VIRTUAL KNOT GROUPS

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ABSTRACT. Virtual knot groups are characterized, and their properties are compared with those of classical knot groups. A coloring theory, generalizing the usual notion of Fox n-coloring, is introduced for virtual oriented links.

1. INTRODUCTION. We recall that a **diagram** D for a classical knot $k \subset \mathbf{R}^3$ is a 4-valent plane graph resulting from a regular projection of k. Extra information, conveyed by a *trompe l'oeil* effect at each vertex, which is then called a **diagram crossing**, makes it possible to reconstruct the knot (Figure 1 (a)).

It is well known that two knots are isotopic if and only if a diagram for one can be transformed into a diagram for the other by a finite sequence of **Reidemeister moves** (see first column of Figure 2). As usual, we regard isotopic knots as the same. Hence we may think of a knot as an equivalence class of diagrams.

In 1996 L. Kauffman introduced the concept of a virtual knot [**Ka1**]. He did this by allowing a third type of crossing, a so-called virtual crossing (Figure 1 (b)), and then generalizing the Reidemeister moves (Figure 2).

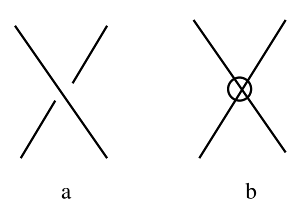

Figure 1: (a) Classical Crossing; (b) Virtual Crossing

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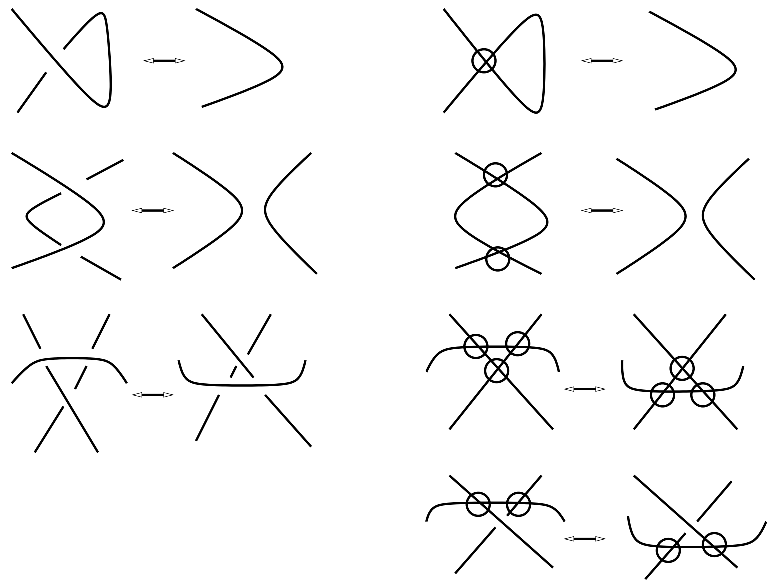

Figure 2: Generalized Reidemeister Moves

A **virtual knot** is now defined to be an equivalence class of diagrams. (Virtual links and oriented virtual knots or links are defined in the obvious way.) A theorem of M. Goussarov, M. Polyak and O. Viro states that if diagrams of a classical knot are equivalent under generalized Reidemeister moves, then they are equivalent under classical Reidemeister moves [**GuPoVi**]. In this sense, the theory of classical knots is embedded in that of virtual knots.

According to [**Ka3**] Kauffman's motivation comes from (1) the study of knots in thickened surfaces of higher genus (classical knot theory being the study of knots in the thickened sphere); (2) the desire to "complete" the set of classical knots in such a way that knots correspond to Gauss codes under a suitable equivalence relation.

We are propelled by a third motivation, one that is group-theoretical. As explained in [**Ka2**],[**Ka3**], a group can be associated to any virtual knot diagram by a generalization of Wirtinger's algorithm: Having first oriented the diagram, one associates a generator to each arc. By an arc we mean a curve in the diagram that begins and ends at classical under-crossings, possibly running through one or more virtual crossings. The relations among the generators are determined by the classical crossings in the usual way (see [**Kaw**], for example). It is clear that reversing the orientation of the diagram does not affect the group. More significantly, equivalent diagrams determine the same group. Hence we can speak of $G(k)$, the **group of a virtual knot** k.

Example 1.1. (Cf. [**Ka2**], [**Ka3**]) The virtual knot k, a modified trefoil, is displayed in Figure 3 together with generators for its group $G(k)$, which has presentation

$$\langle x, y, z \mid xy = zx,\ zx = yz,\ yx = zy\rangle \cong \langle x, y \mid xyx = yxy,\ yxy^{-1} = xyx^{-1}\rangle.$$

It is well-known that $G(k)$ is the group of the 2-twist spun trefoil [**Ze**]. Its commutator subgroup is the cyclic group $\mathbf{Z}/3$ of order three.

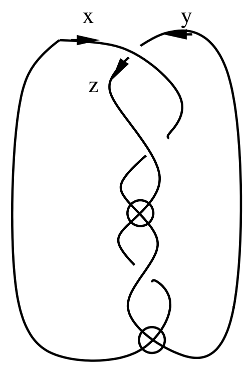

Figure 3: Virtual Knot k

By a theorem of L. Neuwirth [**Ne**], the commutator subgroup of any classical knot group is free whenever it is finitely generated. Hence the group in Example 1.1 is not that of a classical knot. On the other hand, every classical knot group is a virtual knot group. Hence the collection of virtual knot groups is larger than the collection of classical knot groups. We denote the collection of all virtual knot groups by $\mathcal{VK}^1$.

2. CHARACTERIZATION OF VIRTUAL KNOT GROUPS.

We characterize virtual knot groups combinatorially and also topologically. The topological characterization requires the following.

Definition 2.1. Let $B_1^3, \ldots, B_n^3 \subset \mathbf{R}^4$ be a family of mutually disjoint 3-balls, and let $f_1, \ldots, f_n$ be a collection of embeddings $f_i : B^2 \times [0,1] \to \mathbf{R}^4,\ 1 \le i \le n$, with mutually disjoint images such that

$$f_i(B^2 \times [0,1]) \cap \partial B_k^3 = \begin{cases} f_i(B^2, 0), & \text{if } k \equiv i \pmod{n}; \\ f_i(B^2, 1), & \text{if } k \equiv i+1 \pmod{n}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The embedded torus

$$\Big(\bigcup_{i=1}^{n} \partial B_i^3\Big) \cup \Big(\bigcup_{i=1}^{n} f_i(\partial B^2 \times [0,1])\Big) - \Big(\bigcup_{i=1}^{n} f_i(\text{int } B^2 \times \{0,1\})\Big)$$

is called a **ribbon torus** (of n fusions).

Remark. A ribbon torus can be regarded as the result of "piping together" standardly embedded, separated 2-spheres in R^4. The definition of a more familiar notion, that of of a **ribbon 2-knot** (of n fusions), can be recovered from Definition 2.1 by deleting one embedding from the collection $\{f_1, \ldots, f_n\}$. (See [**Kaw**], for example.)

Theorem 2.2. Let G be a group. The following statements are equivalent.

(i) $G \in \mathcal{VK}^1$.

(ii) G has a presentation of the form $\langle x_1, \ldots, x_n \mid x_{i+1} = u_i^{-1} x_i u_i \quad (i \in \mathbf{Z}_n)\rangle$, where $u_1, \ldots, u_n$ are elements of the free group $F(x_1, \ldots, x_n)$.

(iii) G is isomorphic to $\pi_1(\mathbf{R}^4 - T)$, for some ribbon torus T.

Proof. That (i) implies (ii) is an immediate consequence of the Wirtinger algorithm applied to any diagram of a virtual knot with group G.

The equivalence of (ii) and (iii) follows from the theorem of Seifert and Van Kampen. The proof for the analgous statement about ribbon 2-knots can be found in [**Yan**].

We prove that (iii) implies (i). Consider a group presentation as in (ii). By introducing new generators and defining relators, we can assume without any loss of generality that each u_i is one of $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$. For if some u_i is equal to $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$, where $\epsilon_j = \pm 1$, we can replace the generator x_i by $x_{i,0}, \ldots, x_{i,k-1}$, where $x_{i,0} = x_i,\ x_{i,1} = x_{i_1}^{-\epsilon_1} x_{i,0} x_{i_1}^{\epsilon_1},\ \ldots, x_{i,k-1} = x_{i_{k-1}}^{-\epsilon_k} x_{i,k-2} x_{i_{k-1}}^{\epsilon_k}$.

We begin constructing a virtual knot diagram, using the first relation $x_2 = x_{j_1}^{-\epsilon_1} x_1 x_{j_1}^{\epsilon_1}$, as in Figure 4.

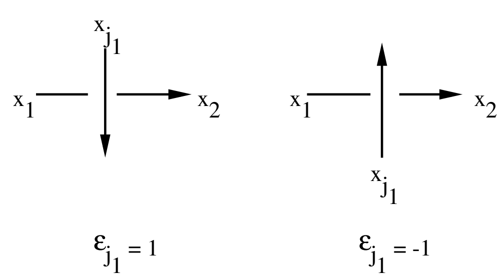

Figure 4: Beginning of Virtual Knot Diagram

We extend the arc labeled x_2 in the same way, using the next relation $x_3 = x_{j_2}^{-\epsilon_2} x_2 x_{j_2}^{\epsilon_2}$. If an arc labeled x_{j_2} has already been drawn, then *we do not draw a second such arc*; we simply continue the existing arc, passing under the arc labeled x_{j_2} in the appropriate direction, and re-emerging as x_3. If an arc labeled x_3 has previously been drawn, then we join the re-emerging arc to it. We introduce virtual crossings as needed.

We repeat the above procedure for each of the arcs labeled $x_3, x_4, \ldots, x_n$. ∎

Corollary 2.3. Assume that G has a presentation of the form $\langle x, y \mid r\rangle$, such that (1) $G/[G,G] \cong \mathbf{Z}$; and (2) the normal closure $\langle\langle x\rangle\rangle$ is equal to G. Then $G \in \mathcal{VK}^1$.

Proof. By introducing a generator a with defining relation $a = yx^{-1}$, and then using the relation to eliminate y, we can assume without any loss of generality that G has a presentation of the form $\langle x, a \mid r \rangle$, where $a \in [G,G]$. The hypothesis (2) implies that the exponent sum of a in r is 1 or -1; we can replace r with an equivalent relation $a = w$, where $w \in F(x,a)$ has zero exponent sum in a. Then $a = \prod_{i=1}^m u_i^{-1} x^{\epsilon_i} u_i$, for some $u_1, \ldots, u_m \in F(x,a)$, and $\epsilon_i = \pm 1,\ 1 \le i \le m$.

Adapting an argument in [**Yaj**], we introduce generators $y_1, \ldots, y_m$ and defining relations $y_i = u_i^{-1} x u_i$, and obtain

$$G \cong \langle x, a, y_1, \ldots, y_m \mid a = \prod_{i=1}^m u_i^{-1} x^{\epsilon_i} u_i,\ y_1 = u_1^{-1} x u_1,\ \ldots,\ y_m = u_m^{-1} x u_m \rangle$$

$$\cong \langle x, a, y_1, \ldots, y_m \mid a = \prod_{i=1}^m y_i^{\epsilon_i},\quad y_1 = u_1^{-1} x u_1,\ \ldots,\ y_m = u_m^{-1} x u_m \rangle$$

We rewrite each word $u_1, \ldots, u_m$, replacing each occurrence of $a^{\pm 1}$ by $(\prod_{i=1}^m y_i^{\epsilon_i})^{\pm 1}$. Finally, we use the first relation to eliminate the generator a. The final presentation satisfies condition (ii) of Theorem 2.2. ∎

We recall that a group G is **residually finite** if for any nontrivial element $g \in G$ there exists a finite-index subgroup $N \lhd G$ such that $g \notin N$. Every classical knot group is residually finite by a theorem stated in [**Th**] (see [**He**] for a proof). However, virtual knot groups need not have this property, as the following example shows.

Example 2.4. The Baumslag-Solitar group $G = \langle x, y \mid x^{-1} y^2 x = y^3 \rangle$ is in $\mathcal{VK}^1$, by Corollary 2.3. It is not residually finite [**Ba**].

Another important property of classical knot groups that fails in the virtual category is coherence. A group is **coherent** if every finitely generated subgroup is finitely presented. By a theorem of P. Scott [**Sc**] the fundamental group of any 3-manifold is coherent. However, in [**Si**] we gave examples of ribbon tori T such that the commutator subgroup of $\pi_1(\mathbf{R}^4 - T)$ is finitely generated but not finitely presented. An explicit example visualized by 3-dimensional cross-sections can be found in [**Ph**].

3. VIRTUAL TANGLE GROUPS. Any Wirtinger presentation for the group of a virtual knot k has the same number of generators as relations. It is well known that if k is a classical knot, then any relation is a consequence of the others. In fact, this property holds for a Wirtinger presentation of the group of any spatial graph (see [**St**], for example.) However, this property can fail when k is a virtual knot. For example, the group in Example 1.1 has no presentation with fewer relations than generators [**Le**]. In this section we will examine some consequences of this failure.

We define a **virtual (n-string) tangle** by diagrams in the obvious way. The **group** $G(t)$ of a virtual tangle t is then defined as for virtual knots, and it can be described by a Wirtinger presentation.

It is convenient to regard a virtual knot diagram as contained the 2-sphere $S^2 = \mathbf{R}^2 \cup \infty$ rather than the plane. This is harmless. Indeed, one can check that the final result of isotoping an arc past ∞ can also be achieved in the plane using generalized Reidemeister moves. Instead of simply pulling the arc over or under the other arcs as we would in the classical case, we introduce virtual crossings and apply generalized Reidemeister moves.

Given a diagram in S^2 for a virtual knot k, we can *cut* k, that is, remove the interior of a small disk neighborhood about a point p on the diagram that is away from any crossing, and thereby obtain a diagram of a virtual 1-string tangle t. It is well known that when k is a classical knot, $G(t) \cong G(k)$. One way to see this is to regard k as a spatial graph with a single vertex, the point p. The Wirtinger relation corresponding to p is $x' = x''$. If we view x' and x'' as generators of $G(t)$ corresponding to the input and output arcs of the tangle t, then since $x' = x''$ is is a consequence of the other Wiritinger relations for the group of the spatial graph, it is also a relation in $G(t)$.

Example 3.1. If we cut a diagram for a virtual knot k, then the group of the resulting virtual 1-string tangle t need not be isomorphic to the group of k. Figure 5 (a) displays a diagram for a virtual knot k with infinite cyclic group. If we cut at the indicated point, then the group of the resulting virtual tangle, indicated in Figure 5 (b), is isomorphic to that of the trefoil. The verification is straightforward.

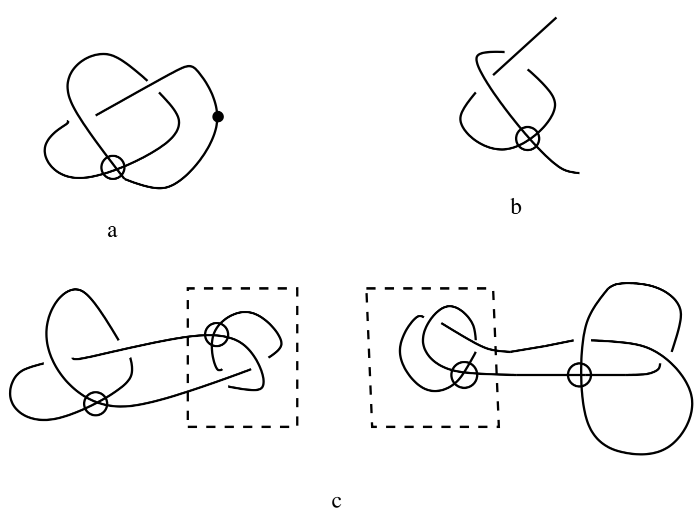

Figure 5: (a) Virtual Knot; (b) Associated 1-String Tangle; (c) Connected Sums

If we attempt to form the connected sum $k\sharp k$ of two copies of k, then the result will depend on the sites where the diagrams are cut and the connection is performed. In Figure 5 (c) we see two connected sums $(k\sharp k)_1$ and $(k\sharp k)_2$: the first has group isomorphic to that of the trefoil, the second has infinite cyclic group. The verification is again routine, and we leave it to the reader.

If we try to deform the diagram for one of the connected sums in Figure 5 (c) into the other by sliding the indicated box along the diagram, a geometric trick for proving that connected sum of oriented classical knots is well defined, then we are forced to apply "forbidden moves" [**GoPoVi**]. These illegal moves appear in Figure 6.

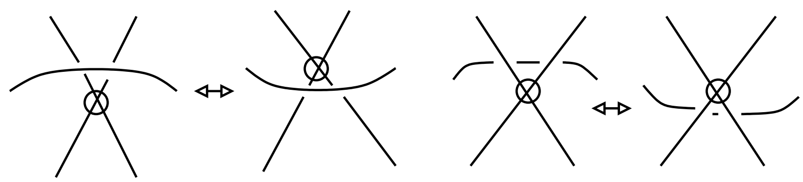

Figure 6: Forbidden Moves

It is interesting to observe that the first forbidden move does not affect the group of the virtual knot. The proof is not difficult. However, using the the generalized Reidemeister moves, the flipping operation and the first forbidden move, we can transform $(k\sharp k)_1$ into $(k\sharp k)_2$. We conclude that the group of the flip of $(k\sharp k)_1$ must be the group of $(k\sharp k)_2$. We have thus arrived at the conclusion of [**GoPoVi**] (see also [**Ka3**]) that the group of a virtual knot and the group of its flip need not be isomorphic. Hence associated to each virtual knot is a *group pair.*

Problem 3.2. Characterize virtual knot group pairs.

4. COLORING VIRTUAL LINK DIAGRAMS. The well-known technique of Fox n-coloring associates a $\mathbf{Z}/n$-module $\mathrm{Col}_{\mathbf{Z}/n}(l)$ to any classical link l. Elements are n*-colorings* of a diagram D of the link: assignments of elements of $\mathbf{Z}/n$ (*colors*) to the arcs of D such that at any crossing, the sum of the colors of the undercrossing arcs is equal to twice the color of the overcrossing. All of this can be done as well for virtual links, with virtual crossings imposing no conditions on the colors. In [**Ka2**] and [**Ka3**] Kauffman notes this fact, and uses it to show that a certain virtual knot k with Jones polynomial equal to 1 is nontrivial. He produces k from a trefoil knot diagram by applying a certain move that does not affect the module of 3-colorings. The conclusion that k is nontrivial then follows from the well-known fact that the trefoil has more 3-colorings than the trivial knot.

In [**SiWi1**] and [**SiWi2**] a generalization of n-coloring for classical oriented links l is introduced. For any positive integer r and any topological group Σ, a space $\mathrm{Col}_{\Sigma,r}(l)$ of

(Σ, r)-colorings is defined. When $r = 2$ the orientation of l is unimportant. When $r = 2$ and Σ is $\mathbf{Z}/n$ with the discrete topology, the space is $\mathrm{Col}_{\mathbf{Z}/n}(l)$.

We will show that the coloring spaces $\mathrm{Col}_{\Sigma,r}(l)$ are also invariants of virtual links. The move that Kauffman applied is the square of a move, we denote by T. We show that for any positive integer r, the iterated move T^r leaves $\mathrm{Col}_{\Sigma,r}(l)$ unchanged.

The spaces $\mathrm{Col}_{\Sigma,r}(l)$, for various r, are all contained in a single space $\mathrm{Col}_{\Sigma,\infty}(l)$ that we define first.

Let Σ be any topological group, and consider the group $\Sigma^{\mathbf{Z}}$ of all bi-infinite sequences (α_j) of elements $\alpha_j \in \Sigma$. The **shift map** $\sigma : \Sigma^{\mathbf{Z}} \to \Sigma^{\mathbf{Z}}$ that sends (α_j) to the sequence (α'_j), where $\alpha'_j = \alpha_{j+1}$, is an automorphism.

Definition 4.1. (cf. [**SiWi2**]) Let Σ be any topological group. Assume that D is a diagram of a virtual oriented link.

(i) A $\Sigma^{\mathbf{Z}}$**-coloring** of D is an assignment of elements $C \in \Sigma^{\mathbf{Z}}$ to the arcs of D such that at any classical crossing

$$(\sigma C_k)C_j = (\sigma C_i)C_k, \tag{3.1}$$

where C_k corresponds to the overcrossing, while C_i, C_j correspond to the undercrossings, and C_j is to the left as we travel in the preferred direction along the arc labeled C_k (see Figure 7).

(ii) A $\Sigma^{\mathbf{Z}}$-coloring is **periodic** if there exists a positive integer r such that $\sigma^r(C) = C$, for every assigned label C. In such a case, we say that the coloring has **period** r.

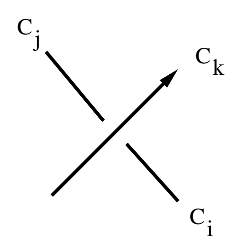

Figure 7: Coloring Convention

The set $\mathrm{Col}_{\Sigma,\infty}(D)$ of $\Sigma^{\mathbf{Z}}$-colorings of D is a closed σ-invariant subspace of $\Sigma^{\mathbf{Z}}$. One can check combinatorially that if D' is a another diagram for l, then $\mathrm{Col}_{\Sigma,\infty}(D)$ and $\mathrm{Col}_{\Sigma,\infty}(D')$ are topologically conjugate; that is, there exists a homeomorphism h between the spaces such that $\sigma \circ h = h \circ \sigma$. Instead, we prefer to sketch the following algebraic argument. The basic ideas come from [**SiWi3**], where the reader can find additional details.

Groups of virtual links are defined via Wirtinger presentations just as they are for virtual knots. Let G be the group of the virtual oriented link l. Consider the homomorphism $\chi : G * \langle x \mid \ \rangle \to \mathbf{Z}$ that maps x and each generator of G to 1. Give the kernel

K the discrete topology, and denote by $\mathrm{Hom}(K,\Sigma)$ the space of homomorphisms ρ, with the compact-open topology. (Intuitively, homomorphisms are close if they agree on many elements of K.) The map $\sigma_x : \mathrm{Hom}(K,\Sigma) \to \mathrm{Hom}(K,\Sigma)$ given by $\sigma_x\rho(a) = \rho(x^{-1}ax)$, for all $a \in K$, is a homeomorphism.

Proposition 4.2. If D is a diagram of a virtual oriented link, then $\mathrm{Col}_{\Sigma,\infty}(D)$ and $\mathrm{Hom}(K,\Sigma)$ are topologically conjugate; that is, there exists a homeomorphism h between them such that $\sigma_x \circ h = h \circ \sigma$.

Proof. It is a consequence of the Reidemeister-Schreier Theorem [**MaKaSo**] that the kernel K has a presentation of the form $\langle a_j, b_j, \ldots \mid r_j, s_j, \ldots\rangle$, where j ranges over $\mathbf{Z}$. Here the number of generator families $a_j, b_j, \ldots$ is finite, as is the number of relator families $r_j, s_j, \ldots$. Also, the relators $r_{j+\nu}, s_{j+\nu}, \ldots$ are obtained from $r_j, s_j, \ldots$ by adding ν to the subscripts of each symbol that appears. The generators of K arise in a natural way. If $x_1, x_2, \ldots$ are the generators of G, then $a_0, b_0, \ldots$ are $x_1x^{-1},\ x_2x^{-1}, \ldots$, respectively. For each j, the symbols $a_j, b_j, \ldots$ represent $x^{-j}a_0x^j, x^{-j}b_0x^j, \ldots$.

A homomorphism $\rho : K \to \Sigma$ consists of an assignment of elements $\alpha_j, \beta_j, \ldots$ to the generators $a_j, b_j, \ldots$, such that each relation $r_j, s_j, \ldots$ becomes the identity element upon substitution. Since a typical Wirtinger relation $cx \cdot bx = ax \cdot cx$ is equivalent to $x^{-j-1}cx^{j+1} \cdot x^{-j}bx^j = x^{-j-1}ax^{j+1} \cdot x^{-j}cx^j$ and hence to $c_{j+1}b_j = a_{j+1}c_j$, for any j, such homomorphisms correspond bijectively to the $\Sigma^{\mathbf{Z}}$-colorings of D. The correspondance is shift-commuting and continuous. ∎

In view of Proposition 4.2 we can define $\mathrm{Col}_{\Sigma,\infty}(l)$ to be $\mathrm{Col}_{\Sigma,\infty}(D)$, where D is any diagram from l. For any positive integer r we define the subspace $\mathrm{Col}_{\Sigma,r}(l)$ of (Σ, r)-colorings in the following way.

The normal subgroup N_r of $G * \langle x \mid \, \rangle$ generated by the rth powers of the generators $x, x_1, x_2, \ldots$, is independent of the diagram D of l, since the generators corresponding to any diagram D' obtained from D by a generalized Reidemeister move are conjugates of the generators corresponding to D. We can describe N_r as the normal closure $\langle\langle x^r, (ax)^r, (bx)^r, \ldots\rangle\rangle$, which is equal to $\langle\langle x^r, x^{-r}(ax)^r, x^{-r}(bx)^r, \ldots\rangle\rangle$. The elements $x^{-r}(ax)^r,\ x^{-r}(bx)^r, \ldots$ can be rewritten as $a_ra_{r-1}\cdots a_1,\ b_rb_{r-1}\cdots b_1,\ \ldots$, and it is now a straightforward matter to check that $N_r \cap K$ is the normal closure in K of the elements $a_{r+j}a_{r+j-1}\cdots a_{j+1},\ b_{r+j}b_{r+j-1}\cdots b_{r+j}, \ldots$, where j ranges over $\mathbf{Z}$. Hence the (discrete) group $K/N_r \cap K$ is an invariant $\hat{K}_r$ of the virtual oriented link l. If l is a classical oriented link, then $\hat{K}_r$ is isomorphic to the fundamental group of the r-fold cyclic cover of S^3 branched over l (see [**SiWi2**]).

Definition 3.3. Let l be a virtual oriented link, Σ a topological group and r a positive integer. A (Σ, r)-**coloring** of l is a homomorphism $\rho : K/N_r \cap K \to \Sigma$. The set

$\mathrm{Hom}(K/N_r \cap K, \Sigma)$ of all such homomorphisms, endowed with the compact-open topology, is dentoted by $\mathrm{Col}_{\Sigma,r}(l)$.

From what we have said, $\mathrm{Col}_{\Sigma,r}(l)$ is an invariant of the virtual oriented link l. We can think of a (Σ, r)-coloring as a $\Sigma^{\mathbf{Z}}$-coloring of a diagram D such that the product in reverse order of any r consecutive coordinates of any assigned label is the identity. Such a coloring is necessarily periodic with period r. (Cf. Definition 4.2 [**SiWi2**].)

Consider the moves T and $\bar{T}$ described in Figure 8. In [**Ka2**] and [**Ka3**] Kauffman observes that the composition T^2 applied to any virtual knot diagram leaves the module of n-colorings unaffected, for any n. The following result is in the spirit of [**Pr**].

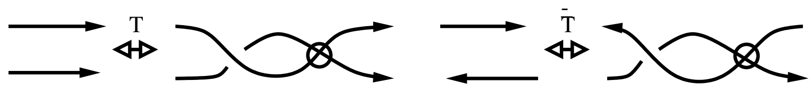

Figure 8: Moves T and $\bar{T}$

Proposition 4.4. Let l, l' be oriented virtual links, Σ a topological group and r a positive integer. If a diagram of l' can be obtained from a diagram of l by a finite sequence of T^r-, $\bar{T}^r$-moves and generalized Reidemeister moves, then $\mathrm{Col}_{\Sigma,r}(l')$ is homeomorphic to $\mathrm{Col}_{\Sigma,r}(l)$.

Proof. It suffices to consider the case that D' is obtained from D by a single T^r-move or a single $\bar{T}^r$-move. We will consider the first, and leave the second case, which is similar, to the reader.

If we have performed a T^r-move at a site of D, then any elements $C, C' \in \Sigma^{\mathbf{Z}}$ assigned to the left-most arcs determine unique labels for each of the new arcs that result from the move. (See Figure 9.) The top and bottom right-most arcs receive the labels C and $(\sigma^r C \cdot \sigma^{r-1} C \cdots \sigma C)^{-1} \cdot \sigma^r C' \cdot (\sigma^{r-1} C \cdot \sigma^{r-2} C \cdots C)$, respectively, which are C and C', in view of Definition 4.3. In this way the (Σ, r)-colorings of D correspond bijectively to those of D', and the result follows. ∎

We can also understand the effect of a T^r-move on the group of a virtual oriented link. Cut the diagram D along the top arc at the site, obtaining a virtual tangle t. Note that the single Wirtinger generator x_0 is replaced by two generators x_0', x_0''; of course, Wirtinger relations for $G(t)$ corresponding to crossings outside of the site must be rewritten using the appropriate choice of x_1', x_1''. Once this has been done, the T^r-move adds the relation $x_1^{-r} x_0' x_1^r = x_0''$.

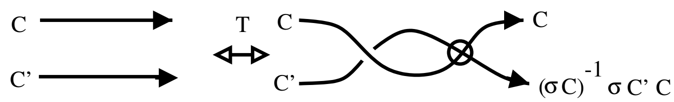

Figure 9: Effect of a T-move on (Σ, r)-colorings

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