KNOTS, LINKS AND REPRESENTATION SHIFTS

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This paper is a summary of a 2-part talk presented by the authors at the Thirteenth Annual Western Workshop on Geometric Topology held in June 1996 at The Colorado College, Colorado Springs, CO. Most of the details can be found in [**SiWi 1**], [**SiWi 2**] and [**SiWi 3**].

Introduction. The group $\pi_1(S^3 - k)$ of a knot k contains an extraordinary amount of information. From combined results of W. Whitten [**Wh**] and M. Culler, C. McA. Gordon, J. Luecke and P.B. Shalen [**CuGoLuSh**] it is known that there are at most two distinct unoriented prime knots with isomorphic groups. Unfortunately, knot groups are generally difficult to use. Knot groups are usually described by presentations, and there is no practical algorithm to decide whether or not two knot groups are isomorphic.

In 1928 J.W. Alexander used homomorphisms (representations) of knot groups onto better understood groups in order to obtain topological invariants. Since then knot group representations have been used effectively by many others. The representations of a given knot group into a fixed finite group have the additional attraction that they are finite in number and so can be tabulated. R. Riley began such a program in [**Ri**].

We take a new approach, examining the representations of the commutator subgroup $K = [\pi_1(S^3 - k), \pi_1(S^3 - k)]$ into a fixed finite group Σ. Although $\mathrm{Hom}(K, \Sigma)$ is often infinite – in fact, uncountable – it has a rich structure that we can understand via symbolic dynamics. In this dynamical system the representations of the knot group $\pi_1(S^3 - k)$ appear (by restricting their domains) as special periodic points. However, the system contains other periodic points and often nonperiodic points, information that can be used to understand more about the structure of the knot exterior and its various covering spaces. The techniques, all algorithmic, apply equally well to links.

1. Representation shifts. Although here we emphasize applications to knot theory, the methods we describe apply in a wide variety of situations.

Definition. [**Si1**] An *augmented group system* (AGS) is a triple $\mathcal{G} = (G,\ \chi,\ x)$ consisting of a finitely presented group G, an epimorphism $\chi : G \to \mathbf{Z}$, and a distinguished element $x \in G$ such that $\chi(x) = 1$.

Two augmented group systems $(G_1,\ \chi_1,\ x_1)$ and $(G_2,\ \chi_2,\ x_2)$ are *equivalent* if there exists an isomorphism $f : G_1 \to G_2$ such that $f(x_1) = x_2$ and $\chi_1 = \chi_2 \circ f$. Equivalent augmented group systems are regarded as the same.

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We can associate an AGS $\mathcal{G}_k$ to any oriented knot k as follows. Let $N(k) = k \times D^2$ be a tubular neighborhood of k. The closure of $S^3 - N(k)$ is the *exterior* of k, and we will denote it by $X(k)$. Let G be the fundamental group $\pi_1(X(k), *)$, where the basepoint is contained in the boundary $\partial X(k)$, and let $x \in G$ be the class of a meridian $m \subset \partial X(k)$ with orientation induced by that of k. Let χ be the abelianization homomorphism that maps x to 1. The uniqueness up to isotopy of tubular neighborhoods ensures that $\mathcal{G}_k$ is well defined.

An AGS $\mathcal{G}_l$ can be associated to any oriented link $l = l_1 \cup \ldots \cup l_d$ by the same general procedure as above. We let x be the class of a meridian of the component l_1. Since the abelianzation of $G = \pi_1(S^3 - l, *)$ is a free abelian group of rank d, there are many choices for χ when the link has more than one component. A natural choice for χ is the "total linking number homomorphism" that maps the class of every oriented meridian to 1.

Definition. [**SiWi1**] Let $\mathcal{G} = (G,\ \chi,\ x)$ be an AGS, and let K be the kernel of χ. Assume that Σ is any finite group. The *representation shift* $\Phi_\Sigma(\mathcal{G})$ is the set of representations $\rho : K \to \Sigma$ together with the mapping $\sigma_x : \Phi_\Sigma(\mathcal{G}) \to \Phi_\Sigma(\mathcal{G})$ defined by

$$(\sigma_x \rho)(a) = \rho(x^{-1} a x) \ \ \forall a \in K.$$

The mapping σ_x is a bijection with inverse $\sigma_{x^{-1}}$. In fact, if we define the topology on $\Phi_\Sigma(\mathcal{G})$ with basis

$$\mathcal{N}_{g_1,\ldots,g_n}(\rho) = \{\rho' \mid \rho'(g_i) = \rho(g_i),\ i = 1, \ldots, n\},$$

where $\rho \in \Phi_\Sigma(\mathcal{G})$ and $g_1, \ldots, g_n \in K$, then σ_x becomes a homeomorphism. Consequently, the pair $(\Phi_\Sigma(\mathcal{G}), \sigma_x)$ is a topological dynamical system (topological space + homeomorphism). We recall that two dynamical systems $(\Phi_1,\ \sigma_1)$ and $(\Phi_2,\ \sigma_2)$ are *topologically conjugate* if there exists a homeomorphism $h : \Phi_1 \to \Phi_2$ such that $\sigma_2 \circ h = h \circ \sigma_1$. Topologically conjugate dynamical systems are regarded as the same.

The main result of [**SiWi1**] is that the topological dynamical system $(\Phi_\Sigma(\mathcal{G}), \sigma_x)$ has the structure of a *shift of finite type*, a special sort of dynamical system that can be completely described by a finite graph Γ. The representations ρ correspond to the bi-infinite paths in Γ. Rather than repeat the proof of this result, we illustrate the algorithm for finding Γ.

Example. Consider the AGS associated to the knot 5_2 in figure 1. The Wirtinger algorithm [**BuZi**], [**Ro**] together with some obvious Tietze transformations produces the following presentation for the group G of the knot.

$$G = \langle x, a \mid x^{-1} a^2 x \cdot a^{-2} \cdot x^{-1} a x \cdot x^{-2} a^{-2} x^2 \rangle.$$

The Reidemeister-Schreier method [**LySc**] gives us a presentation for K:

$$K = \langle a_i \mid a_{i+1}^2 a_i^{-2} a_{i+1} a_{i+2}^{-2} \rangle$$

Here $a_i = x^{-i} a x^i$ and the index i ranges over the integers. J.C. Hausmann and M. Kervaire have termed such a presentation *finite* **Z**-*dynamic* [**HaKe**]. We will think of the relator $a_{i+1}^2 a_i^{-2} a_{i+1} a_{i+2}^{-2}$ as a word $r(a_i, a_{i+1}, a_{i+2})$. Notice that each relator is gotten from $r(a_0, a_1, a_2)$ by "shifting" the subscripts of the generators involved.

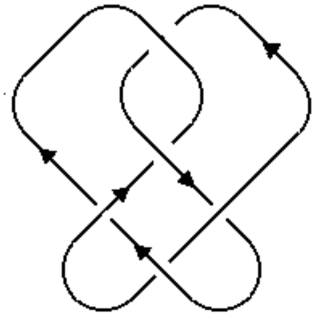

Figure 1

A representation $\rho : K \to \Sigma$ is a function ρ defined on the set $\{a_i\}$ such that $r(\rho(a_i), \rho(a_{i+1}), \rho(a_{i+2}))$ is trivial for all i. When Σ is the cyclic group $\mathbf{Z}/3$, for example, this condition reduces to $\rho(a_{i+2}) \equiv 2\rho(a_i)$ modulo 3. The sequence

$$\ldots\ 1\ 0\ 2\ 0\ 1\ \underline{0}\ 2\ 0\ 1\ 2\ 0\ \ldots$$

describes the representation $\rho : K \to \mathbf{Z}/3$ mapping a_0 to 0, a_1 to 2, etc. The sequence describing $\sigma_x(\rho)$ is simply

$$\ldots\ 1\ 0\ 2\ 0\ 1\ 0\ \underline{2}\ 0\ 1\ 2\ 0\ \ldots$$

Moreover, every representation $\rho : K \to \mathbf{Z}/3$ can be found from the directed graph Γ in figure 2. Any vertex can be regarded as an assignment of values for a_0 and a_1, while the vertex that follows is an assignment for a_1 and a_2, etc. The representation shift $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{5_2})$ has exactly nine elements. Notice that the trivial representation, corresponding to the constant sequence of zeroes, is isolated in the space $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{5_2})$.

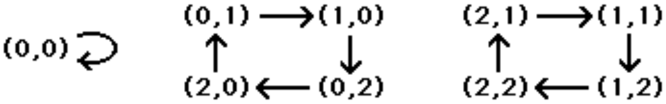

Figure 2

Theorem. [**SiWi3**] Let $\mathcal{G}$ be any AGS and Σ a finite abelian group. Then the representation shift $\Phi_\Sigma(\mathcal{G})$ is finite if and only if the trivial representation is isolated. This will be the case whenever G has infinite cyclic abelianization.

The proof of the theorem makes use of a powerful structure theorem for Markov subgroups, shifts of finite type that are also abelian groups, proved by of B.P. Kitchens [**Ki**]. The theorem can also be proved using algebraic topological techniques of [**Mi**].

When $\mathbf{Z}/3$ is replaced by the symmetric group S_4 in the above example, the representation shift becomes uncountably infinite. Figure 3 contains a detail of the graph describing it. In figure 3 we see two circuits with a common vertex, and hence uncountably many bi-infinite paths. It follows that K contains uncountably many subgroups of index less than or equal to 4. Since the representation shift with $\Sigma = S_3$ can be shown to be finite, K contains uncountably many subgroups of index equal to 4. Further analysis using these techniques shows that K contains uncountably many subgroups of any index exceeding 3.

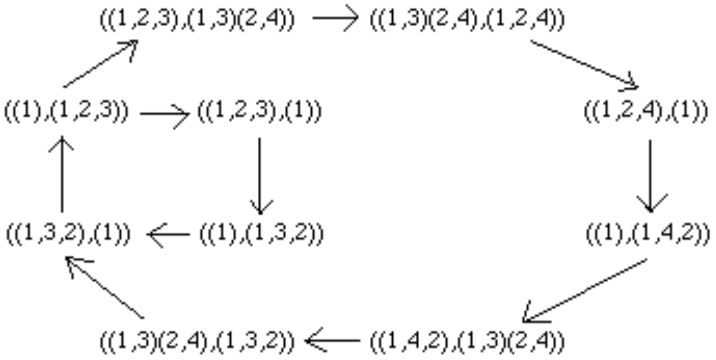

Figure 3

2. Periodic points. Assume that $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS and $\Phi_\Sigma(\mathcal{G})$ is an associated representation shift for some finite group Σ. Although the graph Γ that describes $\Phi_\Sigma(\mathcal{G})$ is not unique, it does contain invariants of $\Phi_\Sigma(\mathcal{G})$, which are necessarily invariants of $\mathcal{G}$.

Definition. A representation $\rho \in \Phi_\Sigma(\mathcal{G})$ is *periodic* if $\sigma_x^r \rho = \rho$ for some positive integer r. In this case, ρ has *period* r.

The representations of period r correspond to the closed paths in Γ having length r.

Example. A knot $k \subset S^3$ is *fibered* if the projection $\partial N(k) = k \times S^1 \to S^1$ extends to a locally trivial fibration $X(k) \to S^1$. By a theorem of J. Stallings (see Theorem 5.1 of [**BuZi**], for example) a knot k is fibered if and only if the commutator subgroup K of its group is finitely generated (and free). In this case, the representation shift $\Phi_\Sigma(\mathcal{G}_k)$ is finite for all finite groups Σ, and so every representation $\rho : K \to \Sigma$ is periodic.

Figure 4 displays the graph describing $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$, the representation shift of the trefoil 3_1. The representation shift $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ of the figure eight knot 4_1 is identical to $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{5_2})$ (see figure 2). Notice that each representation shift contains nine elements. However, $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$ contains two points of period 2 and six of period 6 while $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ contains eight points of period 4. Since representation shifts with different numbers of periodic points for the same period clearly cannot be topologically conjugate, $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1}) \neq \Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$.

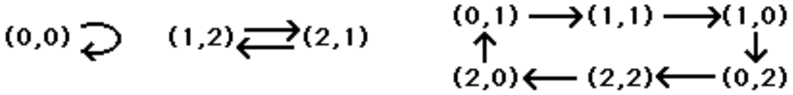

Figure 4

The calculations above show that the trefoil and figure eight knots are different. However, they reveal much more. That there are two representations of period 2 in $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$ other than the trivial one is a consequence of the well-known fact that a trefoil knot diagram can be nontrivially tricolored in exactly six ways [**CrFo**], [**Pr**]; the absence of nontrivial period 2 representations in $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ indicates that the figure eight knot diagram can be tricolored only monochromatically. In general, the number of period 2 points of $\Phi_{\mathbf{Z}/n}(\mathcal{G}_l)$ for any oriented link l tells us how a diagram for l can be n-colored. The periodic points of period greater than 2 correspond to "generalized n-colorings." Details appear in [**SiWi2**].

Another invariant of $\Phi_\Sigma(\mathcal{G})$ that can be calculated from the graph Γ is topological entropy. The *topological entropy* of $\Phi_\Sigma(\mathcal{G})$ is equal to the logarithm of the spectral radius of the adjacency matrix of Γ. We will denote the topological entropy by $h_\Sigma(\mathcal{G})$. It depends only on the AGS $\mathcal{G}$ and target group Σ. In a sense, $h_\Sigma(\mathcal{G})$ is a measure of the average amount of information gained by taking one step in Γ. In the case of the AGS $\mathcal{G}_l$ associated to an oriented link l, the topological entropy is a numerical invariant of l.

In [**Lo**], [**Si2**] entropy invariants for knots were defined using Nielson-Thurston theory for surface homeomorphisms. The entropy invariants above are quite different. The invariants in [**Si2**] are nonzero for most fibered knots. However, if k is any fibered knot, then $\Phi_\Sigma(\mathcal{G}_k)$ is finite and so $h_\Sigma(\mathcal{G}_k) = 0$ for every finite group Σ. These new invariants seem to detect nonfibered knots. In fact, we make the following conjecture.

Conjecture. If k is a nonfibered knot, then $h_\Sigma(\mathcal{G}_k) > 0$ for some finite group Σ.

If the conjecture is true, then it would follow that the commutator subgroup K of the group of any nonfibered knot has uncountably many subgroups of index r whenever r is sufficiently large, a conclusion that we saw in the case of the knot 5_2.

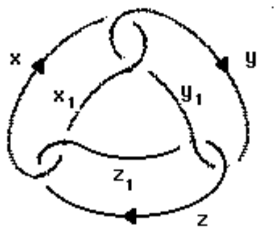

Figure 5

Example. Consider the 3-component link $l = 6^3_1$ as it appears in figure 5 with Wirtinger generators indicated. The group G of the link has presentation

$$\langle x,\ x_1,\ y,\ y_1,\ z,\ z_1 \mid y_1z = zy,\ z_1x = xz,\ x_1y = yx,\ x_1y_1 = yx_1,\ zy_1 = y_1z_1\rangle.$$

Using the first three relators we eliminate the generators $x_1,\ y_1$ and z_1, obtaining

$$\langle x,\ y,\ z \mid xy^{-1}zyz^{-1}yx^{-1}y^{-1},\ zyz^{-1}xz^{-1}x^{-1}zy^{-1}\rangle.$$

The elements $x^{-1}y$ and $x^{-1}z$ vanish under the augmentation homomorphism χ. We use Tietze transformations to replace y and z by xa and xb (introducing new generators a and b). We can then rewrite the presentation for G as

$$\langle x,\ a,\ b,\ \mid a^{-2}b\cdot xab^{-1}ax^{-1},\ ab^{-2}\cdot x^{-1}ba^{-1}bx\rangle.$$

Let a_i and b_i denote $x^{-i}ax^i$ and $x^{-i}bx^i$, respectively. By the Reidemeister-Schreier method the following is a presentation for the kernel K of χ:

$$K = \langle a_i,\ b_i \mid a_{i+1}^{-2}b_{i+1}a_ib_i^{-1}a_i,\ a_ib_i^{-2}b_{i+1}a_{i+1}^{-1}b_{i+1}\rangle.$$

First we will determine $\Phi_{\mathbf{Z}/2}(\mathcal{G}_l)$. Any representation $\rho : K \to \mathbf{Z}/2$ factors through $\mathrm{Hom}(K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/2,\ \mathbf{Z}/2)$, where K^{ab} denotes the abelianization of K. We can get a presentation for $K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/2$ from that of K by allowing all of the generators to commute and by reducing exponents of relators modulo 2. The resulting presentation, expressed in additive notiation, is

$$\langle a_i,\ b_i \mid b_{i+1} = b_i,\ a_{i+1} = a_i\rangle.$$

Clearly, any homomorphism $\rho : K \to \mathbf{Z}/2$ is completely determined by the values $\rho(a_0)$ and $\rho(b_0)$. The representation shift $\Phi_{\mathbf{Z}/2}(\mathcal{G}_l)$ has exactly 4 elements, each a fixed point under σ_x. The topological entropy $h_{\mathbf{Z}/2}(\mathcal{G}_l)$ is zero.

Replacing $\mathbf{Z}/2$ by $\mathbf{Z}/3$ results in a very different representation shift. Consider the presentation for $K^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Z}/3$:

$$\langle a_i,\ b_i \mid a_i - a_{i+1} + b_i - b_{i+1}\rangle$$

The second relator does not appear because it is redundant. Selecting any fixed value for b_0, we can define a representation $\rho : K \to \mathbf{Z}/3$ by assigning arbitrary values for the a_i's; the corresponding values of $b_i,\ i \neq 0$, are uniquely determined. It follows that $\Phi_{\mathbf{Z}/3}(\mathcal{G}_l)$ is a Cartesian product of three full shifts on the symbols of $\mathbf{Z}/3$. (A *full shift* on a finite alphabet $\mathcal{A}$ consists of all sequences $(\alpha_i),\ \alpha_i \in \mathcal{A}$, together with the shift mapping $\sigma : (\alpha_i) \mapsto (\alpha'_i)$, where $\alpha'_i = \alpha_{i+1}$.) In this case the topological entropy $h_{\mathbf{Z}/3}(\mathcal{G}_l)$ is $\log 3$.

Let $\mathcal{G} = (G,\ \chi, x)$ be an AGS and let Σ be any finite group. By a theorem of [**SiWi1**] the topological entropy of a representation shift $\Phi_\Sigma(\mathcal{G})$ does not depend on the choice of distinguished generator x. Hence in the above example, the class x of the meridian of the first component of the link can be replaced by any element that maps to 1 under χ, and the resulting topological entropy will be unchanged. On the other hand, since G^{ab} is free abelian of rank 3 there are infinitely many choices for χ other than the total linking number homomorphism that we employed. Altering χ can change the entropy.

3. Other applications and directions for knots and links. In [**SiWi3**] we proved that for any knot k and finite group Σ, the points of period r in $\Phi_\Sigma(\mathcal{G}_k)$ are in one-to-one correspondence with representations of $\pi_1(\hat{X}_r)$, where $\hat{X}_r$ denotes the r-fold branched cyclic cover of k. Consequently, information, such as the following about the branched cyclic covers of k, is encoded in the symbolic dynamics of the representation shifts.

Theorem. [**SiWi3**] For any finite group Σ, the order of $\mathrm{Hom}(\pi_1\hat{X}_r, \Sigma)$ satisfies a linear recurrence relation.

Corollary. [**SiWi3**] (also proved by W.H. Stevens [**St**]) For any oriented knot k and finite abelian group Σ, there is a positive integer N such that

$$H_1(\hat{X}_{r+N}; \Sigma) \cong H_1(\hat{X}_r; \Sigma)$$

as $\Lambda/(x^r - 1)$-modules, where $\Lambda = \mathbf{Z}[x, x^{-1}]$).

The representation shift techniques used to prove the corollary provide an elementary algorithm for computing the period N. Moreover, they can be used to extend the conclusion of the theorem to orinted links. Details can be found in [**SiWi3**].

We conclude with a brief discussion of work in progress. Let $l = l_1 \cup \ldots \cup l_d$ be an oriented link with group $G = \pi_1(S^3 - l)$. Instead of choosing a homomorphism of G onto the integers, let $\mu : G \to \mathbf{Z}^d$ be the abelianization homomorphism that sends oriented meridians to the standard basis. Let K denote the kernel of μ. For any finite abelian group Σ, the set $\mathrm{Hom}(K, \Sigma)$, which is the same as $\mathrm{Hom}(K^{\mathrm{ab}}, \Sigma)$ admits d commuting automorphisms $\sigma_{x_1}, \ldots, \sigma_{x_d}$ corresponding to the various meridians of the link. The set of representations can be given a topology similar to that of a representation shift so that it becomes a *Z^d-shift of finite type* (see [**LiMa**]). Although $\mathrm{Hom}(K, \Sigma)$ cannot in general be described by a graph, its elements still have an appealing combinatorial description. Representations $\rho : K \to \Sigma$ correspond to labelings of the lattice $\mathbf{Z}^d$ by a finite alphabet $\mathcal{A}(= \Sigma^n)$ that satisfy a finite number of translation-invariant local conditions.

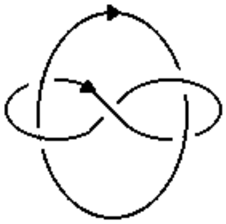

Figure 6

Example. Consider the 2-component link 5_1^2 oriented as as shown in figure 6. The abelianized kernel K^{ab} has presentation

$$\langle \xi_{i,j} \mid \xi_{i,j} - \xi_{i+1,j} - \xi_{i,j+1} + \xi_{i+1,j+1} \rangle,$$

where the indices i, j range over the integers. In this example the alphabet is simply Σ. Representations $\rho : K \to \Sigma$ correspond to Σ-labelings of the lattice $\mathbf{Z}^2$ such that in any 1×1 square the sum of the lower-left and upper-right labels is equal to the sum of the remaining two labels. Here is a particular representation $\rho : K \to \mathbf{Z}/3$.

$$\begin{array}{c}
\vdots \\
\ldots\ 0\ 1\ 2\ 1\ 0\ \ldots \\
\ldots\ 2\ 0\ 1\ 0\ 2\ \ldots \\
\ldots\ 1\ 2\ \underline{0}\ 2\ 1\ \ldots \\
\ldots\ 0\ 1\ 2\ 1\ 0\ \ldots \\
\ldots\ 1\ 2\ 0\ 2\ 1\ \ldots \\
\vdots
\end{array}$$

Here is the representation $\sigma_{x_1}\sigma_{x_2}^2(\rho)$.

$$\begin{array}{c} \vdots \\ \ldots\ 0\ 1\ 2\ \underline{1}\ 0\ \ldots \\ \ldots\ 2\ 0\ 1\ 0\ 2\ \ldots \\ \ldots\ 1\ 2\ 0\ 2\ 1\ \ldots \\ \ldots\ 0\ 1\ 2\ 1\ 0\ \ldots \\ \ldots\ 1\ 2\ 0\ 2\ 1\ \ldots \\ \vdots \end{array}$$

Numerical invariants such as *directional entropy* can be computed for such dynamical systems. Also interesting are the directions in which the $\mathbf{Z}^d$-shift, when restricted, is *expansive*. The latter are related to the geometric invariant of R. Bieri, B.H. Neumann and R. Strebel [**BiNeSt**]. We will discuss these topics in a forthcoming paper.

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