The Formal Definition of a Limit – Part I

Formal Definition of a Limit:

Suppose $f(x)$ is defined for all $x \neq c$ in an open interval containing $c$.

Then $\lim_{x \to c} f(x) = L$ \hspace{1cm} \text{IF:}

for any arbitrarily small value (for any $\varepsilon > 0$) that may be given,

there exists a $\delta > 0$,

such that if $x$ is within distance $\delta$ of $c$ (that is, if $0 < |x - c| < \delta$)

then $f(x)$ will be within distance $\varepsilon$ of $L$ (that is, $|f(x) - L| < \varepsilon$).

General Structure of an Epsilon-Delta Proof

To prove that $\lim_{x \to c} f(x) = L$, we need to prove that given any arbitrarily small value $\varepsilon$, we can find a $\delta$ (this is found in terms of $\varepsilon$) such that if $x$ is within distance $\delta$ of $c$ (if $0 < |x - c| < \delta$) then $f(x)$ will be within distance $\varepsilon$ of $L$ (that is, $|f(x) - L| < \varepsilon$). The mental work of the proof is in finding the correct bounds for $\delta$, and this is found by beginning with the needed conclusion $|f(x) - L| < \varepsilon$ and manipulating it algebraically so that it looks like the condition $0 < |x - c| < \delta$. (You work backwards.)

An Example of an Epsilon-Delta Proof

Prove that $\lim_{x \to 3} 7x = 21$. 
The Method of Epsilon-Delta Proofs Explained as a Narrative

IDEA: Showing that $f(x)$ “tends toward” $L$ as $x$ “tends toward” $c$ using a list of values in a table is not rigorous. How do you prove that $f(x)$ really does tend toward $L$ as $x$ tends toward $c$??

ANSWER: The method of proof is clever and is here modeled on the idea of a lawyer trying to argue his claim in front of a jury.

Lawyer: I claim that $f(x)$ tends toward $L$ as $x$ tends toward $c$.

Jury: Rigorously prove this to us.

Lawyer: I will prove it in the following way. I claim that $f(x)$ is very close to $L$ (as close as you want it to be) if $x$ is close enough to $c$. In particular, you tell me how close you want $f(x)$ and $L$ to be by specifying a very small number $\varepsilon$, and I will tell you that if $x$ and $c$ are “close enough” (by specifying a small number $\delta$ for the distance between them), then $f(x)$ and $L$ will indeed be that close.

Let’s begin with the function $f(x) = 7x$. I claim $\lim_{x \to 3} 7x = 21$.

Jury: The concept for this limit is that as $x$ tends towards 3, then $7x$ tends towards 21. According to your proposed proof, we will tell you how close we require $f(x)$ to be to 21, and you will provide a definition of “close enough” so that as long as $x$ is “close enough” to 3, then $f(x)$ will as be close as we require to 21.

(The jury speak among themselves.)

Jury: We’ve conferred and we require that $f(x)$ be within 0.01 of 21. How close does $x$ need to be to 3?

Lawyer: I will prove that if $x$ is “close enough” to $c$, then $f(x)$ will be within 0.01 of 21, as you require.

(Lawyer scribbles on a sheet of paper.)

Lawyer: “Close enough” means that the distance between $x$ and 3 must be less than 0.142!

Proof: If $0<|x - 3|<0.0142$
then $0<|7x - 21|<(7 \times 0.142)$
and $|7x - 21|<0.0994<0.01$.

Jury: …we’re not fully convinced. Perhaps the function gets as close as 20.99 to 21, but not much closer. We would like you to prove that the function may get as close as a distance of 0.0001 from 21, as long as $x$ is “close enough” to 3.

Lawyer: I’ll do even better than that. For any distance you give me $\varepsilon$, the proof requires that I choose “close enough” as $x$ must be within distance $\varepsilon/7$ of 3.
Proof: If $0 < |x - 3| < \varepsilon \frac{e}{7}$
then $0 < 7|x - 3| < (7 \times \varepsilon \frac{e}{7})$
and $|7x - 21| < \varepsilon.$