing with the abelianization map from K to J produces uncountably many representations from K onto A.

The first statement of Corollary 3.2 also follows from a theorem of B. Kitchens about Markov shifts. Details can be found in Theorem 3.3 of [**SiWi2**]. There it is shown that $(\mathrm{Hom}(K, A), \sigma_x)$ is infinite if and only if the loop in the graph Γ corresponding to the trivial representation is nonisolated. Consequently, if the representation shift is infinite, then we can find a periodic element ρ such that the sequence of groups $\rho(B_j)$ contains the trivial group as well as some nontrivial group.

Conversely, suppose that $(\mathrm{Hom}(K, A), \sigma_x)$ contains a periodic element ρ such that $\{\rho(B_j)\}$ is nonconstant. After replacing ρ by $\sigma_x^k\rho$, for suitable k, we can assume that $\rho(B_0)$ is minimial in $\{\rho(B_j)\}$ with respect to inclusion. Composing each $\rho|_{B_j}$ with the natural projection of A onto $\overline{A} = A/\rho(B_0)$, we obtain a nontrivial periodic representation $\overline{\rho}$ of K into the finite abelian group $\overline{A} = A/\rho(B_0)$. In the graph describing $(\mathrm{Hom}(K, \overline{A}), \sigma_x)$ the loop corresponding to the trivial representation shares a vertex with the loop corresponding to $\overline{\rho}$. Consequently, $(\mathrm{Hom}(K, \overline{A}), \sigma_x)$ is uncountable, and hence K admits infinitely many representations into $\overline{A}$. Since $\overline{A}$ embeds in A, it follows that $(\mathrm{Hom}(K, A), \sigma_x)$ is uncountable. ∎

We conclude this section by showing that for groups of a special class including many nonabelian groups, the existence of infinitely many subgroups with a fixed finite index again implies the existence of uncountably many.

Theorem 3.3. Let G be a finitely presented group, and $\chi : G \to \mathbf{Z}$ an epimorphism. Assume that G has an abelian HNN base contained in the kernel K of χ. Let Σ be a finite group. If $\mathrm{Hom}(K, \Sigma)$ is infinite, then it is uncountable.

Corollary 3.4. Under the hypotheses of Theorem 3.3, if K contains infinitely many subgroups of a fixed finite index r, then it contains uncountably many.

Proof of Theorem 3.3. The main ideas for our proof were supplied by K.H. Kim and F. Roush.

Assume that G has an abelian HNN base. By Lemma 2.2 of [**Yo**], G has a finitely generated abelian base B, and we can use B to construct a finite graph Γ that describes the representation shift $(\mathrm{Hom}(K, \Sigma), \sigma_x)$, as in Section 2.

Since $\mathrm{Hom}(K, \Sigma)$ is infinite, there exists a nonperiodic representation $\rho : K \to \Sigma$. Each image $\rho(B_j)$ is abelian and so can be expressed as the direct sum of its p-primary components. Composing each $\rho|_{B_j}$ with the projection of B_j onto its p-primary component considered as a subgroup of Σ, for fixed p, produces a set of homomorphisms that again defines a representation of K; we can choose p such that the representation will again be nonperiodic. Thus we can assume without any loss of generality that each group $\rho(B_j)$ is an abelian p-group, for some fixed prime p.

If some $\rho(B_i)$ contains all of the subgroups $\rho(B_j)$, then ρ is a nonperiodic representation into $\rho(B_i)$. This means that $\mathrm{Hom}(K, \rho(B_i))$ is infinite and hence uncountable by Corollary 3.2. Then $\mathrm{Hom}(K, \Sigma)$ is uncountable as well.

If the $\rho(B_j)$ are not contained in a single $\rho(B_i)$, then since Σ is finite we can find an integer k for which $\rho(B_k)$ does not contain $\rho(B_{k+1})$. Replacing ρ by $\sigma_x^k \rho$, we may assume henceforth that $\rho(B_0)$ does not contain $\rho(B_1)$.

Let p^n be the maximum exponent of the groups $\rho(B_j)$. First we consider the case $n = 1$. Every group $\rho(B_j)$ is a finite-dimensional vector space over $\mathbf{Z}/p$. We will modify ρ to obtain distinct representations ρ', ρ'' of K into $\rho(B_0)$ with $\rho'|_{B_0} = \rho''|_{B_0}$. This implies that any graph describing the representation shift $(\mathrm{Hom}(K, \rho(B_0)), \sigma_x)$ branches. Hence $\mathrm{Hom}(K, \rho(B_0))$ is infinite, and we are done as above.

We construct ρ' so that it agrees with ρ on B_0. Choose a homomorphism $f_1 : \rho(B_1) \to \rho(B_0)$ such that $f_1(g) = g$ for all $g \in \rho(B_0) \cap \rho(B_1)$, and define $\rho'|_{B_1}$ to be equal to $f_1 \circ \rho|_{B_1}$. Next choose a homomorphism $f_2 : \rho(B_2) \to \rho(B_0)$ such that $f_2(g) = f_1(g)$ for all $g \in \rho(B_1) \cap \rho(B_2)$, and define $\rho'|_{B_2}$ to be equal to $f_2 \circ \rho|_{B_2}$. Continue the procedure for every $j > 0$, and then repeat it for every $j < 0$. Note that by our assumption that $\rho(B_0)$ does not contain $\rho(B_1)$, there is more than one choice for f_1. We construct ρ'' in the same manner as ρ', using a different choice for f_1.

Now suppose that the maximum exponent of the groups $\rho(B_j)$ is p^n, with $n > 1$. Let $p_j : \rho(B_j) \to \rho(B_j)$ be the homomorphism described by $g \mapsto p^{n-1} \cdot g$. (For convenience, we will use additive notation whenever we work in an abelian subgroup.) The set of homomorphisms $p_j \circ \rho|_{B_j}$ determines a nontrivial representation $\rho^\flat$ of K such that each image $\rho^\flat(B_j)$ is an abelian p-group of exponent p. If $\rho^\flat$ is nonperiodic then $\mathrm{Hom}(K, \Sigma)$ is uncountable by the case for $n = 1$, and so we may assume that $\rho^\flat$ is periodic. If some $\rho^\flat(B_i)$ is trivial, then replacing $\rho^\flat|_{B_j}$ by the trivial homomorphism for all $j < i$ yields a nonperiodic representation. If the groups $\rho^\flat(B_j)$ are nontrivial and not identical, then (replacing $\rho^\flat$ by some $\sigma_x^k \rho^\flat$) we can assume that $\rho^\flat(B_0)$ does not contain $\rho^\flat(B_1)$. This is enough to apply our construction for the case $n = 1$.

All that remains is the case in which the groups $\rho^\flat(B_j)$ are identical (and nontrivial). In this case, $\rho^\flat(B_0)$ is contained in $\rho(B_{j-1}) \cap \rho(B_j)$ for all j. We modify ρ to get a new representation $\rho' : K \to \Sigma$ as follows. Let ν_1 be the natural projection of $\rho(B_1)$ onto $\rho(B_1)/\rho(B_0) \cap \rho(B_1)$. Choose a homomorphism $h_1 : \rho(B_1)/\rho(B_0) \cap \rho(B_1) \to \rho^\flat(B_0)$. Since $\rho^\flat(B_0)$ is contained in $\rho(B_0) \cap \rho(B_1)$, the mapping $f_1 : \rho(B_1) \to \rho(B_1)$ given by $f_1(g) = g + h_1 \circ \nu_1(g)$ is an automorphism. Replace $\rho|_{B_1}$ by $f_1 \circ \rho|_{B_1}$. In a similar way, choose any homomorphism $h_2 : \rho(B_2)/\rho(B_1) \cap \rho(B_2) \to \rho^\flat(B_0)$, and let f_2 be the automorphism of $\rho(B_2)$ defined by $f_2(g) = g + h_2 \circ \nu_2(g)$, where ν_2 is the natural projection of $\rho(B_2)$ onto $\rho(B_2)/\rho(B_1) \cap \rho(B_2)$. Replace $\rho|_{B_2}$ by $f_2 \circ \rho|_{B_2}$. Continue this procedure for every positive (but not negative) index j, thereby constructing a new representation. Our initial assumption that $\rho(B_0)$ does not contain $\rho(B_1)$ ensures that the quotient $\rho(B_1)/\rho(B_0) \cap \rho(B_1)$ is nonzero and so h_1 can be chosen nontrivially. Hence the difference of the new representation and ρ is a nonperiodic representation such that the image of every B_j has exponent p. The argument for the case $n = 1$ completes the proof. ∎

Proof of Corollary 3.4. If K contains infinitely many subgroups of index r, then there exists a nonperiodic transitive representation $\rho : K \to S_r$. The proof of Theorem 3.3 shows that the graph Γ describing $\mathrm{Hom}(K, S_r)$ contains two cycles branching from a common vertex corresponding to $\rho|U_0$. The bi-infinite path describing ρ also contains the vertex. Using the path and the two cycles, it is easy to describe uncountably many transitive representations in $\mathrm{Hom}(K, S_r)$. ∎

4. Profinite topologies and topological criteria.

Many of the ideas above can be formulated in the language of profinite topology. Regard the set $\mathcal{N}$ of all finite-index subgroups of G as a basis of neighborhoods of the identity element. If $g \in G$, then the cosets Vg, where $V \in \mathcal{N}$, form a basis of neighborhoods for g. The resulting *profinite* or *Hall* topology makes G into a topological group. A subgroup of G contains an open set if and only if it is both open and closed. If the intersection of all members of $\mathcal{N}$ is trivial (i.e., if G is *residually finite*), then G is a Hausdorff space. Proofs of these statements are not difficult. Details can be found in [**Ha**].

We can give K the relative topology induced by G. A basis $\mathcal{N}'$ of neighborhoods of the identity element in K consists of those finite-index subgroups W such that $W = V \cap K$ for some $V \in \mathcal{N}$. Clearly conjugation by x induces a topological automorphism of K.

If $\rho : K \to S_r$ is any periodic representation, then it follows from Proposition 3.7 of [**SiWi1**] that $W = \{g \in K \mid \rho(g)(1) = 1\}$ is a member of $\mathcal{N}'$. The reason is that ρ extends over $K_{\chi,d} = \ker[\, G \xrightarrow{\chi} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}/d \,]$, where d is the period of ρ, and π is the natural quotient homomorphism. Since $K_{\chi,d}$ has finite index in G, the subgroup $V = \{g \in K_{\chi,d} \mid \rho(g)(1) = 1\}$ also has finite index in G. Intersecting this subgroup with K yields W. Clearly every member of $\mathcal{N}'$ can be obtained in this manner. It follows easily that every periodic representation ρ is continuous.

A topological group is *topologically finitely generated* if it contains a dense subgroup that is finitely generated in the usual sense.

Theorem 4.1. Let K be the kernel of an epimorphism $\chi : G \to \mathbf{Z}$, where G is a finitely presented group. Regard K as a subspace of G, endowed with the profinite topology.

Then K contains at most countably many subgroups of finite index if and only if K is topologically finitely generated.

Proof. Suppose that K contains a dense finitely generated subgroup D. We can choose a finitely generated HNN base B containing D in order to construct various representation shifts. Since each B_j is the image of B_0 by a topological automorphism, each B_j is dense in K. If ρ is a periodic representation, then ρ is a continuous map of K onto a finite discrete space; hence the images $\rho(B_j)$ are constant, equal to $\rho(K)$. By the Corollary 2.4, K contains at most countably many subgroups of finite index.

If K is not topologically finitely generated, then the closure of B is not all of K. As in the proof of Theorem 3.4 of [**Ha**] there exists a a subgroup $V \in \mathcal{N}'$ such that $B \subset V$ and $V \neq K$. By previous remarks there exists a transitive periodic representation $\rho : K \to S_R$, for some R, such that $V = \{g \in K \mid \rho(g)(R) = R\}$. The Substitution Lemma (Lemma 2.3) implies that K contains uncountably many subgroups of finite index. ■

Example 4.2. Let m and n be integers. The Baumslag-Solitar groups $G(m,n) = \langle x, a \mid xa^m x^{-1} = a^n \rangle$, introduced in [**BaSo**], have been a rich source of examples. In [**SiWi1**] we defined $\chi : G(m,n) \to \mathbf{Z}$ to be the epimorphism mapping x to 1 and a to 0. By examining the representation shifts for the kernel $K(m,n) = \langle a_j \mid a_j^m = a_{j+1}^n \rangle$ we proved that $K(m,n)$ has finitely many subgroups of a given finite index if and only if g.c.d.$(m,n) = 1$. (As Theorem 3.3 predicts, $K(m,n)$ has uncountably many subgroups of finite index when g.c.d.$(m,n) \neq 1$.) Hence $K(m,n)$ is topologically finitely generated if and only if m and n are relatively prime.

The group $K(1,2)$ is isomorphic to the group $\mathbf{Z}[1/2]$ of dyadic rational numbers. Since this group is not finitely generated, the phrase "topologically finitely generated" cannot be replaced by "finitely generated" in Theorem 4.1.

A group G is said to be *subgroup separable* or LERF (locally extended residually finite) if given any finitely generated subgroup H and element $g \in G - H$, there is a homomorphism ϕ from G to a finite group such that $\phi(g) \notin \phi(H)$. Equivalently, there exists a subgroup K of finite index in G such that $H \subset K$ but $g \notin K$. Replacing H by the trivial subgroup yields the weaker notion of residually finite. Subgroup separability is an extremely desirable property in geometric topology, and consequently many conjectures involve it. (In [**Ja**] W. Jaco lists three important questions concerning subgroup separability and 3-manifold fundamental groups; the first has been answered negatively by R.G. Burns, A. Karrass and D. Solitar [**BuKaSo**], and the third has been answered affirmatively by D.D. Long and G.A. Niblo [**LoNi**].)

Assume that G satisfies the hypothesis of Theorem 4.1. If G is subgroup separable and K is not finitely generated, then K has a proper subgroup W of finite index in G

containing B. Hence B is not dense in K. We have shown

Corollary 4.3. Assume that G is a finitely presented, subgroup separable group. Let K be the kernel of any epimorphism $\chi : G \to \mathbf{Z}$. If K is not finitely generated, then K contains uncountably many subgroups of finite index.

D.D. Long has introduced a notion that is weaker than subgroup separability [**Lo**]. A group K has the *engulfing property* if for any finitely generated subgroup H of K there exists a proper subgroup of finite index in K containing H. The following theorem provides an answer Question 7.1 of [**SiWi1**].

Theorem 4.4. Let K be the kernel of an epimorphism $\chi : G \to \mathbf{Z}$, where G is finitely presented. The group K contains infinitely many subgroups of some finite index if and only if K has the engulfing property.

Proof. Assume that K contains infinitely many subgroups of some finite index in K. Recall that any finitely generated subgroup of K is contained in a finitely generated HNN base B. In order to prove that K has the engulfing property, it suffices to show that B is contained in some proper subgroup of finite index in K.

By Corollary 2.4 there exists a representation ρ of K onto a finite group Σ such that the sequence of subgroups $\rho(B_j)$ is nonconstant. Without loss of generality we can assume that $\rho(B_0)$ is minimal in $\{\rho(B_j)\}$ with respect to inclusion, and hence is a proper subgroup of Σ. Enumerate the cosets of $\rho(B_0)$ in Σ as $1, 2, \ldots, R$, with $\rho(B_0)$ corresponding to 1. The group Σ acts transitively on the set by (right) multiplication. Then B is contained in $\{g \in K \mid \rho(g)(1) = 1\}$, a subgroup of index $R > 1$ in K.

Conversely, assume that K has the engulfing property. Let B be any finitely generated HNN base. We can find a transitive representation $\rho : K \to S_r$, for some $r > 1$, such that B is contained in $\{g \in K \mid \rho(g)(1) = 1\}$. By Corollary 2.4 the group K has infinitely many subgroups of some finite index. ∎

We can give K a finer topology by choosing the set of *all* finite-index subgroups to be a basis of neighborhoods for the identity element. Of course, this is just the profinite topology for K. With this topology every representation of K into a finite discrete group is continuous. The proof of the following theorem is similar to that of Theorem 4.1.

Theorem 4.5. Let K be the kernel of an epimorphism $\chi : G \to \mathbf{Z}$, where G is a finitely presented group. Endow K with the profinite topology. Then K contains only finitely many subgroups of any finite index if and only if K topologically finitely generated.

Corollary 4.6. Under the hypothesis of Theorem 4.5, the group K has the engulfing property if and only if it is not topologically finitely generated.

It is easy to see that if K is an arbitrary group containing only finitely many subgroups of any finite index, then K need not be topologically finitely generated. Consider, for example, the free product K of cyclic groups $\mathbf{Z}/p_i$, where p_i ranges over all primes.

The condition that a group contains only finitely many subgroups of any finite index is desirable in many situations. Residually finite groups with this condition, for example, are known to be hopfian. (A group is hopfian if any homomorphism of the group onto itself is an isomorphism.) The result for finitely generated groups is due to Malcev, and the proof in [**LySc**] can be adapted. More recently, J. Dixon, E. Formanek, J. Poland and L. Ribes proved that if a group contains only finitely many subgroups of any finite index, then its profinite completion is determined by the set of isomorphism classes of its finite quotients. The reader is referred to [**DiFoPoRi**] (see also [**Sc**]) for this result as well as several attractive corollaries.

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Dept. of Mathematics and Statistics, Univ. of South Alabama, Mobile, AL 36688
email: silver@mathstat.usouthal.edu, williams@mathstat.usouthal.edu