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RANDOM LATTICE TANGLES

BY

Irina P. Teneva

A Thesis

Submitted to the Graduate Faculty of the
University of South Alabama
in partial fulfillment of the
requirements for the degree of

Master of Science

in

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ABSTRACT

Teneva, Irina P. M.S., University of South Alabama, August 1999. Random Lattice Tangles. Chair of Committee: Susan G. Williams.

We define a new mathematical model for random knotting and linking and investigate some of its topological and statistical properties. A random lattice tangle is a planar diagram constructed by tiling the plane with square diagrams of four simple types according to some probability measure μ. Generic properties of lattice tangles depend on the measure μ, and this may allow us to model a variety of chemical and physical processes.

We show that every knot and link can occur in a lattice tangle. In fact, we prove that with only a few restrictions on the probability measure every knot and link will appear infinitely often in a randomly chosen tiling. We use results and techniques from percolation theory to analyze the large-scale structure of lattice tangles. Finally, we define periodic lattice tangles and describe an algorithm that can be used to explore similar questions about this new construction.

INTRODUCTION

We define a *lattice tangle* to be an infinite diagram (or graph) in $\mathbf{R}^2$ constructed by tiling the plane with the unit square tiles shown in Figure 0.1. Type 1 and Type 2 of these are called *crossing tiles*, and the other two are *smoothing tiles*. A block segment of a lattice tangle is shown in Figure 0.2. We regard a lattice tangle diagram as representing a collection of bounded closed curves and unbounded arcs lying in a thickened plane; a crossing indicates where one curve passes over another.

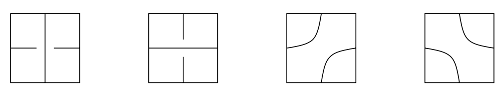


Figure 0.1: Tangle tiles.

Our original inspiration for this construction came from the traditional Celtic knots and links which may be designed using crossings and smoothings in a similar way. Celtic knots are always alternating and very symmetric, so in their construction many restrictions are put on which tiles are used where. In contrast, lattice tangles fill the entire plane, and there is no restriction on placement of the tiles. We may suppose that the tiles have been chosen randomly, according to some probability distribution. Our aim is to study the topological and statistical properties of the resulting lattice tangles.

Lattice tangles may be used to simulate some physical or chemical process. For example, they may be an appropriate model for knotting and tangling of a fibrous material. Other models have been devised for random knotting and linking. For example, Sumners and Whittington [**SuWh**] model random knots using self-avoiding walks in $\mathbf{R}^3$. Their motivation was to study knotting of DNA molecules and polymer chains. A lattice tangle model will have the advantage of producing a planar diagram. This is important since knots and links are most commonly studied using planar diagrams because of their simplicity and the fact that information about the 3-dimensional object can be recovered from the diagram.

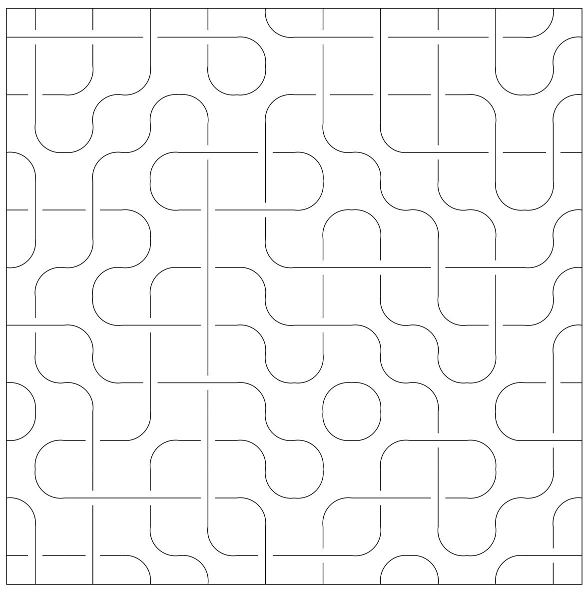


Figure 0.2: Segment of a lattice tangle.

Many of the techniques we use come from percolation theory, a body of mathematics that was developed to model various physical processes such as flow of liquid in a porous medium, spread of disease in a population, and ferromagnetism. Percolation theory is concerned with the existence of a critical phenomenon which marks a change in the global connectivity of a system. In bond percolation theory, we begin with a set of points (e.g. $\mathbf{Z}^2$) and suppose that bonds are placed at random between neighboring points (the definition of neighboring may vary from one model to another). Site percolation on the plane may be described as a random tiling with black and white tiles, each corresponding to an integer point of $\mathbf{R}^2$. Lattice tangles are similar in spirit but more complex topologically than either site or bond percolation. For more background on percolation theory see [**BuKe1**], [**BuKe2**], [**Gr1**], [**Kes2**].

1. KNOTS AND LINKS IN TANGLES

A *knot* is a simple closed curve in space. Think of it as a knotted string with its two ends glued together. Two knots are *equivalent* if one can be deformed into the other without cutting or passing through the string. A *link* is a finite collection of non-intersecting knots. Since a knot is a special kind of link (a link with one component), we will use the term link to include any knot. A (*regular*) *link diagram* is a regular projection of the link on the plane with the crossing information resolved (that is, the diagram shows which strand crosses over). More detailed knot theory information and definitions can be found in [**Mu**].

It is known that there are infinitely many links. We will show that all of them can appear in a lattice tangle. To do this, we will use a theorem of Alexander (c.f. [**Mu, 209**]) which says that every link can be represented by a diagram which is the closure of some braid. A *braid* or *n-braid* β is simply a stacking of braid components σ_k and σ_k^{-1}, $k \in \{1, \ldots, n-1\}$ (Figure 1.1).

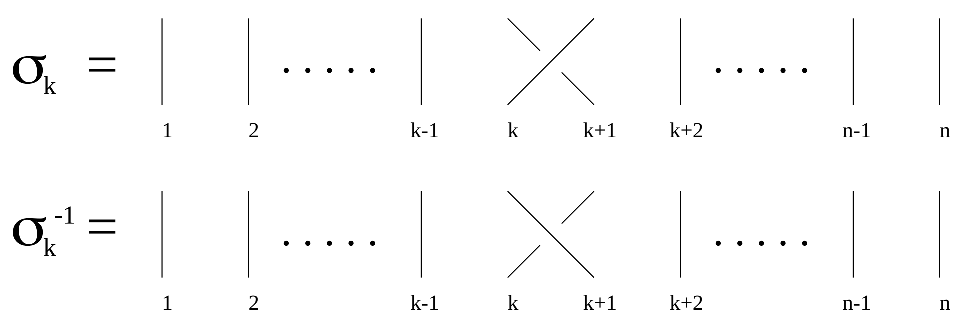


Figure 1.1: Arbitrary braid components σ_k and σ_k^{-1}.

The *closure* $\bar{\beta}$ of a braid is obtained by connecting its inputs and outputs as shown in Figure 1.2.

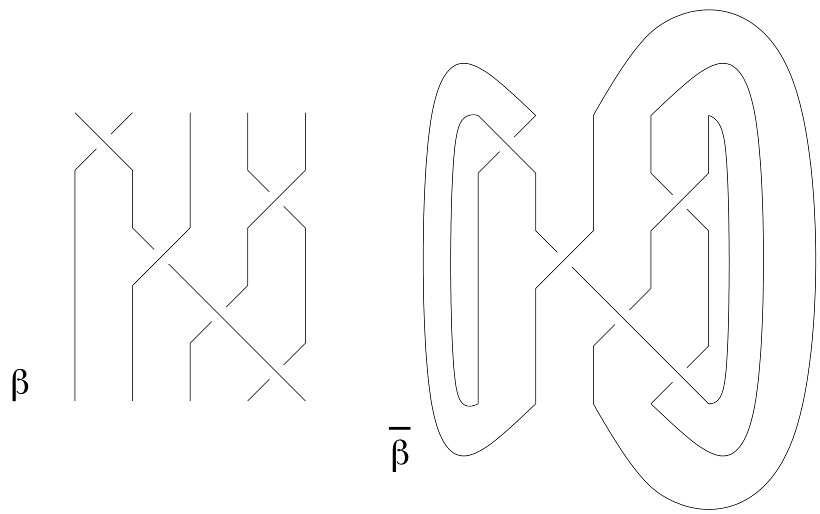


Figure 1.2: Braid β and its closure $\bar{\beta}$.

Proposition 1.1: Any link can be constructed using the set of tiles in Figure 0.1.

Proof: For the moment, consider our set of tiles as blocks with sides of length 2. Then in every tile the endpoints of each arc correspond to points of $\mathbf{Z}^2$ with one even and one odd coordinate. For every odd integer n, connecting all points $(a, b) \in \mathbf{Z}^2$ such that $a + b = n$ gives us diagonal lines with equations $y = -x + n$ (See Figure 1.3).

Now consider arbitrary components σ_k and σ_k^{-1} of an n-braid β (Figure 1.1). If we think of the input and output strands of these components as corresponding to integer points on two adjacent diagonals constructed above, it is easy to see that any braid component can be obtained by using Type 1 tile for the crossing in σ_k^{-1}, Type 2 tile for the crossing in σ_k, and the smoothing tiles for the other line segments.

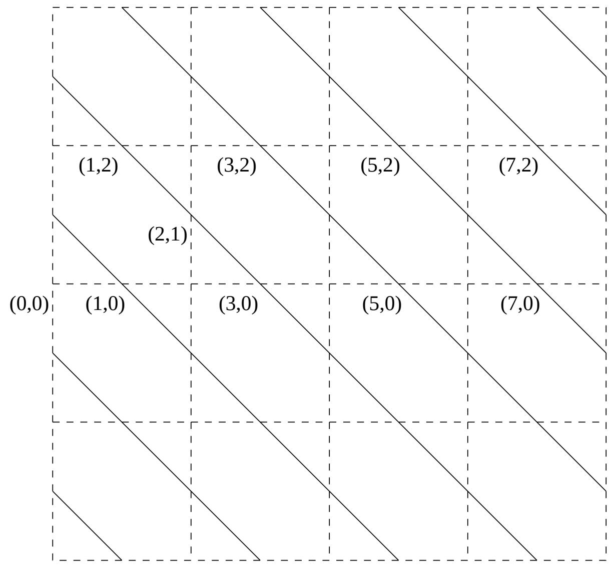


Figure 1.3: Lines $y = -x + n$ for n odd.

From here it is easy to construct the closure $\bar{\beta}$ of a braid β using the smoothing tiles (see example in Figure 1.4).

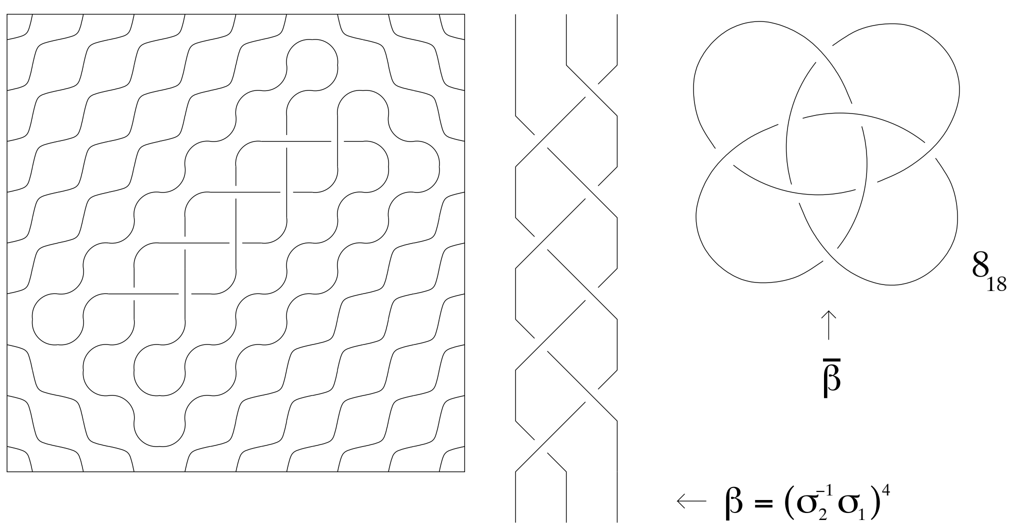


Figure 1.4: A knot constructed using the tiles in Figure 0.1.

Now we apply Alexander's Theorem to obtain the desired result. ■

Next we will introduce a probabilistic structure on the set of lattice tangles. Formally, we regard a lattice tangle as an element $x = (x_{\mathbf{n}})$ of $\mathcal{A}^{\mathbf{Z}^2}$, where $\mathcal{A}$ is the set (or *alphabet*) of tiles in Figure 0.1 and $x_{\mathbf{n}}$ is the tile in the $\mathbf{n}^{\text{th}}$ coordinate ($\mathbf{n} = (n_1, n_2) \in \mathbf{Z}^2$). Following the usual terminology in percolation theory we will sometimes refer to an element of $\mathcal{A}^{\mathbf{Z}^2}$ as a *configuration.* We give $\mathcal{A}$ the discrete topology. Then $\mathcal{A}^{\mathbf{Z}^2}$ is a compact space with the product topology. We have a family of translation or *shift* maps $\sigma = \{\sigma^{\mathbf{m}} : \mathbf{m} \in \mathbf{Z}^2\}$ acting on $\mathcal{A}^{\mathbf{Z}^2}$ by $(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}$.

We want to model a situation where crossings and smoothings occur "randomly" according to a probabilistic scheme which may vary from one application to another. To do this, we need a probability measure μ defined on $\mathcal{A}^{\mathbf{Z}^2}$. We will assume throughout that μ is invariant under σ. For many of our results we take μ to be a *Bernoulli measure.* That is, there are nonnegative real numbers $p_1, \ldots, p_4$, $p_1 + \cdots + p_4 = 1$, such that the probability of the i^{th} tile type occuring at any specific coordinate is p_i, independent of the tile types at the other coordinates. Thus, for any finite set $F \subset \mathbf{Z}^2$, the *cylinder set* defined by specifying the tile type of each $\mathbf{n} \in F$ has probability equal to the product of the probabilities of the tile types.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathcal{A}^{\mathbf{Z}^2}$. A measure μ on $\mathcal{B}$ is *ergodic* with respect to σ if every σ-invariant set E in $\mathcal{B}$ (i.e. $\sigma^{\mathbf{n}} E = E$ for all $\mathbf{n} \in \mathbf{Z}^2$) has measure 0 or 1. We say μ is *strongly mixing* if

$$\lim_{|\mathbf{n}| \to \infty} \mu(\sigma^{\mathbf{n}} A \cap B) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{B}$, and *weakly mixing* if

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} |\mu(\sigma^{(i,j)} A \cap B) - \mu(A)\mu(B)| = 0$$

for all $A, B \in \mathcal{B}$.

Clearly, strongly mixing implies weakly mixing, which in turn implies that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} \mu(\sigma^{(i,j)} A \cap B) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{B}$. The last expression implies that μ is ergodic. For if we let $A = B$ be a σ-invariant set then $\sigma^{\mathbf{n}} A = A, \forall \mathbf{n} \in \mathbf{Z}^2$. Hence $\mu(\sigma^{\mathbf{n}} A \cap A) = \mu(A)$ and so

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} \mu(\sigma^{(i,j)} A \cap A) = \lim_{n\to\infty} \frac{n^2 \mu(A)}{n^2} = \mu(A).$$

But $\mu(A) = (\mu(A))^2$ implies that $\mu(A) = 0$ or 1. Since this is true for any σ-invariant set $A \in \mathcal{B}$, μ is ergodic.

In fact, the above definitions and implications remain valid even if we consider only sets in a semi-algebra $\mathcal{S}$ which generates $\mathcal{B}$ ($\mathcal{S}$ generates $\mathcal{B}$ means that $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{S}$). Since the collection $\mathcal{S}$ of all cylinder sets of $\mathcal{A}^{\mathbf{Z}^2}$ is a basis for the topology on $\mathcal{A}^{\mathbf{Z}^2}$, $\mathcal{S}$ is a semi-algebra which generates $\mathcal{B}$.

The following two results are well known.

Lemma 1.2: If μ is a Bernoulli measure on $\mathcal{A}^{\mathbf{Z}^2}$, then μ is strongly mixing (and therefore ergodic).

Proof: If A, B are cylinder sets of $\mathcal{A}^{\mathbf{Z}^2}$, let F_A and F_B be the sets of fixed coordinates of A and B, respectively. Let N be the greatest distance between points in F_A and F_B. Then, for all $\mathbf{n} \in \mathbf{Z}^2$ with $|\mathbf{n}| \geq N$, $(F_A + \mathbf{n}) \cap F_B = \emptyset$. So, $\mu(\sigma^{\mathbf{n}} A \cap B) = \mu(A)\mu(B)$ if $|\mathbf{n}| \geq N$. Thus,

$$\lim_{|\mathbf{n}|\to\infty} \mu(\sigma^{\mathbf{n}} A \cap B) = \mu(A)\mu(B)$$

which implies that μ is strongly mixing. ∎

Proposition 1.3: Assume μ is ergodic. If C is a finite configuration (or pattern) of tiles and if C has a positive probability of occurrence, then it occurs infinitely often in almost every tiling x in $\mathcal{A}^{\mathbf{Z}^2}$.

Proof: Let E be the set of configurations $x \in \mathcal{A}^{\mathbf{Z}^2}$ in which C occurs finitely many times, and let D be the set of x in which C occurs at a certain fixed location. Then

$\mu(D) > 0$ since μ is σ-invariant. If $x \in E$ then by the Ergodic Theorem (c.f. [**Pe**, 30])

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} \chi_D(\sigma^{(i,j)}x) = \int_{\mathcal{A}^{\mathbf{Z}^2}} \chi_D \, d\mu = \mu(D)$$

for almost every configuration x. (Here χ_D is the characteristic function of D.) But if $x \in E$ then $\chi_D(\sigma^{\mathbf{n}}x) = 0$ for $\mathbf{n}$ sufficiently large and so

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} \chi_D(\sigma^{(i,j)}x) = 0.$$

Hence E must have measure 0. ∎

Corollary 1.4: Assume that μ is a Bernoulli measure and that all tile types occur with positive probability. Then, with probability one, a randomly chosen tiling contains an infinite number of copies of any link.

This result follows directly from Proposition 1.3 after we use the facts that any link can be constructed using the tiles in $\mathcal{A}$ (from Proposition 1.1) and that μ is also ergodic (applying Lemma 1.2).

2. LARGE-SCALE STRUCTURE OF LATTICE TANGLES

We expect the typical appearance of random lattice tangles to change when we change the probability measure. In the next two sections, we will be mostly interested in a Bernoulli measure μ, but some of the results hold for any ergodic measure.

Some interesting questions arise in connection with the probability assigned to each tile. What is the probability that the tangle component containing the origin is bounded (i.e. it is a closed curve)? When do we have unbounded tangle components, and what can we say about their number? Notice that the type of crossing is of no importance in answering these questions. We begin with an easy special case.

Proposition 2.1: Suppose μ is an ergodic measure with the property that one of the smoothing tiles (Type 3 or 4) occurs with probability 0. Then almost every configuration $x \in \mathcal{A}^{\mathbf{Z}^2}$ has infinitely many unbounded tangle components.

Proof: Assume that the set S of configurations which contain no Type 3 tiles has measure one. In this set, for the individual tiles we have at most four types of arcs with respect to their direction: top$\leftrightarrow$bottom, right$\leftrightarrow$left, top$\leftrightarrow$right, and left$\leftrightarrow$bottom (Figure 2.1a).

To have a closed tangle component it is necessary to have a right$\leftrightarrow$bottom and a top$\leftrightarrow$left arc. But configurations with such arcs are not in S (that is, they are in a set of measure 0). Also, for a configuration $x \in S$, in an $n \times n$ box we will have exactly $2n$ arcs (see example in Figure 2.1b). This follows from the fact that such a configuration will have no closed components. Hence, in a tiling of the entire plane, there are infinitely many unbounded tangle components.

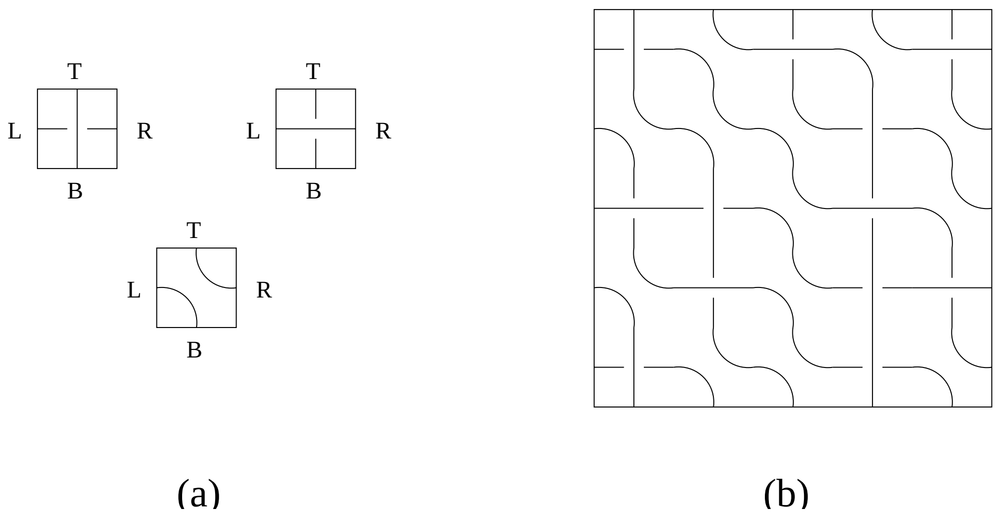


Figure 2.1: Tangle with no Type 3 tiles.

The argument is symmetric for Type 4 tiles. ∎

We will now use new lattice objects, which we call mazes, to answer questions about some topological properties of lattice tangles. A *maze* M is a tiling of the plane with the unit square tiles in Figure 2.2. We will call the set of these tiles $\mathcal{M}$. The diagonal line segments in the first two tiles are called *wall segments*.

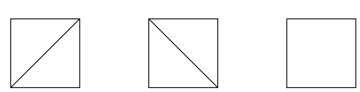
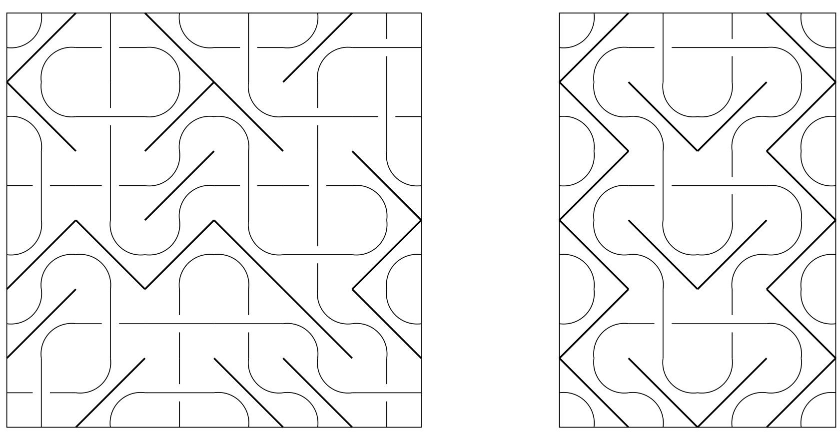


Figure 2.2: The set $\mathcal{M}$ of maze tiles.

Let M be a maze, and let W be the set of wall segments in M. A *wall* is a connected component of W. We define the *maze interior*, denoted by $\text{int}(M)$, to be $\mathbf{R}^2 - W$. A *chamber* is a bounded component of int(M), while a *corridor* is an unbounded component. Note that for each lattice tangle there is a corresponding maze. The wall segments in the maze tiles can be thought of as separations for the smoothing tiles, and the blank tile can correspond to both types of crossing tiles. If we superpose the two sets of tiles using this correspondence, the resulting construction will be a maze with a tangle in its chambers and corridors (see Figure 2.3a). The correspondence between tangles and mazes can be described by a map $f : \mathcal{A}^{\mathbf{Z}^2} \to \mathcal{M}^{\mathbf{Z}^2}$ where $\mathcal{M}^{\mathbf{Z}^2}$ is the space of mazes. Then a measure μ on $\mathcal{A}^{\mathbf{Z}^2}$ induces a measure $\bar{\mu}$ on $\mathcal{M}^{\mathbf{Z}^2}$ by $\bar{\mu}(E) = \mu(f^{-1}(E))$.

It is obvious that a chamber in the maze M will contain only bounded tangle components. So, if all maze components are bounded (i.e. chambers) then all tangle components are bounded. Unfortunately, knowing when M has an unbounded component (i.e. corridor) does not tell us anything about the type of tangle components. A corridor of M may contain bounded tangle components. In fact, it may contain *only* bounded tangle components (see example in Figure 2.3b).



(a) (b)

Figure 2.3: Superposition of maze and tangle.

The following definitions are adaptations from site percolation to our maze construction, and so many similar conclusions can be drawn and the proofs can be modified for our purposes. Our next three results are modeled on analogous results of Burton and Keane [**BuKe1**] with the assumption that the measure μ is only ergodic.

If A is a chamber or a corridor, let $\bar{A}$ be the union of A and all wall segments adjacent to A. Two components A and B of $\mathrm{int}(M)$ are called *neighbors* if $\bar{A} \cap \bar{B} \neq \emptyset$. A *corridor ribbon* is the union of a corridor A and the bounded components of $\mathbf{R}^2 - A$ (that is, walls and chambers surrounded by A). A *chamber ribbon* is an infinite set of connected chambers. A finite set of such chambers is called a *rock*. Note that from the definition of chamber ribbon it follows that rocks cannot be neighbors of chamber ribbons. We say that a wall C in a maze M is *enclosed* by another wall, C', if C is contained in a bounded component of $\mathbf{R}^2 - C'$, and we write $C \prec C'$. A wall is called *inessential* if it is enclosed by another wall, and *essential* otherwise. A maze M is an *essential realization* if each wall is enclosed by an essential wall, and an *infinite cascade* if each wall is inessential (and hence bounded).

We will see that a maze is either an essential realization or an infinite cascade. Moreover, if it is essential, then corridor and chamber ribbons are "parallel" to each other in a topological sense. That is, we can think of the maze as stripes of corridor and/or chamber ribbons each one of which has at most two neighbors. Notice that for each $m, n \in \{0, 1, \ldots\} \cup \{\infty\}$, the set of configurations with m corridor ribbons and n chamber ribbons is a σ-invariant set. Thus for every ergodic measure μ there is a unique pair (m, n) for which this set has measure one.

Lemma 2.2: A maze M is either an essential realization or an infinite cascade. If μ is an ergodic measure, then one of these two possibilities occurs with probability one.

Proof: Assume M is not an essential realization. Then there exist distinct bounded walls $C_1, C_2, \ldots$ such that $C_1 \prec C_2 \prec C_3 \prec \cdots$. Suppose $\mathbf{x} \in \mathbf{R}^2$ is not enclosed by C_n for all n. That is, $\mathbf{x} \in \mathrm{ext}(C_n)$ for all n. Let P_1 be a path from $\mathbf{x}$ to

C_1. Then there exist $P_1, P_2, P_3, \ldots, P_n$ such that $P_n \subset P_{n-1} \subset \cdots \subset P_1$. Hence $|P_n| < |P_{n-1}| < \cdots < |P_1|$ where $|P_k|$ is the length of the path from $\mathbf{x}$ to C_k. Since the shortest distance between points on two disconnected wall segments is $\frac{\sqrt{2}}{2}$,

$$|P_2| < |P_1| - \frac{\sqrt{2}}{2},\ |P_3| < |P_2| - \frac{\sqrt{2}}{2} < |P_1| - \frac{2\sqrt{2}}{2}, \ldots.$$

So, $|P_n| < |P_1| - (n-1)\frac{\sqrt{2}}{2}$ which is not possible for large n. This means that M is an infinite cascade. The second statement follows from the fact that the set of essential realizations is shift-invariant. ■

Theorem 2.3: For a set of configurations x of measure 1, each rock in x has exactly two corridor ribbons as neighbors. (This means that rocks do not affect the general layout of corridor ribbons.)

Proof: If x is a random maze in $\mathcal{M}^{\mathbf{Z}^2}$, let E be the event that there exists a rock with at least three corridor ribbons as neighbors in x. (A rock cannot have only one corridor ribbon as a neighbor since in that case the rock will be part of the ribbon by definition.) Let E_k be the event that there is such a rock in the block $B_k = [-k,k] \times [-k,k]$. Then $E = \cup_k E_k$. Since the number of tiles is finite, there are only finitely many ways w ($w = 3^{4k^2}$, to be precise) to tile B_k. Let $\{P_1, P_2, \ldots, P_w\}$ be the set of all pictures obtained by tiling B_k, and let $E_{k,m} = E_k \cap P_m$. That is, $E_{k,m}$ is the event that there is a rock with at least three corridor ribbons as neighbors in B_k and the picture in B_k is P_m. Then $E_k = \cup_m E_{k,m}$.

Suppose $P(E) > 0$. (Since E is an invariant set and μ is ergodic, $P(E) > 0$ implies that $P(E) = 1$, but this fact is not important for this proof.) Then $P(E_k) > 0$ for some k, and hence $P(E_{k,m}) > 0$ for some m. By the Ergodic Theorem,

$$P(E_{k,m}) = \int_{\mathcal{M}^{\mathbf{Z}^2}} \chi_{E_{k,m}}\, d\mu = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} \chi_{E_{k,m}}(\sigma^{(i,j)}x) \geq 2\epsilon$$

for almost every maze $x \in \mathcal{M}^{\mathbf{Z}^2}$.

Fix $x \in \mathcal{M}^{\mathbf{Z}^2}$ for which the above expression holds. Then there exists $N \in \mathbf{N}$ such that for all $n \geq N$

$$\sum_{i,j=0}^{n-1} \chi_{E_{k,m}}(\sigma^{(i,j)}x) \geq \epsilon n^2.$$

Notice that

$$\sum_{i,j=0}^{n-1} \chi_{E_{k,m}}(\sigma^{(i,j)}x)$$

is the number of points (i,j) in $[0,n-1] \times [0,n-1] \subseteq \mathbf{Z}^2$ such that $\sigma^{(i,j)}x \in E_{k,m}$. In other words, this sum represents the number of $2k \times 2k$ boxes in $[-k, k+n-1] \times [-k, k+n-1]$ (Figure 2.4a) containing a rock with at least three corridor ribbons as neighbors.

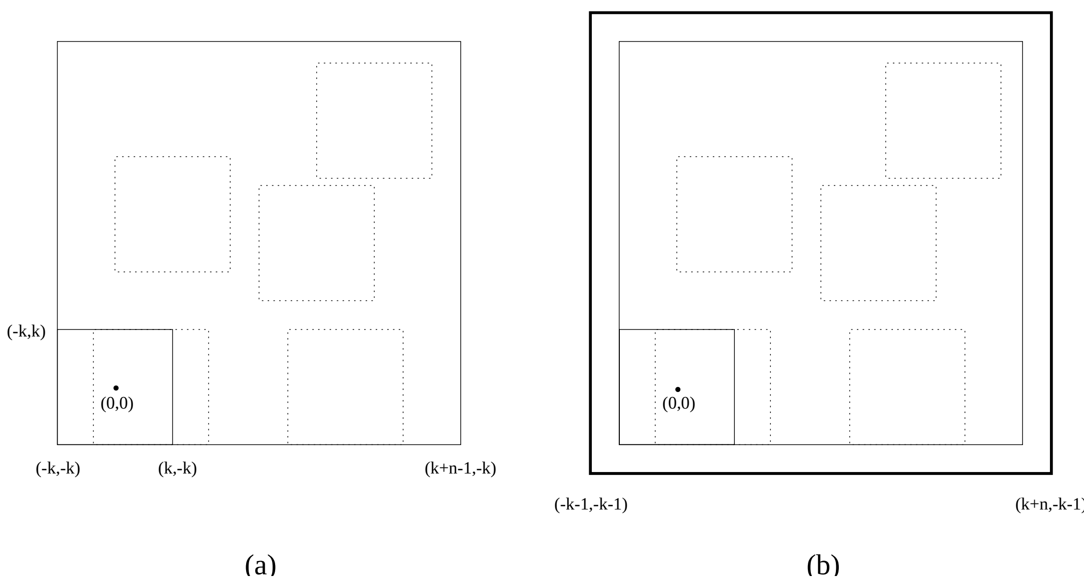


Figure 2.4: Copies of P_m in a larger block.

It is tempting to think that this number is an overcount because we may have such a rock in two different $2k \times 2k$ boxes. If this is the case, since these boxes are identical (the picture in them is P_m) and one is a shift of the other, there must be two (or more) such rocks in $[-k, k] \times [-k, k]$ (Figure 2.5).

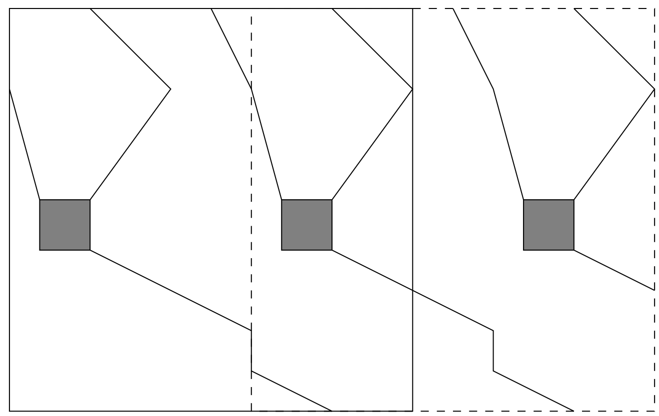


Figure 2.5: Example with intersecting copies of P_m.

Hence we may be undercounting, not overcounting.

We choose to look at a slightly larger box to make sure that the wall segments of the corridor ribbons which are neighbors of these rocks intersect the boundary of the box. So, consider the block $B = [-k-1, k+n] \times [-k-1, k+n]$ (Figure 2.4b). Certainly, the number of rocks with at least three corridor ribbons as neighbors in B is not less than ϵn^2 since $[-k, k+n-1] \times [-k, k+n-1] \subset B$. We will show that this is impossible for large n.

For each of these rocks there are at least three wall segment paths from the rock to integer points on the boundary of B. Starting from the bottom left corner and going in the clockwise direction along the boundary of B, call the second of these points a *central point*. It is possible for two rocks to have wall segment paths with the same end points, but the central point for each rock cannot be the central point for another rock (see example in Figure 2.6). This fact follows from the Jordan Curve Theorem.

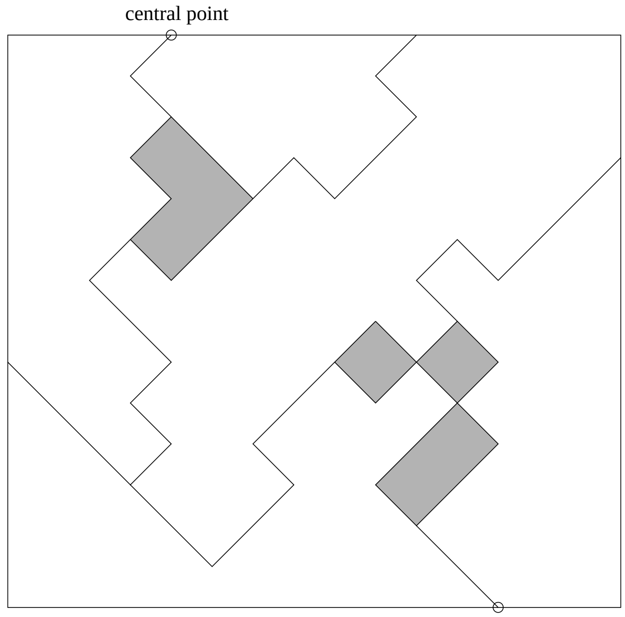


Figure 2.6: Rocks with three corridor ribbons as neighbors.

This implies that the central points for the rocks in B are distinct. Hence there are at least ϵn^2 of them on the boundary of B. Since k is fixed and there are $4(n+2k+1)$ integer points on the boundary of B, this statement does not make sense for large n. ■

Theorem 2.4: For a set of configurations x of measure one, the complement of any ribbon has at most two components.

We will omit the proof of Theorem 2.4 since it uses ideas similar to those in Theorem 2.3.

An interesting application is to consider the case where both crossing tiles have zero probability of occurrence. (This means that in the corresponding maze construction there is zero probability of seeing the blank tile.) Then, with probability one, a lattice tangle consists of non-overlapping closed curves and arcs. In this case each chamber of the maze corresponds to a closed curve in the tangle, and each corridor corresponds to an unbounded arc. From Theorems 2.3 and 2.4 it follows that if we see infinitely many unbounded arcs with probability one, then we may order them as A_n, $n \in \mathbf{Z}$, so that each A_i lies between A_{i-1} and A_{i+1}.

Using the next theorem we will describe in more detail the maze structure in relation to probablilities assigned to the maze tiles and then, by making a connection to lattice tangles, we will draw some conclusions about the types and number of tangle components. In the proofs of the earlier results we used techniques which were modifications from site percolation where our measure μ was only assumed to be ergodic. The proof of the next theorem, however, makes use of some known bond percolation results which rely on the fact that the assignments of open and closed bonds to elements of the edge set are independent of each other. Thus, we will restrict ourselves to a Bernoulli measure μ.

On the square lattice, (ordinary) bond percolation is defined in the following way. Let $\mathbf{L}^2$ be the square lattice with $\mathbf{Z}^2$ as the set of vertices and $\mathbf{E}^2$ as the set of edges between neighboring points in $\mathbf{Z}^2$ (i.e. points which are distance 1 from each other). We write $\mathbf{L}^2 = (\mathbf{Z}^2, \mathbf{E}^2)$. With probability p we assign a bond (or an *open bond*) to an edge in $\mathbf{L}^2$, and no bond (or a *closed bond*) with probability $1-p$, where these assignments are made independently of each other. An *open (closed) cluster* in $\mathbf{L}^2$ is a collection of connected open (closed) bonds. A *configuration* is an assignment of open or closed bonds to the edges of $\mathbf{L}^2$.

The main interest in bond percolation (as well as in site percolation) is the existance of a *critical probability*, p_c, above or below which we are more or less likely to have an unbounded open cluster. For bond percolation on $\mathbf{L}^2$ it is known that if $p > p_c$ then almost every configuration x will contain an unbounded open cluster, and if $p \leq p_c$ then, with probability one, all clusters in x are bounded. We define the *dual lattice* of $\mathbf{L}^2$ to be the lattice with $\{\mathbf{x} + (\frac{1}{2}, \frac{1}{2}) : \mathbf{x} \in \mathbf{Z}^2\}$ as a vertex set and a set of edges between neighboring points (same neighboring relation as before). Notice that if e is an edge in $\mathbf{L}^2$ connecting the points (x_1, y_1) and (x_2, y_2), then there is a corresponding edge e' in the dual of $\mathbf{L}^2$ which connects the points $(x_1 + \frac{1}{2}, y_1 + \frac{1}{2})$ and $(x_2 - \frac{1}{2}, y_2 - \frac{1}{2})$. If x is a configuration, we obtain the dual of x

by assigning a closed (open) bond to e' if e corresponds to an open (closed) bond. Thus, the probability of an open bond in the dual of x is $1 - p$. It is easy to see that for every bounded open cluster in the dual there is a path of open bonds in x which forms a closed curve and vice versa. In the example in Figure 2.7, the maze M is indicated by solid lines and its dual by dashed lines. The heavier dashed lines indicate a closed curve of open bonds in the dual that surrounds a bounded open cluster in M.

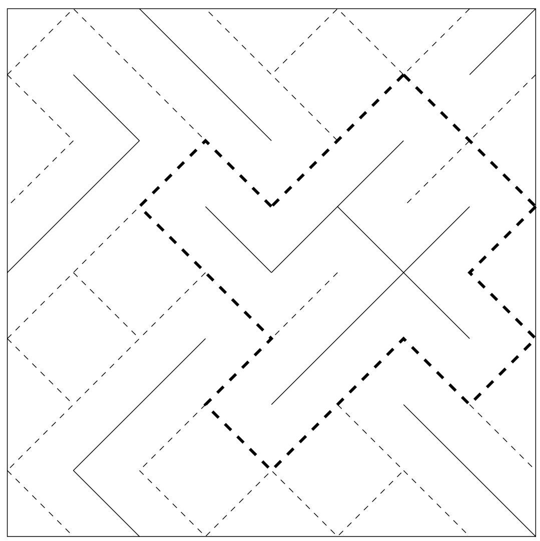


Figure 2.7: A segment of a maze and its dual.

Theorem 2.5: Suppose the maze tiles in Figure 2.2 occur independently with probabilities p, q, and r, respectively, where $q = p$ and $r = 1 - 2p$. Let A be the event that M contains exactly one corridor and B be the event that all components of the interior of M are chambers. Let $P_p(A)$ and $P_p(B)$ be the probabilities of A and B, respectively, with respect to p. Then (a) if $p < \frac{1}{2}$, $P_p(A) = 1$, and (b) if $p = \frac{1}{2}$, $P_p(B) = 1$. (Note that $p = q$ and $p + q + r = 1$ imply $p \leq \frac{1}{2}$.)

Part (b) is essentially contained in [**Gr1**] in a different guise, as we explain below.

Proof: To each maze $M \in \mathcal{M}^{\mathbf{Z}^2}$ we associate *even* and *odd* bond configurations described the following way: An *even bond* is a line segment connecting points (m, n) and $(m+1, n \pm 1)$ in $\mathbf{Z}^2$ where $m+n$ is even. Let M_e be the subgraph of M consisting only of the even bonds in M. The probability that M_e will have an open bond between two points (m, n) and $(m+1, n+1)$ (where $m+n$ is even) is equal to the probability that M has a tile of Type 1 in this position, namely p. Similarly, the probability of an open bond between (m, n) and $(m+1, n-1)$ is equal to the probability that there is a Type 2 tile in this position in M. Since the tiles in M are placed independently of each other, we can regard M_e as an element (*even configuration*) of an *even bond percolation* where the probability of an open bond is p and the probability of a closed bond is $1-p$. *Odd bonds*, *odd configurations* M_o, and *odd bond percolation* are defined similarly (see example in Figure 2.8).

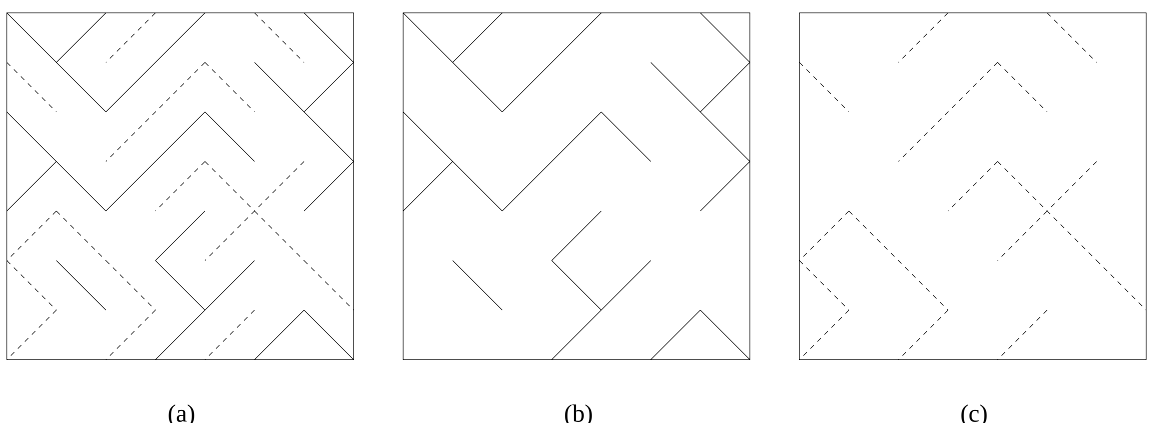


Figure 2.8: (a)M (b)M_e (c)M_o

The even bond configuration is defined on the lattice $\mathbf{L}_e^2 = (S, T)$ where $S = \{(m, n) : m+n \text{ is even }\}$ is the set of vertices and T is the set of edges between neighboring points in S (Two points $\mathbf{x}, \mathbf{y} \in S$ are neighbors if $|\mathbf{x} - \mathbf{y}| = \sqrt{2}$). Likewise, the odd bond configuration is an element of the bond percolation on $\mathbf{L}_o^2 = (S', T')$ with $S' = \{(m, n) : m+n \text{ is odd }\}$ as the set of vertices and T' as the set of edges using the same neighboring relation (Figure 2.9).

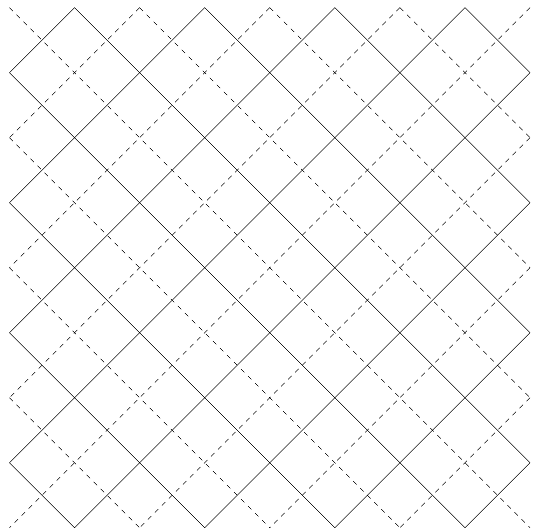


Figure 2.9: $\mathbf{L}_e^2$, $\mathbf{L}_o^2$

Let M'_e and M'_o be the duals of M_e and M_o. (Notice that the set of open bonds in M_e (M_o) is a subset of the set of open bonds in M'_o (M'_e).) Let p' be the probability of an open bond in M'_e or M'_o. Then $p' = 1 - p$.

(a) Assume $p < \frac{1}{2}$. Then $p' > \frac{1}{2}$. It is known that the critical probability p_c of bond percolation on $\mathbf{L}^2$ is $\frac{1}{2}$ [**Kes1**]. Hence, with probability one, M'_e contains exactly one unbounded cluster of open bonds [**Gr2**]. But such a cluster corresponds to a corridor in M_e. A similar argument applies to the odd configuration case since the probabilities of even and odd bonds are the same. Thus M_o also has one corridor with probability one. It is obvious that even and odd bond configurations have no common end points. Suppose M (which is a superposition of M_e and M_o) has more than one corridor. Then there is at least one unbounded wall in M containing only odd or only even bonds as wall segments. But this implies that either M_e or M_o has more than one corridor which is a contradiction. Then it suffices to show that M has a corridor.

Since both M_e and M_o have a corridor, neither is an infinite cascade. Hence they are essential realizations (applying Lemma 2.2). If C is a bounded essential wall M_e (i.e. one that is enclosed by another wall), then either C is an essential wall in M or C is enclosed by a bounded essential wall of M_o. In either case, we have an essential wall, call it C', in M (see Figure 2.10). Pick a point $\mathbf{x}$ not enclosed by C' so that the distance between $\mathbf{x}$ and C' is less than $\frac{\sqrt{2}}{2}$. This condition ensures that $\mathbf{x}$ is not enclosed by another odd or even wall. Now we can find an infinite path in M through $\mathbf{x}$. Therefore, M contains exactly one corridor.

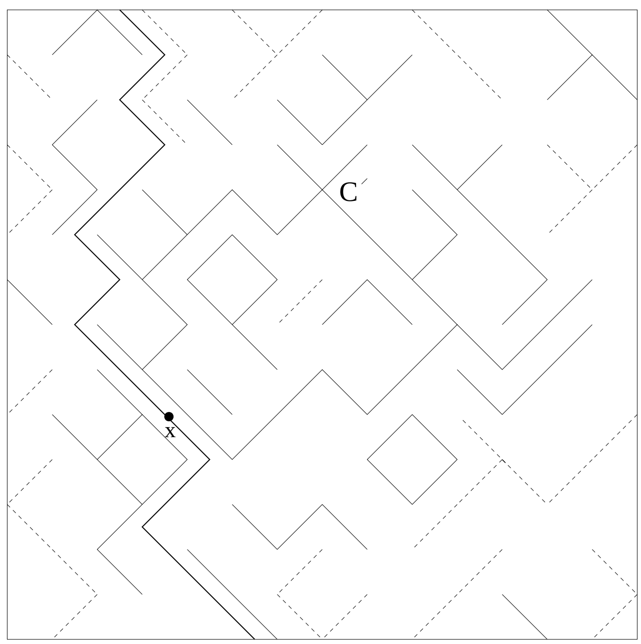


Figure 2.10: Infinite path in M.

(b) If $p = \frac{1}{2}$, $p' = \frac{1}{2}$. Then, with probability one, all open clusters in M_e' and M_o' are bounded. This means that all components of M_e and M_o are chambers. Hence, with probability one, all components of M are chambers. ■

Using (b) we can conclude that assigning probability $\frac{1}{2}$ to each of the smoothing tiles will result in only bounded tangle components. In fact, Kesten proved that for bond percolation on $\mathbf{Z}^2$ the critical phenomenon occurs when the sum of the probabilities of open horizontal and vertical bonds is one [**Kes2**]. Translated for the lattice tangles problem, this means that as long as the probabilities of the smoothing tiles add up to one, all tangle components will be bounded with probability one.

Our maze and tangle model is equivalent to a deterministic lattice model for the scattering of particles first proposed by Ruijgrok and Cohen [**RuCo**]. This model involves randomly placing 2-sided mirrors in either diagonal orientation at integer points of a square lattice. Light moving along the edges of the lattice is reflected by the mirrors. Notice that this light traces essentially the same path as the tangle components inside a maze. This mirror model is a simplification of Ehrenfest wind-tree model [**Eh**], which in turn is a simplification of the classical Lorentz gas model [**Lo**] in which a particle is scattered elastically by randomly placed spheres in $\mathbf{R}^3$.

The main problem regarding the mirror model is to find the probability that a beam of light will return to its starting point, or equivalently the probability that a tangle component starting at a given point will be bounded. Grimmett [**Gr1**] showed that this probability is one in the case where the probability of occurrence of each type of mirror is $\frac{1}{2}$. Bunimovich and Troubetzkoy [**BunTr**] extended this to the case where the probabilities of the two mirror types add up to one. If either mirror type occurs with probability 0 we see easily, as in Proposition 1.1, that the probability of return is zero. The case where both mirror types occur with some positive probability is still an open problem. Recent numerical evidence ([**Co**], [**ZiKoCo**]) suggests that the probability of return in this case is one. See [**Gr2**] for further discussion of this problem and [**Qu**] for related results.

3. PERIODIC TANGLES

Consider an $m \times n$ block B tiled randomly. Tile the entire plane by repeating B horizontally and vertically. The resulting lattice tangle, T_B, will be called a *periodic* lattice tangle (Figure 3.1).

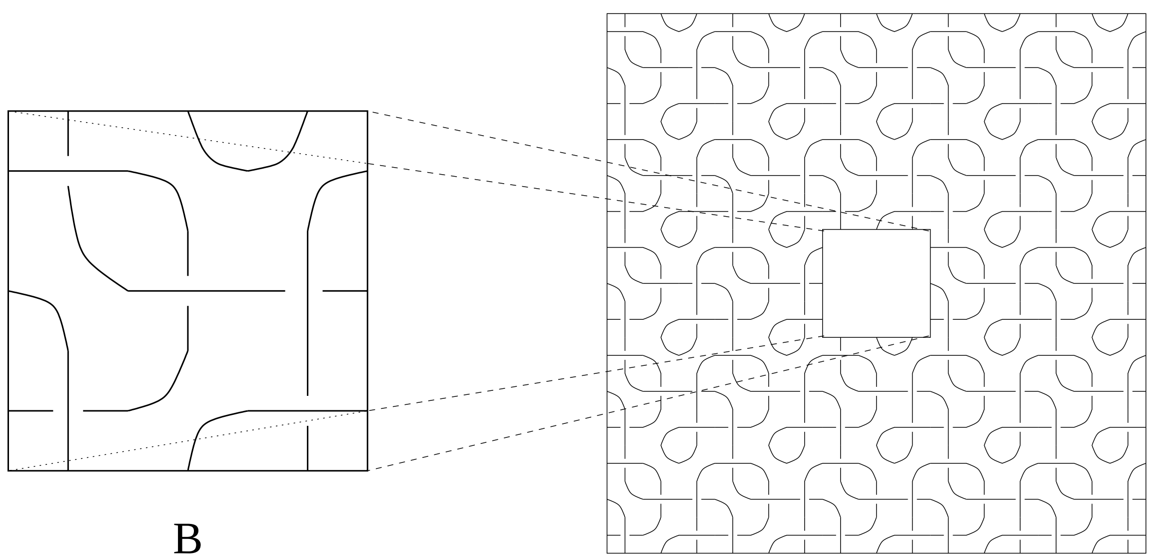


Figure 3.1: Block B and (a segment of) the resulting periodic tangle.

As in Section 2, we can answer some questions about the type of the components of such tangles. However, this construction allows us to do it on the finite level, i.e. by examining only the starting block B.

The following is an algorithm for deciding what type of components (bounded or unbounded) T_B will have:

Step 1: The boundary of B consists of two horizontal and two vertical lines. Assign labels $h_1, h_2, \ldots, h_m$, and $v_1, v_2, \ldots, v_n$ to B as shown in Figure 3.2. Notice that these labels correspond to points of intersection between the bottom and left boundary lines of B and the tangle components. Since opposite boundary lines will be identified in T_B, it makes sense to assume the same labeling for the points on the right and top boundary line as for those of the left and the bottom, respectively.

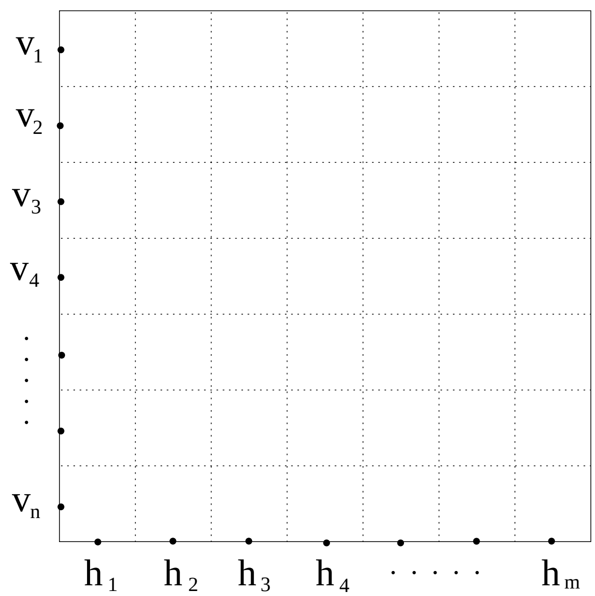


Figure 3.2: Assignment of vertices to B.

Step 2: Starting from h_1, follow the tangle component until you reach another vertex. This completes an oriented *path segment*. Move to the opposite boundary line (to the vertex labeled the same way as the last one). Continue until you get back to h_1 and thus closing the path. Pick a vertex which has not been used yet. Repeat the process obtaining another closed path. Continue until all vertices have been used.

Step 3: For each closed path, construct a graph, G_k, with vertex set the set of vertices used in this path. Between any two vertices, place an edge if there is a single path segment connecting them. The edges are labeled according to the points of output. If e is the edge from vertex a to vertex b where b lies on the left boundary line, then e receives the label l. Similarly, the edge is labeled with r, u, or d if b is on the right, top, or bottom boundary line, respectively (See Figure 3.3 for an example). For each graph G_k, let $|l|, |r|, |u|, |d|$ be the number of edges labeled with l, r, u, d, respectively.

Proposition 3.1: G_k corresponds to a bounded component in T_B if and only if $|l| = |r|$ and $|u| = |d|$.

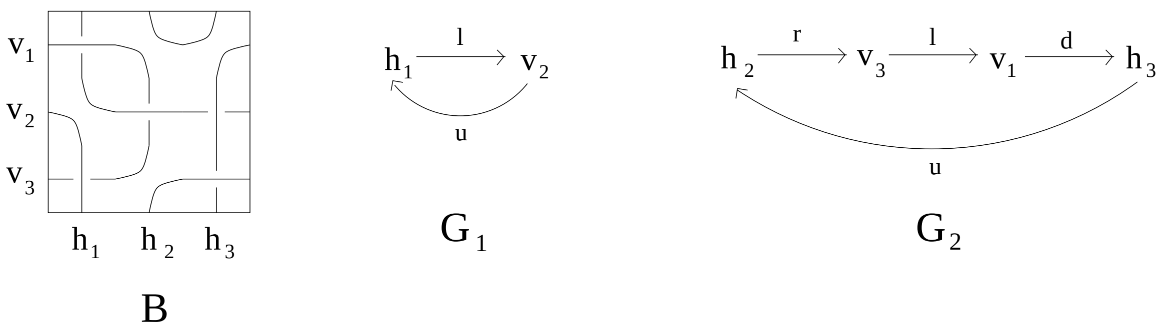


Figure 3.3: Example of a block B and the graphs obtained from it.

Step 4: Up to this point we have only counted components which intersect the boundary of B. To complete the process, identify all (bounded) components that lie inside B.

It will be very difficult and time-consuming to complete this algorithm by hand for large blocks B, and so a computer program can be of use here. We used such a computer program to generate some results for blocks up to size 20×20 from which we were able to form Conjecture 3.2.

Proof of Proposition 3.1: Assume $|l| = |r|$ and $|u| = |d|$. For all $j \in \mathbf{Z}$, let L_j (level j) be the horizontal strip in the plane which consists of infinitely many copies of B such that the bottom boundary line of B lies on the line $y = jn$ (assuming B is an $n \times m$ box). The vertices of G_k correspond to points on the lines separating these strips, while edges labeled with u or d correspond to moves to the level above or below, respectively. Clearly, if $|u| = |d|$ there will be no vertical change in position as we trace an arc from one point on the boundary until its first return. (See example in Figure 3.4).

An analogous argument (considering vertical strips) shows that if $|l| = |r|$, there will be no horizontal change in position. Hence $|u| = |d|$ and $|l| = |r|$ imply zero total change and the desired result follows.

Now assume that $|u| \neq |d|$ or $|l| \neq |r|$ for some graph G_k. Define a *circuit* along G_k to be a walk on the graph which starts and ends at one vertex and goes only once through the rest of the vertices. Then if we trace the path segments in T_B corresponding to one circuit along G_k, there will be a positive horizontal and/or vertical distance between the starting and ending points. Tracing the next sequence of path segments will result in the same displacement. Thus, the tangle component corresponding to G_k will be unbounded. ■

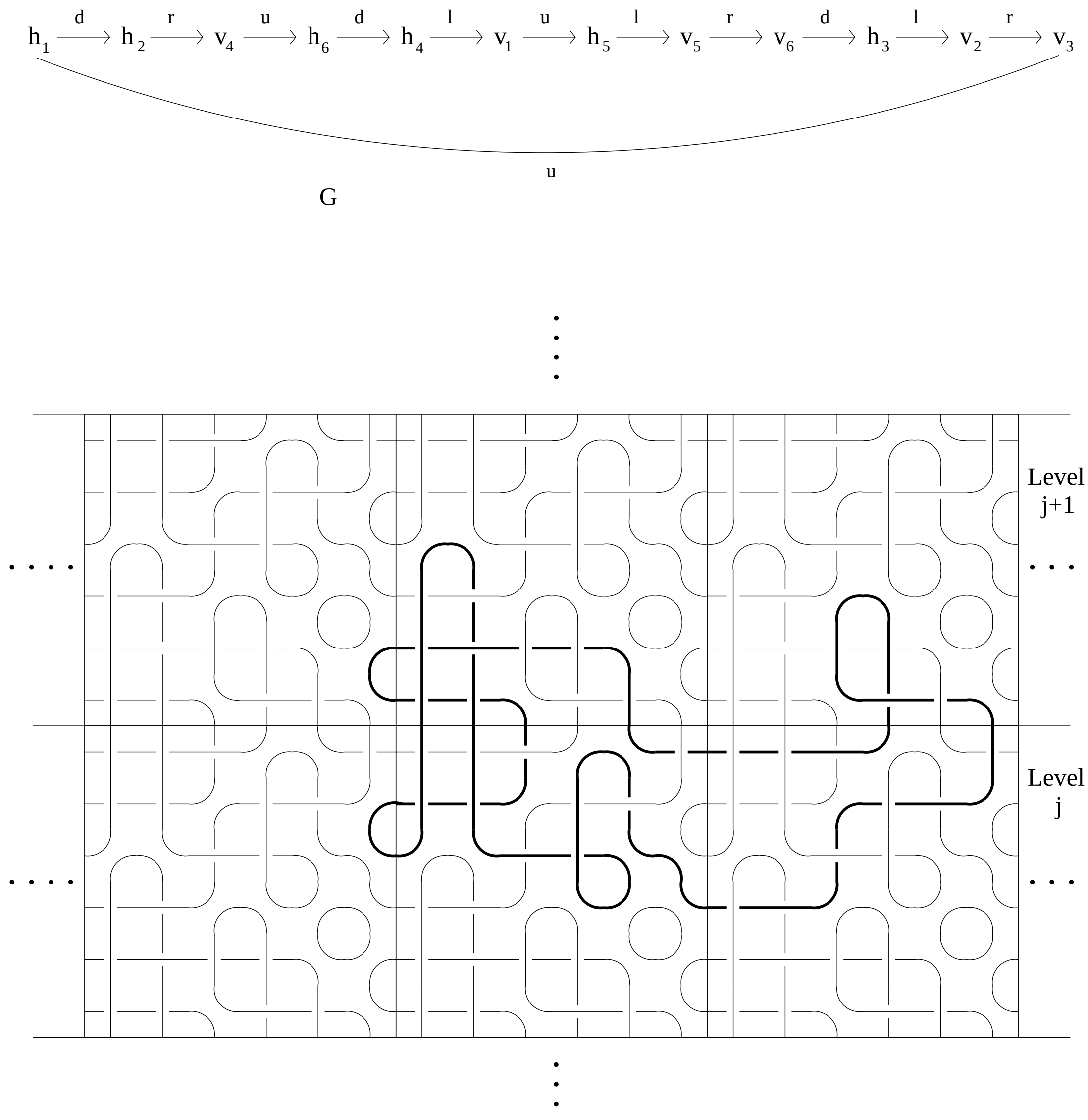


Figure 3.4: Graph G and the bounded component corresponding to it.

The last argument suggests that an unbounded component of T_B has a "slope." If C_k is the unbounded component corresponding to G_k, we define the *slope* of C_k, slope(C_k), to be $\frac{|u|-|d|}{|r|-|l|}$. In this case, "slope" is defined in global terms. It would seem natural to speak of a vertical unbounded tangle component when $|r| - |l| = 0$ but $|u| \neq |d|$. For example, from the block B in Figure 3.5 we obtain the graphs G_1, G_2, and G_3 (Figure 3.6) which correspond to three unbounded tangle components, C_1, C_2, and C_3.

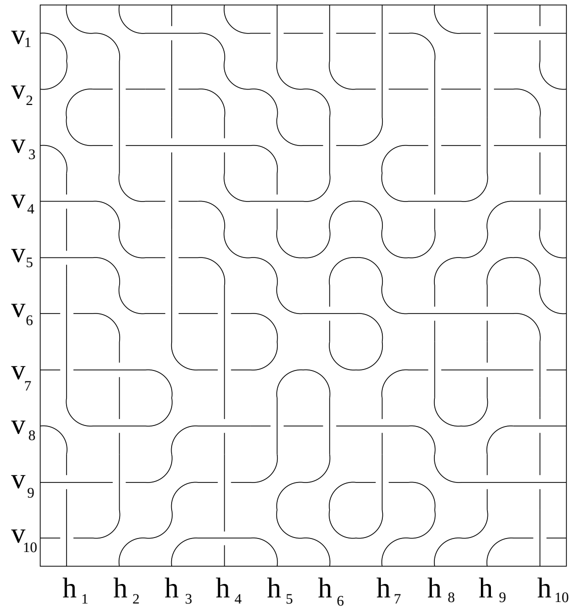


Figure 3.5: Random block B.

It is easy to verify that slope$(C_1) = -1$, C_2 is a vertical unbounded component, and slope$(C_3) = 0$ (hence C_3 is a horizontal unbounded component). A segment of the resulting periodic tangle T_B is shown on Figure 3.7 where gray, black, and dotted lines are used for the components C_1, C_2, and C_3, respectively.

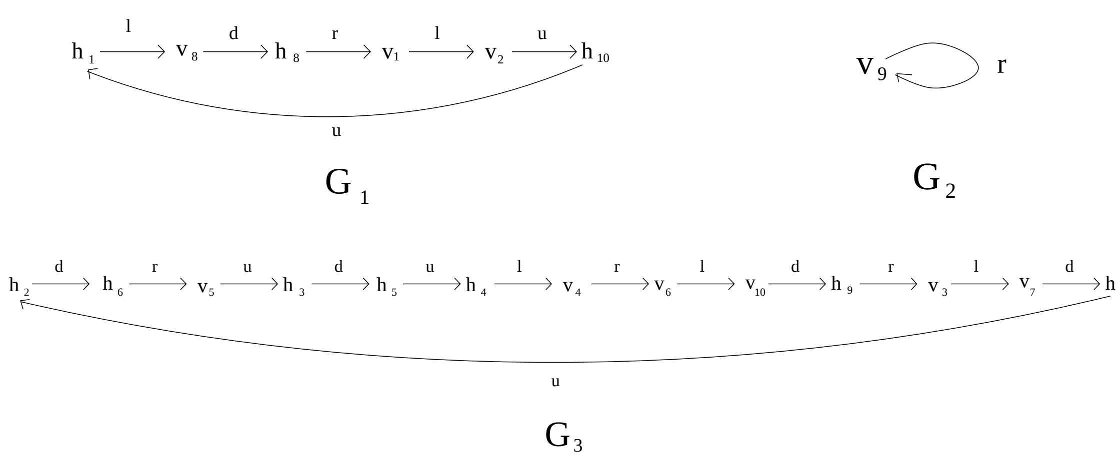


Figure 3.6: Graphs G_1, G_2, and G_3 obtained from B.

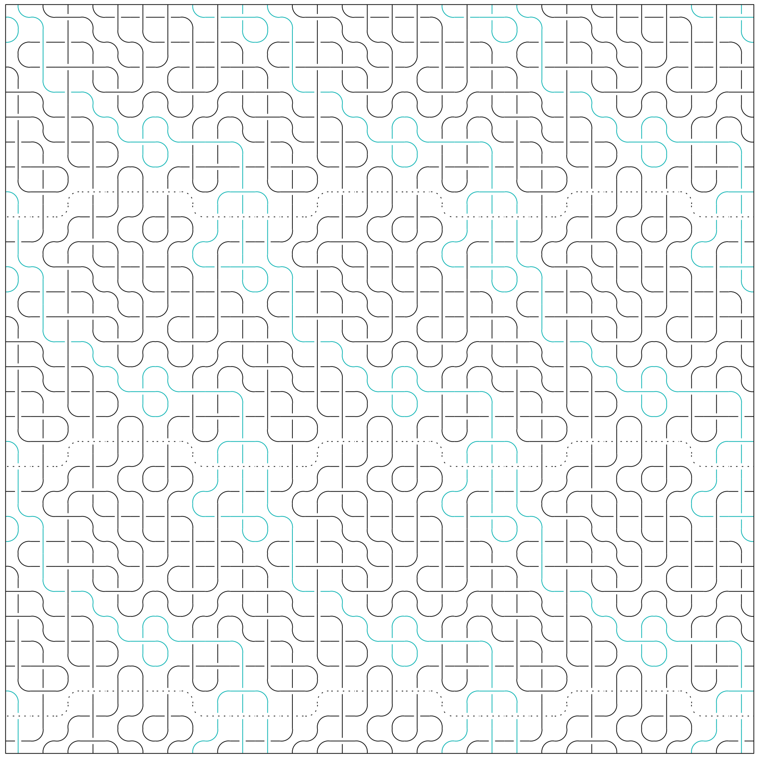


Figure 3.7: Periodic tangle with unbounded components.

In Section 2 we explored the idea of types of tangle components in relation to particular probability assignments. For periodic tangles, we can put such questions in more quantitative form using ratios that reflect the relative frequencies of bounded and unbounded components.

If T_B is a periodic tangle constructed with the $m \times n$ block B, let N be the number of graphs (obtained using the above algorithm) corresponding to unbounded components of T_B, M_1 the number of graphs corresponding to bounded components in T_B, and M_2 the number of bounded components that lie inside the block B (from Step 4 of algorithm). Set $M = M_1 + M_2$. Let $R_M(B) = \frac{M}{M+N}$ and $R_N(B) = \frac{N}{M+N}$. T_B has only bounded (unbounded) components if $R_M(B)$ ($R_N(B)$) is equal to one. This is the idea that we will use in the statement of the following conjecture.

Fix a choice of tile probabilities, as we did for a Bernoulli measure on $\mathcal{A}^{\mathbf{Z}^2}$. For each m and n we can define a measure $\mu_{m,n}$ on the set of $m \times n$ blocks B by choosing the tiles independently according to the given probabilities. The expected value of R_M with respect to this measure is given by

$$E(R_M) = \sum R_M(B)\mu_{m,n}(\{B\})$$

where the sum is taken over all $m \times n$ blocks B.

Conjecture 3.2: If the tiles of Types 3 and 4 (smoothing tiles) both occur with positive probability, then $E(R_M) \to 1$ as $m, n \to \infty$.

In other words, the bigger the block B is, the more likely it is to see mostly bounded components in T_B. With this restatement, we can see that Conjecture 3.2 supports the one stated at the end of Section 2 which says that if tile Types 3 and 4 occur with positive probability then almost every configuration contains only bounded tangle components.

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