Twisted Alexander Polynomials and Representation Shifts

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Abstract

For any knot, the following are equivalent. (1) The infinite cyclic cover has uncountably many finite covers; (2) there exists a finite-image representation of the knot group for which the twisted Alexander polynomial vanishes; (3) the knot group admits a finite-image representation such that the image of the fundamental group of an incompressible Seifert surface is a proper subgroup of the image of the commutator subgroup of the knot group.

Keywords: Knot, knot group, twisted Alexander polynomial, representation shift [1]

1 Introduction

Fibered knots have been studied in settings of complex algebraic geometry and dynamical systems as well as topology. Their relative simplicity accounts in part for their appeal. Among the "simplest hyperbolic knots" catalogued by P. Callahan, J. Dean and J. Weeks [1], those hyperbolic knots with 6 or fewer ideal tetrahedra in their complements, an overwhelming majority are fibered.

Each of the three conditions listed in the abstract is known to imply that the knot k is not fibered. We conjecture that conversely, each of these conditions holds for every nonfibered knot. That (3) implies (2) we derive

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from a result of Friedl and Vidussi [4]. The equivalence of (1) and (3) follows from our earlier work [16]. In [16] we also stated a variant of the above conjecture, given below as Conjecture 2.2. A weaker conjecture, closely related to one in [6], was recently proved by Friedl and Vidussi [7]. (See Theorem 2.4 below.)

Section 2 presents basic notions of representation shifts and twisted Alexander polynomials. The main result appears in Section 3.

Many of our results about knot complements generalize in a straightforward manner for arbitrary 3-manifolds.

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2 Fibered knots, rep shifts and twisted homology

We recall that a knot $k \subset \mathbb{S}^3$ is fibered if its exterior admits a locally trivial fibration over the circle. By combined theorems of Neuwirth [14] and Stallings [20], this condition is equivalent to the requirement that the commutator subgroup π' of the knot group $\pi = \pi_1(\mathbb{S}^3 \setminus k)$ is finitely generated. Then π' is in fact isomorphic to the fundamental group of a minimal-genus Seifert surface for k.

If k is fibered, then π', being finitely generated, has countably many subgroups of finite index. Equivalently, the infinite cyclic cover X_∞ of k has countably many finite covers. However, for nonfibered knots the situation is different. For every known example, the number of finite covers is uncountable. The authors conjectured this for all nonfibered knots [17], and proved it for nonfibered knots of genus one [19].

For any knot, the finite-index subgroups of π' can be found using representation shifts. We sketch the basic idea, referring the reader to [15] [16] [17] for details.

Given a finite group Σ, consider the space $\Phi_\Sigma = \mathrm{Hom}(\pi', \Sigma)$ of homomorphisms $\rho : \pi' \to \Sigma$. The topology is the compact-open topology, where π' and Σ are discrete. Conjugation in π by a meridian x induces a homeomorphism σ_x described by

$$\sigma_x \rho(a) = \rho(x^{-1} a x),\ \forall a \in \pi'.$$

The pair (Φ_Σ, σ_x), called a *representation shift*, is a dynamical system that, up to topological conjugacy, is independent of the choice of meridian. Consequently, the topological entropy h_Σ, a measure of complexity of σ_x, is an invariant of k and Σ.

We say $\rho \in \Phi_\Sigma$ has *period* r if $\sigma_x^r \rho = \rho$. The set Fix σ_x^r of points with period r can be identified with $\mathrm{Hom}(\pi_1 M_r, \Sigma)$, where M_r denotes the r-fold cyclic cover of $\mathbb{S}^3$ branched over k; in fact, when Σ is abelian, Fix σ_x^r and $\mathrm{Hom}(\pi_1 M_r, \Sigma)$ are isomorphic as abelian groups. The following proposition is proved in [17] (see also [19] for a sketch of the proof).

Proposition 2.1. *For any knot k, the following statements are equivalent.*

(1) The infinite cyclic cover of k has uncountably many finite covers.

(2) The representation shift Φ_Σ is uncountable, for some finite group Σ.

(3) The topological entropy h_Σ is positive, for some finite group Σ.

(4) $\lim_{r\to\infty} \frac{1}{r} \log |\mathrm{Hom}(\pi_1 M_r, \Sigma)|$ *is positive, for some finite group Σ.*

The following conjecture of [17] proposes a characterization of fibered knots.

Conjecture 2.2. *A knot k is fibered iff $h_\Sigma = 0$ for every finite group Σ.*

Remark 2.3. As noted above, the forward implication is known. Conjecture 2.2 can be restated as: k is nonfibered iff any of the equivalent conditions of Proposition 2.1 holds.

It is well known that the Alexander polynomial Δ_k of a fibered knot is a monic polynomial with degree equal to twice the genus of the knot. While useful, such a fibering obstruction is far from complete. The Alexander polynomial of any fibered knot is also the Alexander polynomial of infinitely many nonfibered knots. Twisted Alexander polynomials, introduced by X.S. Lin [11] in 1990, provide more sensitive fibering obstructions.

The Alexander polynomial Δ_k is the 0th characteristic polynomial of $H_1 X_\infty$, regarded as a $\mathbb{Z}[t, t^{-1}]$-module. The module has a square presentation matrix, and Δ_k can be computed as the determinant of this matrix.

Let $\gamma : \pi \to \mathrm{GL}(n, R)$ be a homomorphism, where R is a Noetherian UFD. One can define twisted homology of X_∞ and subsequently a twisted Alexander polynomial $\Delta_{k,\gamma}$, well defined up to a unit in $R[t, t^{-1}]$. The classical Alexander polynomial Δ_k arises from the trivial homomorphism. We refer the reader to [10] for the general definition of twisted Alexander polynomial. Here we describe an equivalent way to think about $\Delta_{k,\gamma}$ in the case that the image of γ is finite.

Assume that γ is a homomorphism from π to a finite group Σ. Without loss of generality, we can assume that Σ is a group of permutation matrices in $\mathrm{GL}(n, \mathbb{Z})$. Let $\tilde{X}$ be the n-fold cover of X_∞ determined by γ. Shapiro's

Lemma implies that the $\mathbb{Z}[t, t^{-1}]$-module $H_1\tilde{X}$ is isomorphic to the twisted first-homology group of X_∞. It has a square presentation matrix [18] with determinant equal to the twisted Alexander polynomial $\Delta_{k,\gamma}$.

The knot group π acts on $\{1, \ldots, n\}$, and by restriction, so does the commutator subgroup π'. Let $\mathcal{O}_\gamma$ denote the number of orbits of $\{1, \ldots, n\}$ under the action of π'. It is immediate that $H_0\tilde{X} \cong \mathbb{Z}^{\mathcal{O}_\gamma}$ (cf. Lemma 4.3 of [4]) and consequently its 0th characteristic polynomial is $(t-1)^{\mathcal{O}_\gamma}$.

If k is fibered, then $\Delta_{k,\gamma}$ is monic for any finite-image representation γ ([2], see also [9], [3]). A computation in section 2 of [9] implies that the related Wada invariant $W(t)$ has degree $2gn - n$, where g is the genus of k. By Theorem 4.1 of [10],

$$\Delta_{k,\gamma} = W(t) \cdot (t-1)^{\mathcal{O}_\gamma}.$$

Hence the degree of $\Delta_{k,\gamma}$ is $2gn - n + \mathcal{O}_\gamma$. The following theorem of Friedl and Vidussi was established earlier by them for genus-1 knots [6].

Theorem 2.4. *[7] A knot k is fibered iff for every finite-image representation γ of the knot group, the twisted Alexander polynomial $\Delta_{k,\gamma}$ is monic and has degree $2gn - n + \mathcal{O}_\gamma$.*

3 Finite covers and twisted polynomials

We recall that a group G is *subgroup separable* if for any proper subgroup H and element $g \in G \setminus H$, there exists a finite-image representation $\gamma : G \to \Sigma$ such that $\gamma(g) \notin \gamma(H)$.

The representation γ "separates" the subgroup H from the element g. The following condition can be regarded as a weak form of subgroup separability for knot groups. It requires only that the fundamental group of an incompressible Seifert surface can be separated from *some* element of the commutator subgroup.

Definition 3.1. The group π of a knot k is *weakly subgroup separable* if it admits a finite-image representation such that the image of the fundamental group of an incompressible Seifert surface for k is properly contained in the image of the commutator subgroup π'.

Any incompressible Seifert surface S determines an HNN decomposition $(B; U, V, \phi)$ of π with *stable letter* x. The stable letter is represented by an oriented meridian of k, while the *base* B is the fundamental group of $\mathbb{S}^3$ split along S, a relative cobordism between two copies $S_\pm$ of S; the split meridian provides a base path $p : I \to \partial B$ from S_- to S_+. Then $U = \pi_1(S_-, p(0))$ and $V = \pi_1(S_+, p(1))$. Choose $p(0)$ to be the base point of B and π. There is an isomorphism $\phi : U \to V$ corresponding to re-identification of S_- and S_+. The group π is isomorphic to the amalgamated free product $\langle B, x \mid x^{-1}ux = \phi(u)\ (u \in U)\rangle$. The natural map $B \to \langle B, x \mid x^{-1}ux = \phi(u)\ (u \in U)\rangle$ is an embedding of B into π'. (See [13] for details.)

The condition of Definition 3.1 can be paraphrased by saying that the *amalgamating subgroup* U of the HNN decomposition can be separated in π from some element of π'. The following result shows that the condition of Definition 3.1 is independent of the incompressible Seifert surface S, and that it is equivalent to a condition of [4].

Proposition 3.2. *Assume that S and S' are incompressible Seifert surfaces for a knot k. If $\pi_1 S$ can be separated in π from some element of π', then it can be separated in the fundamental group of $\mathbb{S}^3$ split along S from some element. Furthermore, $\pi_1 S'$ can be separated in π from some element of π'.*

Proof. Assume that $\pi_1 S$ can be separated in π from some element of π'; that is, assume that there exists a finite-image representation $\gamma : \pi \to \Sigma$ such that $\gamma(\pi_1 S)$ is a proper subgroup of $\gamma(\pi')$. The restriction $\gamma|_{\pi'}$ is periodic with period (not necessarily least) equal to the order of $\gamma(x)$ in Σ. By Corollary 2.4(ii) of [16], π' has uncountably many subgroups of finite index.

On the other hand, Corollary 2.4(ii) implies that if π' has uncountably many subgroups of finite index, then for any HNN decomposition $(B; U, V, \phi)$ with stable letter x, there exists a periodic finite-image representation $\rho : \pi' \to \Sigma$ such that the sequence of subgroups $\rho(x^{-j}Bx^j)$ is not constant. After conjugation by a suitable power of x, we can assume that some element of $\rho(B)$ is not in $\rho(xBx^{-1})$.

We apply this first to the HNN decomposition corresponding to S. Since $U = \pi_1 S$ lies in $xBx^{-1} \cap B$, $\rho(\pi_1 S)$ is properly contained in $\rho(B)$. Proposition 5.1 of [17] ensures that ρ extends to a representation $\bar\rho : \pi \to \bar\Sigma$, where $\bar\Sigma$ is a finite overgroup of Σ, such that $\bar\rho(\pi_1 S)$ is a proper subgroup of $\bar\rho(B)$.

A similar argument, letting $(B; U, V, \phi)$ be the HNN decompostion corresponding to S', shows that $\pi_1 S'$ can be separated in B from some element, an element that is in π'. Hence it can be separated in π from that element of π'. □

Remark 3.3. Corollary 2.4(ii) of [16] and the above argument show that the group π of a knot k is weakly subgroup separable iff the infinite cyclic cover of k has uncountably many finite covers.

Clearly, if the group of k is weakly subgroup separable, then k is nonfibered. On the other hand, if the group of every nonfibered knot k is weakly subgroup separable, then Conjecture 2.2 is true.

A theorem of D.D. Long and G.A. Niblo [12] together with a result of D. Gabai [8] imply that the group of any genus-1 nonfibered knot is weakly subgroup separable. Friedl and Vidussi prove a more general version of this result for 3-manifolds in [4]. See [19] for a direct proof of our special case.

Theorem 3.4. *Let $k \subset \mathbb{S}^3$ be a knot. The following are equivalent.*

(1) The infinite cyclic cover of k has uncountably many finite covers;

(2) the twisted Alexander polynomial $\Delta_{k,\gamma}$ vanishes for some finite-image representation γ;

(3) the group of k is weakly subgroup separable.

Proof. We will prove that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

The implication (1) $\Longrightarrow$ (3) has already been established (see Remark 3.3).

Assume that π admits a finite-image representation $\gamma : \pi \to \Sigma$ such that the image of an incompressible Seifert surface of k is a proper subgroup of $\gamma(\pi')$. Let n be the order of Σ. If we regard Σ as a group of permutation matrices in $\mathrm{GL}(n, \mathbb{Z})$ via the right regular Cayley representation, then by Proposition 3.2 and the proof of Theorem 4.2 [4], the corresponding twisted Alexander polynomial $\Delta_{k,\gamma}$ vanishes. Hence (3) $\Longrightarrow$ (2).

Finally, assume that $\gamma : \pi \to \Sigma$ is a finite-image representation such that $\Delta_{k,\gamma}$ vanishes. As above, we assume that Σ is a group of permutation matrices in $\mathrm{GL}(n, \mathbb{Z})$, for some n. Its elements permute the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$. Let r be the order of $\gamma(x)$. Then the restriction $\rho = \gamma|_{\pi'}$ has period r; that is, $\sigma_x^r \rho = \rho$. Consider the associated n-fold cover $\tilde{X}_\infty$ of X_∞.

The homology of $\tilde{X}_\infty$ is a finitely generated $\mathbb{Z}[t, t^{-1}]$-module, and its 0th characteristic polynomial is equal to $\Delta_{k,\rho} = 0$. We may assume that $\tilde{X}_\infty$ is connected; otherwise, since its homology is the direct sum of contributions from each connected component, we can replace $\tilde{X}_\infty$ by a connected component for which the homology has vanishing 0th characteristic polynomial. The fundamental group of $\tilde{X}_\infty$ is the stabilizer of some basis vector e_i; that is, the set of all $a \in \pi'$ such that $\rho(a)(e_i) = e_i$.

Let $q : \hat{X}_\infty \to \tilde{X}_\infty$ be the finite m-fold cover such that $\pi_1 \hat{X}_\infty$ is the intersection N of kernels

$$N = \ker\, \rho \cap \ker\, \sigma_x \rho \cap \cdots \cap \ker\, \sigma_x^{r-1} \rho.$$

Since ρ has period r, the subgroup N is invariant under conjugation by x. Hence the homology of $H_1\hat{X}_\infty$ is a finitely generated $\mathbb{Z}[t, t^{-1}]$-module.

Let p a prime that does not divide m. Let $\tau : H_1(\tilde{X}_\infty; \mathbb{Z}/p) \to H_1(\hat{X}_\infty; \mathbb{Z}/p)$ be the transfer homomorphism induced by the chain map that takes each i-chain to the sum of its preimages. Since $\tau \circ q$ is multiplication by m, which is relatively prime to p, the composition $\tau \circ q$ is injective and hence so is τ. Consequently, $H_1(\tilde{X}_\infty; \mathbb{Z}/p)$ is a submodule of $H_1(\hat{X}_\infty; \mathbb{Z}/p)$, and hence the 0th characteristic polynomial of $H_1(\hat{X}_\infty; \mathbb{Z}/p)$ vanishes.

The Structure Theorem for finitely generated modules over a PID implies that $H_1(\hat{X}_\infty; \mathbb{Z}/p)$ contains at least one summand isomorphic to $(\mathbb{Z}/p)[t, t^{-1}]$. Consider the sequence of epimorphisms:

$$N = \pi_1\hat{X}_\infty \to H_1(\hat{X}_\infty; \mathbb{Z}) \to H_1(\hat{X}_\infty; \mathbb{Z}/p) \to (\mathbb{Z}/p)[t, t^{-1}].$$

Regarding $(\mathbb{Z}/p)[t, t^{-1}]$ as a direct sum of countably many copies of $\mathbb{Z}/p$, we see immediately that there exist uncountably many homomorphisms $h : N \to \mathbb{Z}/p$. For each homomorphism, ker h is normal in N and hence normal in π'. Consider the short exact sequence

$$0 \to \frac{N}{\text{ker } h} \to \frac{\pi'}{\text{ker } h} \to \frac{\pi'}{N} \to 1.$$

Since $N/\text{ker } h \cong \mathbb{Z}/p$ and π'/N is finite, each quotient $\pi'/\text{ker } h$ is finite. We can choose uncountably many h such that the quotient groups $\pi'/\text{ker } h$ are isomorphic to the same finite group Σ. Hence π' has uncountably many distinct homomorphisms to Σ, and so (2) $\implies$ (1).

□

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