

VIRTUAL TANGLES AND A THEOREM OF KREBES

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ABSTRACT. Let t be a virtual 2-string tangle embedded in a virtual knot or link k . If a prime p divides the determinants of both the numerator and the denominator closure of t , then p divides the determinant of k .

An n -string tangle consists of n disjoint arcs properly embedded in the 3-ball. Two tangles are regarded as the same if there is an isotopy of the 3-ball that takes one tangle to the other while fixing each point of the boundary.

Tangles, like classical knots, are represented by diagrams. Two diagrams represent the same tangle if and only if they are related by a finite sequence of Reidemeister moves. From a 2-string tangle t one obtains a link $n(t)$, its *numerator* closure, by joining the top ends and the bottom ends of the tangle. The *denominator* closure $d(t)$ is similarly formed by joining the left-hand ends and the right-hand ends. See Figure 1.

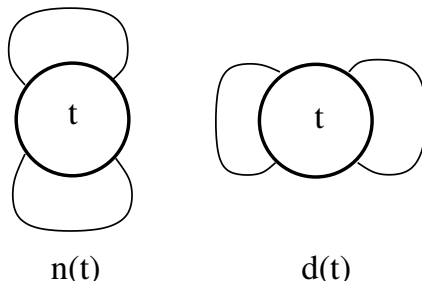


Figure 1: Diagrams of $n(t)$ and $d(t)$

In [Kr] D. Krebs studied the manner in which a diagram of a 2-string tangle t can sit inside a diagram for a link l . He proved that if a positive integer n divides the determinants of both $n(t)$ and $d(t)$, then n also divides the determinant of l . (See below for definitions.) We extend Krebs's result for n prime to the category of virtual tangles and links. Our proof, which employs Fox coloring, is both short and elementary.

Motivated by several considerations, L. Kauffman introduced the notion of a virtual link in 1996 [Ka1]. We briefly review the definitions involved. The reader might consult [Ka2] for details. Recall that a diagram for a classical link is a 4-regular graph in the plane with information at each vertex indicating how the link crosses itself when viewed from

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a fixed perspective. The decorated vertex is called a crossing. A *virtual link diagram* is defined in the same way. However, such a diagram may contain another type of crossing, a “virtual crossing.” Classical crossing and virtual crossing conventions appear in Figure 2. In some respects virtual crossings are treated as though they are not there. For example, the arcs of a virtual link diagram are defined to be the maximal connected components, just as they are for classical link diagrams, regardless of the virtual crossings that they might contain.

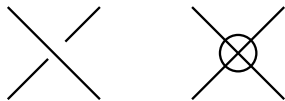


Figure 2: Classical Crossing and Virtual Crossing

Two virtual link diagrams are equivalent if one can be obtained from the other by a finite sequence of (classical) Reidemeister moves or “virtual Reidemeister moves.” Virtual Reidemeister moves are described in Figure 3. By a generalized Reidemeister move we mean either a classical or virtual Reidemeister move [Ka3].

A *virtual link* is an equivalence class of diagrams. We define *virtual tangle diagram* and *virtual tangle* in the same manner.

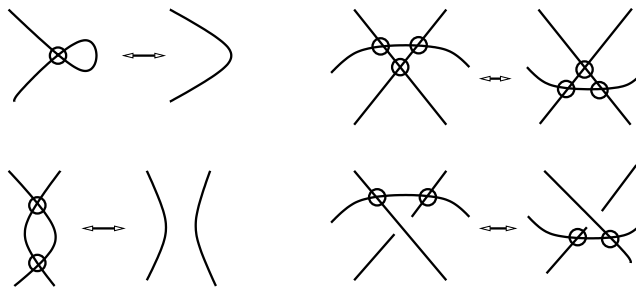


Figure 3: Virtual Reidemeister Moves

Virtual knots generalize classical ones, in the sense that classical diagrams are equivalent under (classical) Reidemeister moves whenever they are equivalent under generalized Reidemeister moves. A proof can be found in [GuPoVi] (see also [Ka3]).

Consider any virtual link or tangle diagram \mathcal{D} , and let p be a prime. A (Fox) p -coloring of \mathcal{D} is an assignment of elements (*colors*) of the cyclic group \mathbf{Z}/p to the arcs of \mathcal{D} in such a way that at any classical crossing twice the color of the overcrossing arc is equal to the sum of the colors of the two undercrossing arcs. No requirement is imposed at virtual crossings. Of course, we can always p -color \mathcal{D} so that each arc receives the same color. Such monochromatic p -colorings are said to be *trivial*.

Any sum of two p -colorings of \mathcal{D} is again a p -coloring of \mathcal{D} . We denote the vector space of p -colorings of \mathcal{D} by $\text{Col}_p(\mathcal{D})$. Whenever \mathcal{D} is a virtual 2-string tangle diagram,

we define $\text{Col}_p(\mathcal{D}, \partial\mathcal{D})$ to be the subspace of $\text{Col}_p(\mathcal{D})$ consisting of those p -colorings that assign zero to each of the four ends of \mathcal{D} . Clearly there is an exact sequence

$$0 \rightarrow \text{Col}_p(\mathcal{D}, \partial\mathcal{D}) \xrightarrow{i} \text{Col}_p(\mathcal{D}) \xrightarrow{r} (\mathbf{Z}/p)^4,$$

where i is the inclusion and r the restriction homomorphism. It is a straightforward matter to verify that spaces associated to equivalent diagrams are isomorphic (see [Ka2] for the case of a virtual link diagram).

For any diagram \mathcal{D} of a virtual link l we see that $\text{Col}_p(\mathcal{D})$ is the direct sum of the 1-dimensional subspace of trivial p -colorings and the subspace $\text{Col}_p^0(\mathcal{D})$ consisting of p -colorings that assign zero to a fixed (but arbitrary) arc of \mathcal{D} . Notice that \mathcal{D} admits a nontrivial p -coloring exactly when $\text{Col}_p^0(\mathcal{D})$ is nonzero.

In order to determine $\text{Col}_p^0(\mathcal{D})$, we label the arcs of \mathcal{D} by distinct variables x_i . Each classical crossing determines a linear equation of the form $x_i + x_j - 2x_k = 0$. The system of equations can be described by a matrix M that is square, say $r \times r$. Since \mathcal{D} admits p -colorings for every prime, it follows that the determinant of M vanishes. More interesting is the absolute value of the gcd of the $(r-1) \times (r-1)$ subdeterminants of M , called the *determinant* of l and denoted by $\text{Det}(l)$. As in the case of classical links the determinant of a virtual link is independent of the diagram. Moreover, it is easy to see from the system of equations that $\text{Col}_p^0(\mathcal{D})$ is nonzero exactly when p divides $\text{Det}(l)$. We summarize our discussion, generalizing a well-known result for classical links.

Lemma 1. Let p be a prime. A virtual link l has a nontrivial p -coloring if and only if p divides $\text{Det}(l)$.

We will say that a virtual 2-string tangle t *embeds* in a virtual link l if some diagram of t extends to a diagram of l .

Theorem 1. Let t be a virtual 2-string tangle embedded in a virtual link l . If a prime p divides $\text{Det}(n(t))$ and $\text{Det}(d(t))$, then p also divides $\text{Det}(l)$.

Proof. Let \mathcal{D} be a diagram for the tangle t that extends to a diagram $\bar{\mathcal{D}}$ for the link l . We consider two cases.

Case 1. $\text{Col}_p(\mathcal{D}, \partial\mathcal{D})$ is nontrivial.

We extend a nonzero member of $\text{Col}_p(\mathcal{D}, \partial\mathcal{D})$ over $\bar{\mathcal{D}}$, assigning zero to each of the remaining arcs. By Lemma 1 the prime p divides $\text{Det}(l)$.

Case 2. $\text{Col}_p(\mathcal{D}, \partial\mathcal{D})$ is trivial.

Let $n(\mathcal{D})$ be the numerator closure of a diagram for t . In any nontrivial p -coloring of $n(\mathcal{D})$ the colors of the top and bottom arcs must be distinct; for if the colors were

identical, then subtracting the common color from that of every arc would produce a nontrivial element of $\text{Col}_p(\mathcal{D}, \partial\mathcal{D})$. Since p divides $\text{Det}(n(\mathcal{D}))$, we can consider a nontrivial p -coloring of $n(\mathcal{D})$. By first subtracting a trivial p -coloring so that the bottom arc is colored zero, and then by multiplying the p -coloring, we find that there exist p -colorings of $n(\mathcal{D})$ such that the color of the top arc is arbitrary while the color of the bottom arc is zero.

The diagram \mathcal{D} can now be p -colored in such a way that the top ends receive the same arbitrary color α while the bottom arcs are colored zero. A similar argument using the hypothesis that p divides $\text{Det}(d(\mathcal{D}))$ shows that we can also p -color \mathcal{D} so that the right-hand ends receive the same arbitrary color β while the left-hand are colored zero. Adding these colorings together, and then adding an arbitrary color γ to every arc produces a p -coloring of \mathcal{D} in which the end arcs are colored subject only to the condition that the alternating sum of colors around the end arcs is zero. See Figure 4.

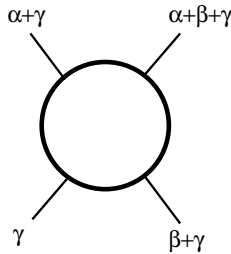


Figure 4: p -Coloring \mathcal{D}

Assume now that \mathcal{D} is contained in a diagram $\bar{\mathcal{D}}$ of any virtual link l . Let \mathcal{D}' be the complementary 2-tangle of \mathcal{D} in $\bar{\mathcal{D}}$. Since the homogeneous system of equations determining p -colorings of any virtual 2-string tangle consists of two more equations than the number of variables, we can find a nontrivial p -coloring of \mathcal{D}' satisfying the additional condition that the alternating sum of colors around its end arcs is zero. By what we have said above we can extend the coloring across \mathcal{D} . Hence $\bar{\mathcal{D}}$ has a nontrivial p -coloring, and so p divides $\text{Det}(l)$. ■

Case 2 of the above proof is vacuous for classical tangles and links. It is not difficult to see that Lemma 2.2(g) of [Pr1] implies that for any diagram \mathcal{D} of a classical 2-string tangle and any prime p , the dimension of the image $r(\text{Col}_p(\mathcal{D}))$ never exceeds 2. We are grateful to J. Przytycki for informing us about the lemma and providing us with its proof [Pr2]. The proof, which uses the Goeritz matrix, will appear in [JaPr].

For virtual tangles and links, case 2 of the above proof is not vacuous. Figure 5 is a diagram \mathcal{D} of a virtual 2-string tangle t with labels representing arbitrary elements of $\mathbf{Z}/3$. From it we see that both $n(\mathcal{D})$ and $d(\mathcal{D})$ can be nontrivially 3-colored. (In fact, $\text{Det}(n(t)) = 3, \text{Det}(d(t)) = 0$.) Since the colors of the end arcs uniquely determine the remaining colors, we find that $\text{Col}_3(\mathcal{D}, \partial\mathcal{D}) = 0$. In this example, the dimension of

$r(\text{Col}_3(\mathcal{D}))$ is 4. This phenomenon is related to the fact that Wirtinger presentations of virtual knot groups need not contain redundant relators as they do for classical knots. The fact is explored in [SiWi].

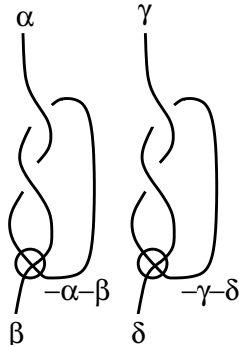


Figure 5: Virtual 2-string tangle t

References.

- [GoPoVi] M. Goussarov, M. Polyak and O. Viro, “Finite type invariants of classical and virtual knots,” preprint: math.GT/1981/9810073.
- [JaPr] F. Jaeger and J. H. Przytycki, “A non-commutative version of the Goeritz matrix of a link,” in preparation.
- [Ka1] L. H. Kauffman, Talks at MSRI Meeting in January 1997, AMS Meeting at University of Maryland, College Park in March 1997.
- [Ka2] L. H. Kauffman, “Virtual knot theory,” preprint.
- [Ka3] L. H. Kauffman, “An introduction to virtual knot theory,” preprint.
- [Kaw] A. Kawauchi, A Survey of Knot Theory, Birkhäuser Verlag, Berlin 1996.
- [Kr] D. A. Krebs, “An obstruction to embedding 4-tangles in links,” to appear in *J. Knot Theory Ramifications*.
- [Pr1] J. H. Przytycki, “3-Coloring and other elementary invariants of knots,” in Knot Theory, Banach Center Publications 42, Warsaw 1998.
- [Pr2] J. H. Przytycki, Private communication, 02.12.98.
- [SiWi] D. S. Silver and S. G. Williams, “Virtual knot groups,” preprint.

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