

## Introduction to Symbolic Dynamics

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ABSTRACT. We give an overview of the field of symbolic dynamics: its history, applications and basic definitions and examples.

### 1. Origins

The field of symbolic dynamics evolved as a tool for analyzing general dynamical systems by discretizing space. Imagine a point following some trajectory in a space. Partition the space into finitely many pieces, each labeled by a different symbol. We obtain a *symbolic trajectory* by writing down the sequence of symbols corresponding to the successive partition elements visited by the point in its orbit. We may ask: Does the symbolic trajectory completely determine the orbit? Can we find a simple description of the set of all possible symbolic trajectories? And, most important, can we learn anything about the dynamics of the system by scrutinizing its symbolic trajectories? The answers to these questions will depend not only on the nature of our dynamical system, but on the judicious choice of a partition.

Hadamard is generally credited with the first successful use of symbolic dynamics techniques in his analysis of geodesic flows on surfaces of negative curvature in 1898 [Ha]. Forty years later the subject received its first systematic study, and its name, in the foundational paper of Marsten Morse and Gustav Hedlund [MH]. Here for the first time symbolic systems are treated in the abstract, as objects in their own right. This abstract study was motivated both by the intrinsic mathematical interest of symbolic systems and the need to better understand them in order to apply symbolic techniques to continuous systems. However, a further impetus was given by the emergence of information theory and the mathematical theory of communication pioneered by C.E. Shannon [Sh].

Symbolic dynamics has continued to find application to an ever-widening array of continuous systems: hyperbolic diffeomorphisms, maps of the interval, billiards, complex dynamics and more. At the same time it contributes to, and finds inspiration in, problems arising in the storage and transmission of data, as we will see in Brian Marcus's chapter. Computer simulations of continuous systems necessarily involve a discretization of space, and results of symbolic dynamics help us

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1991 *Mathematics Subject Classification*. Primary 37B10.

*Key words and phrases*. Symbolic Dynamics.

The author was supported in part by NSF Grant #0071004.

understand how well, or how badly, the simulation may mimic the original. And symbolic dynamics *per se* has proved a bottomless source of beautiful mathematics and intriguing questions.

There are two excellent texts on symbolic dynamics. *An Introduction to Symbolic Dynamics and Coding*, by Douglas Lind and Brian Marcus [LM], has the more modest prerequisites (for example, no prior knowledge of topology or measure theory is assumed), while B. Kitchens's more compact *Symbolic Dynamics: One-sided, Two-sided and Countable State Markov Shifts* [Ki] assumes basic first-year graduate mathematics. For the most part, we have followed the notation and terminology of [LM] in this survey. Also highly recommended is the collection [BMN] of survey articles from the 1997 Summer School on Symbolic Dynamics and its Applications in Frontera, Chile. The selection of topics is largely complementary to that of this short course.

## 2. Two simple examples

Consider the unit interval  $I = [0, 1)$  and the map  $f$  that sends  $x \in I$  to  $\{2x\}$ , the fractional part of  $2x$ . We are interested in the orbit  $x, f(x), f^2(x) = f(f(x)), \dots$ . If we wanted to trace this orbit on a computer screen we might begin by resolving the interval into  $2^{10}$  pixels. However, we will content ourselves with a much cruder discretization of space: we will break  $I$  into just two parts,  $I_0 = [0, \frac{1}{2})$  and  $I_1 = [\frac{1}{2}, 1)$ . We assign to  $x$  a symbolic trajectory  $x_0x_1x_2\dots$  where  $x_i$  is 0 or 1 according as  $f^i(x)$  is in  $I_0$  or  $I_1$ . A little consideration will show that the expression  $.x_0x_1x_2\dots$  is simply a binary expansion of the number  $x$ . Hence  $x$  is completely determined by its symbolic trajectory. We see here an exchange of spatial information for time series information mediated by dynamics: We can recover the complexity of the continuum  $I$  from our crude 2-element partition, provided that we observe the evolution of the system for all time.

What symbolic trajectories will appear in this scheme? All binary sequences except those that end in  $111\dots$ . This awkward exception can be removed by working instead with closed intervals  $I = [0, 1]$ ,  $I_0 = [0, \frac{1}{2}]$  and  $I_1 = [\frac{1}{2}, 1]$ , and mapping sequences to points instead of the other way around. Beginning with a binary sequence  $x_0x_1x_2\dots$ , we can assign to it the unique point

$$x = \bigcap_{i=1}^{\infty} f^{-i}(I_{x_i})$$

that has that symbolic itinerary. Then, for example,  $\frac{1}{2}$  will arise from two symbolic trajectories, corresponding to the two binary expansions  $\frac{1}{2} = .1000\dots = .0111\dots$ . It is common practice in the application of symbolic techniques to sacrifice strict one-to-one correspondence for a simpler description of the set of symbolic trajectories.

Our map  $f$  now has a very pleasant symbolic representation. If  $x = .x_0x_1x_2\dots$  then  $f(x) = \{2x\} = .x_1x_2x_3\dots$ . We shift the symbolic sequence to the left and lop off the initial symbol. The key to the utility of symbolic dynamics is that the dynamics is given by a simple coordinate shift. Dynamic properties that might have seemed elusive in the original setting now become transparent. For instance, we can immediately identify the points of period 3 (that is, with  $f^3(x) = x$ ). They are the eight points with repeating symbolic trajectories  $x_0x_1x_2x_0x_1x_2\dots$ . Note that the set of points with symbolic representation beginning with some fixed

initial string  $x_0x_1\dots x_n$  is the dyadic interval  $[k/2^{n+1}, (k+1)/2^{n+1}]$ , where  $k = x_0 \cdot 2^n + x_1 \cdot 2^{n-1} + \dots + x_n$ . The orbit of a point is dense in  $I$ , visiting every interval no matter how small, if and only if its symbol sequence contains all possible finite strings of 0's and 1's.

As a variation on the first example, consider the map  $g(x) = \{\gamma x\}$  on  $I$ , where  $\gamma = (1 + \sqrt{5})/2$  is the golden mean. We let  $I_0 = [0, \frac{1}{\gamma}]$  and  $I_1 = [\frac{1}{\gamma}, 1]$ . Since  $\gamma = 1 + \frac{1}{\gamma}$ , we have  $g(I_0) = I$  and  $g(I_1) = I_0$ . A point that lands in  $I_1$  under some iterate of  $g$  must move to  $I_0$  at the next iteration. In fact, it is not hard to see that the set of symbolic trajectories is exactly the set of binary sequences that do not contain the string 11.

The symbolic trajectory  $x_0x_1x_2\dots$  corresponds to a series expansion  $x = x_0\gamma^{-1} + x_1\gamma^{-2} + x_2\gamma^{-3} + \dots$ . Expansions of numbers with respect to a non-integer base  $\beta$  are called *beta expansions*. There is a very interesting literature relating dynamic properties of symbolic systems obtained by beta expansions to the number-theoretic properties of beta. The chapter by C. Frougny in [BMN] provides an up-to-date survey.

### 3. Full shifts and subshifts

We let  $\mathcal{A}$  denote a symbol set or *alphabet*, which for now we assume to be finite. The (two-sided) *full  $\mathcal{A}$ -shift* is the dynamical system consisting of the set of biinfinite symbol sequences, together with the *shift map*  $\sigma$  that shifts all coordinates to the left. More formally, our space is

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$$

and the map  $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$  satisfies  $(\sigma x)_i = x_{i+1}$ . If  $\mathcal{A} = \{0, 1, \dots, n-1\}$  we call  $\mathcal{A}^{\mathbb{Z}}$  the full  $n$ -shift.

The advantage of working with biinfinite sequences is that the shift map is invertible. However, we may also consider the one-sided  $\mathcal{A}$ -shift  $\mathcal{A}^{\mathbb{N}}$ , with the truncating shift map described in the previous section. These arise naturally as symbolic representations of noninvertible maps like the map  $x \rightarrow \{2x\}$  on  $I$ . For simplicity we state most of our definitions for two-sided shifts; the one-sided analogue is generally clear.

We often think of an element of  $\mathcal{A}^{\mathbb{Z}}$  as a time series, with  $x_0$  representing the present location or state of our trajectory,  $(x_i)_{i < 0}$  its past history and  $(x_i)_{i > 0}$  its future. The action of the shift map is like a tick of the clock, moving us one step into the future.

We consider two points of  $\mathcal{A}^{\mathbb{Z}}$  to be close to one another if they agree on a large central block  $x_{-n} \dots x_n$  of coordinates. To be more concrete, we can define the distance between distinct points  $x$  and  $y$  to be  $d(x, y) = 2^{-n}$  where  $n$  is the smallest integer with  $x_{-n} \neq y_{-n}$  or  $x_n \neq y_n$ . This is a metric, and induces the product topology on  $\mathcal{A}^{\mathbb{Z}}$ . The map  $\sigma$  and its inverse are continuous: if  $x$  and  $y$  agree on their central  $2n+1$  coordinates, then  $\sigma x$  and  $\sigma y$  agree at least on their central  $2n-1$  coordinates.

A *subshift* or *shift space* is a closed subset of some full shift  $\mathcal{A}^{\mathbb{Z}}$  that is invariant under the action of  $\sigma$ . For example, the set of binary sequences that do not contain the string 11 is a subshift of the 2-shift. It is closed because its complement is open: if a sequence contains 11 then every sequence sufficiently close to it does as well. This subshift is often called the *golden mean shift*, in part because of its

connection to the golden mean beta expansion. More generally, let  $\mathcal{F}$  be any set of finite strings (also called *words* or *blocks*) of symbols of  $\mathcal{A}$ . The set of sequences that do not contain any word of  $\mathcal{F}$  is a subshift  $X_{\mathcal{F}}$  of  $\mathcal{A}^{\mathbb{Z}}$ . In fact, every subshift is of this type, as an easy topological argument will show.

If  $X_{\mathcal{F}}$  is determined by a finite set  $\mathcal{F}$  of “forbidden” words, we call  $X_{\mathcal{F}}$  a *(sub)shift of finite type*, or SFT for short. This is the most fully studied class of symbolic dynamical systems, and the one that has been exploited most in the analysis of general dynamical systems. The systems originally considered by Hadamard were of this type.

For any subshift  $X$  we will denote the set of words of length  $n$  that appear in some element of  $X$  by  $\mathcal{B}_n(X)$ , the *allowed  $n$ -blocks* of  $X$ . If  $\mathcal{F}$  is a set of words of length not exceeding  $m$ , then the SFT  $X_{\mathcal{F}}$  is characterized by its set of allowed  $m$ -blocks:  $\mathcal{B}_m(X)$  is the set of  $m$ -blocks over the alphabet  $\mathcal{A}$  that do not contain a word of  $\mathcal{F}$ , and a sequence  $x \in \mathcal{A}^{\mathbb{Z}}$  is in  $X_{\mathcal{F}}$  if and only if all of its  $m$ -blocks are in  $\mathcal{B}_m(X)$ . A shift of finite type that is determined by its  $m$ -blocks is an  $(m-1)$ -step SFT. The idea behind this terminology is that we must look back  $m-1$  steps in our symbolic sequence to see which symbols we are allowed to write next. The golden mean shift is a 1-step SFT, with allowed 2-blocks 00, 01 and 10. If we allow consecutive 1’s, but no strings of three in a row, we get a 2-step SFT.

A simple example of a subshift that is not of finite type is the *even shift* first studied by B. Weiss [We]. It consists of all binary strings in which two 1’s are always separated by an even number of 0’s. Its set of forbidden words is  $\mathcal{F} = \{101, 10001, 1000001, \dots\}$ . A variation on this theme is the *prime gap shift*, in which two 1’s are separated by a prime number of 0’s.

A point of notation is in order before we close this section. We have been speaking of “the” shift map  $\sigma$  on an arbitrary subshift  $X$ . In careful parlance, two maps are not the same if they have different domains. A shift space is really a pair  $(X, \sigma_X)$ , where  $X$  is a closed subset of some  $\mathcal{A}^{\mathbb{Z}}$  invariant under the coordinate shift on that particular full shift, and  $\sigma_X$  is the restriction of that coordinate shift to  $X$ . In these notes we use  $X$  as a shorthand for the pair  $(X, \sigma_X)$ , but often the map  $\sigma_X$  is singled out instead.

#### 4. Coding and isomorphism

The term *code* is variously used in symbolic dynamics and related fields for maps of different sorts between symbolic systems, or from a general dynamical system to a symbolic one. For example, the one-sided golden mean shift might be described as a coding of the map  $x \rightarrow \{\gamma x\}$  on the interval. It should be noted that in coding and information theory, a mapping may be called an *encoder*, and its image a *code*.

Within symbolic dynamics we are naturally interested in maps that preserve, at least to some extent, the topology and dynamics of the shift space. We want nearby points to go to nearby points, and if  $x$  is sent to  $y$  then its shift  $\sigma x$  should go to  $\sigma y$ . A *homomorphism* from one subshift to another is a continuous map  $\phi$  that commutes with the shift, that is, for which  $\phi \circ \sigma = \sigma \circ \phi$ . An onto homomorphism is traditionally called a *factor map*, a term also used in ergodic theory, although the term *quotient map* would be more consistent with usage in other areas of mathematics. An isomorphism (invertible homomorphism) from one subshift to another is also called

a *topological conjugacy*, or simply a *conjugacy*, as is usual in the general theory of dynamical systems.

We can define a factor map  $\phi$  from the golden mean shift to the even shift as follows: map  $x = (x_i)$  to  $y = (y_i)$  where  $y_i = 1 - (x_i + x_{i+1})$  for all  $i$ . Since each 1 in  $x$  is immediately preceded and followed by a 0, the 0's in  $y$  are produced in pairs. Clearly  $\sigma\phi(x) = \phi(\sigma x)$ . The map is continuous because it is given by a local rule, so that a central block of  $y$  is determined by a slightly longer central block of  $x$ . In general, a *sliding block code* from a subshift  $X$  to a subshift  $Y$  is a map  $\phi$  given by a local rule  $(\phi(x))_i = \Phi(x_{i-m} \dots x_{i+a})$ , where  $m$  and  $a$  are integers with  $-m \leq a$  and  $\Phi$  is a map from the  $(m+a+1)$ -blocks of  $X$  to the symbols of  $Y$ . The numbers  $m$  and  $a$ , usually taken to be nonnegative, are called respectively the *memory* and *anticipation* of the code. An argument using the compactness of  $X$  yields the Curtis-Hedlund-Lyndon theorem: every homomorphism between subshifts is given by a sliding block code.

Of particular interest are the *higher block codes*. We can define a homomorphism from any subshift  $X$  into the full  $\mathcal{B}_n(X)$ -shift by the sliding block code  $\Phi(x_0x_1 \dots x_{n-1}) = [x_0x_1 \dots x_{n-1}]$ . Here we use the square brackets to emphasize that the enclosed block is being treated as a single symbol in a new alphabet. Thus when  $m = 2$ , the sequence  $\dots x_{-1}x_0x_1x_2 \dots$  is sent to

$$\dots [x_{-1}x_0][x_0x_1][x_1x_2] \dots$$

This homomorphism is clearly one-to-one. Its image is the *n-block presentation* of  $X$ , denoted  $X^{[n]}$ .

Higher block presentations provide an important technical tool in symbolic dynamics. If  $X$  is an  $m$ -step shift of finite type, then  $X^{[m]}$  is a 1-step SFT: the allowed 2-blocks of  $X^{[m]}$  are the blocks  $[x_0 \dots x_{m-1}][x_1 \dots x_m]$  where  $x_0 \dots x_m$  is an allowed  $(m+1)$ -block of  $X$ . Thus every SFT is conjugate to a 1-step SFT. Also, if  $\phi$  is a sliding block code with memory  $m$  and anticipation  $l$  as described above, it induces a sliding block code  $\psi$  of zero memory and anticipation from  $X^{[m+l]}$  to  $Y$  given by the block map  $\Psi([x_{i-m} \dots x_{i+l}]) = \Phi(x_{i-m} \dots x_{i+l})$  from symbols (1-blocks) of  $X^{[m+l]}$  to symbols of  $Y$ . By this device we are often able to reduce general arguments about sliding block codes to the case of one-block codes  $(\phi(x))_i = \Phi(x_i)$ .

## 5. Graphs and matrices

Recall that a 1-step SFT  $X$  is characterized by its set of allowed 2-blocks, that is, by a list of which symbols may follow which in our symbol sequences. We may represent such a system by a directed graph: the vertices are the symbols of the alphabet  $\mathcal{A}$  and there is an edge from  $a$  to  $b$  if and only if the word  $ab$  is allowed. Each element  $(x_i)$  of  $X$  corresponds to a biinfinite walk on the graph, following edges from one vertex to the next. Conversely, a (finite) directed graph  $G$  with no parallel edges determines a 1-step SFT  $\hat{X}_G$  with alphabet equal to the set of vertices of  $G$ . We call this the *vertex shift* associated with  $G$ . The graph of the golden mean shift is shown in figure 1. From here on, *graph* will always mean a directed graph.

A graph  $G$  with  $n$  vertices is conveniently described by giving its *adjacency matrix*, the  $n \times n$  matrix  $A = (a_{ij})$  where  $a_{ij}$  is the number of edges from the  $i$ th vertex to the  $j$ th. Thus every vertex shift corresponds to a square matrix  $A$  of 0's

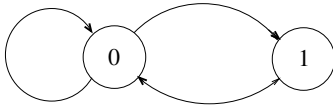


FIGURE 1. Vertex graph of the golden mean shift

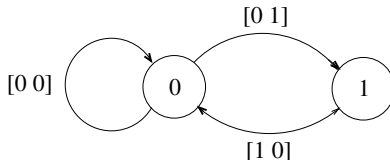


FIGURE 2. 2-block presentation of the golden mean shift

and 1's, sometimes called the *transition matrix* of the vertex shift. The transition matrix for the golden mean shift is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Vertex shifts capture the constraints of a 1-step SFT in an appealingly simple way. Graphs of this sort are used in the field of stochastic processes to describe Markov chain models. The vertices are states of a system, and an edge from  $a$  to  $b$  is labeled by the probability of transition from state  $a$  to state  $b$ , which is assumed to be stationary and independent of previous states. The absence of an edge from  $a$  to  $b$  indicates zero transition probability. Then the vertex shift given by the graph is the underlying topological space supporting the Markov chain. For this reason 1-step shifts of finite type are also called *topological Markov chains*.

Even if the graph  $G$  has parallel edges, we can still view it as representing a 1-step SFT if we take the set of edges of  $G$ , instead of the vertices, as the symbol set. The *edge shift*  $X_G$  is the set of biinfinite walks on the edges of  $G$ , that is, the set of edge sequences  $(x_i)$  such that the terminal vertex of  $x_i$  is the initial vertex of  $x_{i+1}$  for all  $i$ . The edge shift is also denoted by  $X_A$ , where  $A$  is the adjacency matrix of  $G$  as before.

Edge shifts allow a more efficient representation in many cases. For example, the full  $n$ -shift is given as a vertex shift by the  $n \times n$  matrix of 1's, and as an edge shift by the  $1 \times 1$  matrix  $(n)$ . (The full shift is a shift of finite type, with empty set of forbidden words.) Not every 1-step SFT is an edge shift. For example, there is no graph with two edges that represents the golden mean shift. However, if  $X$  is the vertex shift with graph  $G$  then its 2-block presentation  $X^{[2]}$  can be naturally identified with the edge shift of  $G$  by identifying an edge from vertex  $a$  to vertex  $b$  with the 2-block  $[ab]$ . Hence every shift of finite type is conjugate to an edge shift. In figure 2 we show the 2-block presentation of the golden mean shift represented as an edge shift.

It can be seen that a subshift that is conjugate to a shift of finite type is itself of finite type. However, the homomorphic image of an SFT may not be an SFT, as we can see by the factor map from the golden mean shift to the even shift. The class of subshifts that are factors of SFT are called the *sofic* shifts. This term, derived from the Hebrew word for *finite*, was coined by B. Weiss [We]. Suppose  $Y$  is a

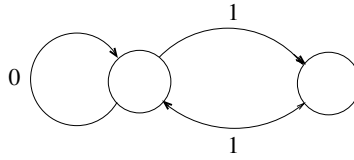


FIGURE 3. Graph of even shift

factor of an SFT  $X$ . By replacing  $X$  with a higher block presentation if necessary, we can assume that  $X$  is an edge shift  $X_G$  and that the factor map is a 1-block code. That is, it takes an edge sequence  $x = (x_i) \in X$  to  $y = (\Phi(x_i)) \in Y$  where  $\Phi$  is a map from edges of  $G$  to symbols of  $Y$ . If we label each edge  $e$  of  $G$  with the symbol  $\Phi(e)$ , then the image of an edge sequence under the sliding block code is the sequence of edge labels. Hence any sofic shift may be represented by a labeling of the edges of a directed graph with symbols (not necessarily distinct) from some alphabet  $\mathcal{A}$ . As with SFT, the elements of  $Y$  correspond to biinfinite walks in this graph.

The 2-block code from the golden mean shift to the even shift described in the previous section produces the edge labeling in figure 3. It is easy to see that the prime gap system cannot be represented by a finite labeled graph, so it is not a sofic shift.

Sofic systems have a natural connection to automata theory. We think of the vertices of the graph as internal states of a machine, and the label on an edge from  $v$  to  $v'$  as the instruction that the machine, when in state  $v$ , should go to state  $v'$  if that label is read as input. A *nondeterministic finite-state automaton* (NFA) is just such a directed, edge-labeled graph, with one or more designated *initial states* and *accepting states*. The automaton is said to *accept* or *recognize* a word  $b_1 \dots b_n$  if this word labels a path from an initial state to an accepting state. The set of all words accepted by an NFA is a *regular language*. In this terminology, the set of allowed blocks of all lengths in a sofic system  $Y$  is a regular language given by a NFA in which all the states are both initial and accepting. For more on connections between automata theory and symbolic dynamics, see [BP] or [BMN].

## 6. Invariants

One of the most basic questions we may ask about symbolic dynamical systems is how we can tell when two subshifts are conjugate. Even for the simplest class of systems, the shifts of finite type, a complete and effective classification remains elusive.

On the other hand, we have many ways of telling that two subshifts are different. By an *invariant* of conjugacy we mean any quantity or mathematical object that we can assign to subshift that remains unchanged when we replace the subshift by a conjugate one. There are several well-known invariants that can be defined for dynamical systems in general, and others that apply only to shift spaces, or to the smaller class of SFT. We list a few:

**6.1. Periodic point count.** If  $f$  is a homeomorphism of a compact space  $X$ , we denote by  $Fix_n(X)$  the set of all  $x \in X$  with  $f^n(x) = x$ . We will call these the period  $n$  points of the dynamical system. Note that with this terminology, a period

$n$  point is also a period  $m$  point if  $m$  is a multiple of  $n$ . We distinguish the smallest such  $n$  by calling it the least period of  $x$ . A conjugacy between dynamical systems preserves periods of points, so conjugate shifts have the same cardinality of period  $n$  points for every  $n$ .

As we observed before, for a shift space the period  $n$  points are those sequences  $(x_i)$  with  $x_{i+n} = x_i$  for all  $i$ . If the alphabet  $\mathcal{A}$  has  $r$  symbols then there are at most  $r^n$  period  $n$  points.

It is a well-known fact that in a graph with adjacency matrix  $A$ , the number of walks of length  $n$  from vertex  $i$  to vertex  $j$  along the edges of the graph is just the  $ij$  entry of  $A^n$ . For an edge shift  $X_A$  the period  $n$  points correspond to walks of length  $n$  with the same initial state and final state  $i$ . Hence the number of period  $n$  points is easily computed as the trace, or sum of the diagonal entries, of  $A^n$ .

If the number  $p_n(X)$  of period  $n$  points of  $X$  is finite for all  $n$ , the periodic point count can be encoded in the Artin-Mazur *zeta function* of  $X$ . This is the power series

$$\zeta_X(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} t^n\right).$$

For the SFT  $X_A$  given by an  $r \times r$  transition matrix  $A$ , this function takes the remarkably simple form

$$\begin{aligned} \zeta_A(t) &= \frac{1}{\det(I - tA)} \\ (6.1) \qquad &= \frac{1}{t^r \chi_A(t^{-1})}, \end{aligned}$$

where  $\chi_A$  is the characteristic polynomial of  $A$ . This formula, known as the Bowen-Lanford formula, can be verified by putting  $A$  in its Jordan normal form and using the trace result cited above. More generally, it follows from a theorem of Manning [Ma] that every sofic system has rational zeta function.

**6.2. Topological entropy.** Topological entropy was defined in [AKM] for general compact dynamical systems, in analogy with the concept of measure theoretic entropy developed earlier by Shannon and by Kolmogorov and Sinai. We will not give the general definition, which is fairly involved, since there is a much simpler formulation for the special case of symbolic dynamical systems. This formulation can be motivated in terms of information theory.

Think of an allowed  $n$ -block of a subshift  $X$  as the information we would gain by observing our symbolic dynamical system for  $n$  ticks of the clock. If  $X$  is the full 2-shift, the  $n$ -block may be any one of the  $2^n$  binary strings of length  $n$ , so by recording what particular  $n$ -block occurs we gain  $n$  bits of information, or one bit per symbol. However, if  $X$  is the golden mean shift then not all binary strings can occur. We gain less information from observing a particular block since there are many blocks that we could have ruled out in advance. The number  $N_n$  of binary strings of length  $n$  with no consecutive 1's satisfies the Fibonacci recurrence relation

$$N_{n+2} = N_{n+1} + N_n,$$

since we can form an allowed  $(n+2)$ -block either by tacking a 0 onto an allowed  $(n+1)$ -block or by putting 01 after an allowed  $n$ -block. The number  $N_n$  is the  $(n+2)$ -th Fibonacci number, and grows asymptotically as  $C\gamma^n$  where  $C$  is a constant and  $\gamma = (1 + \sqrt{5})/2$  is the golden mean. We may say that the amount of

information we gain by observing a particular  $n$ -block of the golden mean shift is about  $\log_2(C\gamma^n) = n \log_2 \gamma + \log_2 C$  bits, or roughly  $\log_2(\gamma)$  bits per symbol.

We define the (topological) *entropy* of a shift space  $X$  to be the limit

$$h(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N_n$$

where  $N_n = N_n(X)$  is the number of allowed  $n$ -blocks of  $X$ . We can describe  $h(X)$  as the per-symbol information rate of the shift, or as the exponential growth rate of the number of  $n$ -blocks. That this limit exists (and is equal to the infimum of the sequence) can be established from the observation that  $N_{m+n} \leq N_m \cdot N_n$ . Whether we use natural or base 2 logarithms is a matter of personal proclivity; with the natural log we are measuring information in *nats* instead of bits.

If  $Y$  is a factor of  $X$  given as the image of a sliding  $m$ -block code, then the number of  $n$ -blocks of  $Y$  cannot exceed the number of  $(n+m)$ -blocks of  $X$ . Thus

$$\begin{aligned} (6.2) \quad h(Y) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log N_n(Y) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log N_{n+m}(X) \\ &= \left( \lim_{n \rightarrow \infty} \frac{n+m}{n} \right) h(X) = h(X). \end{aligned}$$

Since conjugate shifts are factors of one another, they have equal entropy.

We see immediately that the entropy of the full  $r$ -shift is  $\log r$ , since there are  $r^n$  words of length  $n$ . For a shift of finite type  $X_A$  given by a transition matrix  $A$ , the number of allowed  $n$ -blocks is the sum of the entries of  $A^n$ . By a result known as the Perron-Frobenius theorem, every square nonnegative matrix  $A$  has a nonnegative real eigenvalue  $\lambda_A$  (the *Perron-Frobenius eigenvalue*) that is greater than or equal to the modulus of every other eigenvalue of  $A$ . For the golden mean shift  $\lambda_A = \gamma$ . It can be shown that in general,  $h(X_A) = \log \lambda_A$ .

**6.3. Algebraic invariants for shifts of finite type.** As we have seen, for a shift of finite type  $X_A$  the number of period  $n$  points, the zeta function and the entropy can all be simply expressed in terms of algebraic invariants of the transition matrix  $A$ . We can see from the invariance of the zeta function that if  $X_A$  and  $X_B$  are conjugate SFT then  $A$  and  $B$  must have the same characteristic polynomial, up to some factor  $t^k$ . In fact a stronger statement is true: the Jordan forms of the invertible parts of  $A$  and  $B$  must be the same.

Another useful algebraic invariant is the *Bowen-Franks group*. If  $A$  is an  $r \times r$  transition matrix, its Bowen-Franks group is the quotient of  $\mathbb{Z}^r$  by its image under the matrix  $I - A$ :

$$BF(A) = \mathbb{Z}^r / \mathbb{Z}^r(I - A).$$

If  $X_A$  is conjugate to  $X_B$  then the Bowen-Franks groups of  $A$  and  $B$  must be isomorphic [BF]. This condition is easily checked by computing the elementary divisors of  $A$  and  $B$ . The Bowen-Franks group is, in fact, invariant under flow equivalence of SFT, a weaker equivalence than conjugacy.

One of the most sought-after goals in symbolic dynamics has been a complete and effective classification of shifts of finite type in terms of their transition matrices. In 1973 R. Williams [Wi] introduced two important equivalence relations on the set of square matrices over the non-negative integers, *shift equivalence* and *strong shift equivalence*. He showed that finite type shifts  $X_A$  and  $X_B$  are conjugate if

and only  $A$  and  $B$  are strong shift equivalent. Strong shift equivalence implies shift equivalence; the converse statement became known as the shift equivalence conjecture or Williams conjecture. The importance of the conjecture lies in the fact that latter equivalence is more tractable: in fact, it is known to be decidable [KR1]. Jack Wagoner’s chapter relates the developments that led in 1997 to a counterexample to the shift equivalence conjecture by K.H. Kim and F. Roush [KR2] following joint work with Wagoner, and outlines the current state of affairs.

Crucial to the solution of the shift equivalence problem was the study of the group of automorphisms, or self-conjugacies, of a shift of finite type. Bob Devaney’s chapter will take us a step beyond Hadamard’s inspiration by showing how the automorphism group of a shift can encode information from other dynamical settings, in this case families of complex polynomial maps.

## 7. Wider vistas

There are several ways in which we can relax our notion of symbolic dynamical system to get a larger class of systems. One is to allow a countable alphabet in place of a finite one. This makes the shift space noncompact, which introduces some complications, for example in finding an appropriate definition of entropy. The theory of countable state topological Markov chains—vertex shifts on a graph with countably many vertices but only finitely many edges entering or leaving each edge—is of particular interest. A good introduction to this topic is Chapter 7 of [Ki].

We could instead choose our symbol set to be a compact group such as the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Although this takes us far from the original idea of a symbolic dynamical system as a discretization of space, if our map is still a coordinate shift map then some of the spirit and techniques remain. We will encounter shift spaces with alphabet  $\mathbb{T}$  in Doug Lind’s chapter.

The study of a dynamical system  $(X, \sigma)$  is really the study of the behavior of  $X$  under the iterates  $\sigma^n$  of  $\sigma$ . We may describe this as an action of the group  $\mathbb{Z}$  on  $X$ : for every  $n \in \mathbb{Z}$  we have a coordinate shift map  $\sigma^n$  that shifts all coordinates by  $n$ , with  $\sigma^{m+n} = \sigma^m \circ \sigma^n$ . In general, an *action* of a group  $G$  by homeomorphisms on a space  $X$  is a map that takes each  $g \in G$  to a homeomorphism  $f_g$  of  $X$  in such a way that  $f_{gh} = f_g \circ f_h$ . Another way to broaden our notion of symbolic dynamical systems is to consider actions by other discrete infinite groups in place of  $\mathbb{Z}$ . One of the most exciting currents in symbolic dynamics is development of the theory of  $\mathbb{Z}^d$ -actions. Elements of a  $\mathbb{Z}^d$  symbolic dynamical system are  $d$ -dimensional arrays  $(x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$  of symbols, and for each  $\mathbf{m} \in \mathbb{Z}^d$  there is a shift map  $\sigma^{\mathbf{m}}$  that shifts all coordinates by  $\mathbf{m}$ . Doug Lind’s chapter is a survey of these multidimensional systems. As you will see, there are very nice results for special class of multidimensional systems with algebraic structure, but the study of general  $\mathbb{Z}^d$ -actions involves substantial complications not found in the one-dimensional case.

We can think of an element of a two-dimensional symbolic dynamical system as a tiling of the plane by unit square tiles of different colors, where the colors are simply our alphabet of symbols. A translation of the tiling by an integer vector  $\mathbf{m}$  gives another tiling that represents a coordinate shift of the original. To bring the techniques of symbolic dynamics to bear on the general problem of tiling the plane with tiles of various shapes we need to allow general translations in the plane. In Robbie Robinson’s chapter, which examines not just planar but  $d$ -dimensional

tilings, we will be working in the setting of  $\mathbb{R}^d$  actions by translation of Euclidean space.

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