1. Work the review sheets for midterms 1, 2 and 3.

2. Determine the dimensions of a rectangle of largest area that can be inscribed in a semicircle of radius r. Draw a picture, label and name your variables, show all your work.

\[
\text{Area} = 2wh. \quad \text{Constraint: } h^2 + w^2 = r^2. \quad \text{Solve for } w, \ w = \sqrt{r^2 - h^2}. \quad \text{We need to maximize } A(h) = 2h\sqrt{r^2 - h^2} - \frac{2h^2}{\sqrt{r^2 - h^2}}. \quad \text{We find the critical points by setting the last expression equal to zero, multiplying by the denominator and dividing by 2 yields: } (r^2 - h^2) - h^2 = 0. \quad \text{Hence } h^2 = \frac{r^2}{2} \text{ and } h = w = \sqrt{2/2} \cdot r.
\]

3. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. Draw a picture, label and name your variables, show all your work.

4. Evaluate the indefinite integrals:
   (a) \( \int \frac{t\sqrt{t} + \sqrt{t}}{t^2} \, dt \)
   (b) \( \int \sec^2(2t) \, dt \)
   (c) \( \int e^{-3x} - 5e^{2x} \, dx \)
   (d) \( \int \frac{1}{1 + x^2} \, dx \)
   (e) \( \int \frac{1}{\sqrt{1 - x^2}} \, dx \)
   (f) \( \int \sec(\pi t) \, \tan(\pi t) \, dt \)

5. Use finite approximations to estimate the area under the graph of the function \( f(x) = \frac{1}{x} \) between 1 and 3.
   (a) using four rectangles of equal length and right-hand endpoints.
   (b) using four rectangles of equal length and left-hand endpoints.
   (c) using four rectangles of equal length and midpoints.

We have \( x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3. \ \Delta x_k = .5 = 1/2. \)
For part (c) we take the midpoints: \( c_1 = 1.25 = 5/4, c_2 = 1.75 = 7/4, c_3 = 2.25 = 9/4, c_4 = 2.75 = 11/4. \)

The Riemann sum is therefore \( (4/5 + 4/7 + 4/9 + 4/11) \cdot 1/2. \) Approximately 1.09.

6. Write the sums without sigma notation.
   (a) \( \sum_{k=1}^{6} \frac{2^k}{2k - 1} \)
   (b) \( \sum_{k=0}^{4} (-1)^k \frac{k^2}{2k + 1} \)
   (c) \( \sum_{k=-1}^{3} \frac{(-1)^k}{2k + 3} \)

7. Express the sums in sigma notation.
   (a) \( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \)
   (b) \( 1 - \frac{2}{3} + \frac{4}{5} - \frac{8}{7} + \frac{16}{9} \)
   (c) \( 1 - 5 + 9 - 13 + 17 - 21 + 25 \)

8. For the function \( f(x) = x^2 + 1, \) find a formula for the Riemann sum obtained by dividing the interval \( [0, 2] \) into \( n \) equal subintervals and using right-hand endpoints for each \( c_k. \) Then take the limit as \( n \to \infty \) to calculate the area under the curve.

Partition: \( x_0 = 0, x_1 = 2/n, x_2 = 4/n, ..., x_k = 2k/n, ..., x_n = 2n/n = 2 \) and \( \Delta x_k = 2/n. \)
Right-hand endpoints implies that $c_k = x_k$. Hence $f(c_k) = (2k/n)^2 + 1$. The Riemann sum is

$$\sum_{k=1}^{n} f(c_k) \Delta x_k = \sum_{k=1}^{n} \left( \left( \frac{2k}{n} \right)^2 + 1 \right) \cdot \frac{2}{n} = \frac{8}{n^3} \sum_{k=1}^{n} k^2 + 2 \sum_{k=1}^{n} 1 = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} + 2 \rightarrow_{n \to \infty} \frac{8}{3} + 2 = \frac{14}{3}.$$

9. Express the limit $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{1-c_k} \Delta x_k$ as a definite integral, where $P$ is a partition of $[2,3]$.

$$\int_{2}^{3} \frac{1}{1-x} \, dx.$$

10. Evaluate the integrals.

\[(a) \int_{1}^{4} \frac{t \sqrt{t} + 1}{t^2} \, dt \quad (b) \int_{0}^{\pi/4} \sec^2(t) \, dt \quad (c) \int_{0}^{1} \frac{1}{1+x} \, dx \quad (d) \int_{-1}^{1} \frac{x^3}{1+x^2} \, dx \quad (e) \int_{0}^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx \quad (f) \int_{0}^{1} 2^{t} \, dt.\]

11. Find the total area between the region and the $x$-axis.

\[(a) y = x^3 - x, \quad -1 \leq x \leq 1.\]

$y = x(x-1)(x+1)$. Therefore the function crosses the $x$-axis at $-1, 0, 1$. Between $-1$ and 0 the function is positive and between 0 and 1 the function is negative. The area is therefore equal to

$$\int_{-1}^{0} (x^3 - x) \, dx - \int_{0}^{1} (x^3 - x) \, dx = x^4/4 - x^2/2\big|_{0}^{1} - [x^4/4 - x^2/2]\big|_{0}^{1} = \frac{1}{4} + \frac{1}{2} - (\frac{1}{4} - \frac{1}{2}) = 1/2.$$

\[(b) y = -x^2 - 2x, \quad -2 \leq x \leq 2.\]

$y = -x(x + 2)$. The function crosses the $x$-axis at 0 and at $-2$. Between $-2$ and 0 the function is positive, between 0 and 2 the function is negative. The area is therefore equal to

$$\int_{-2}^{0} -x^2 - 2x \, dx - \int_{0}^{2} -x^2 - 2x \, dx = -x^3/3 - x^2\big|_{0}^{2} - [-x^3/3 - x^2]\big|_{0}^{2} = (-\frac{8}{3} + 4) - (-\frac{8}{3} - 4) = 8.$$

\[(c) y = \cos x, \quad 0 \leq x \leq \pi.\] The function crosses at $\pi/2$.

Area $= \int_{0}^{\pi/2} \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \, dx = \sin x\big|_{0}^{\pi/2} - \sin x\big|_{\pi/2}^{\pi} = 2.$

12. Find $dy/dx$.

\[(a) y = \int_{0}^{x} \frac{t \sqrt{t} + 1}{t^2} \, dt \quad (b) y = \int_{-x}^{x} e^{-t^2} \, dt \quad (c) y = \int_{-x}^{x} e^{-t^2} \, dt \quad (d) y = \int_{0}^{\sin x} \frac{1}{\sqrt{1-t^2}} \, dt.\]

13. Review all quizzes and exams.

You need to know the mean value theorem for the integral, partitions, sigma notation, setting up Riemann sums, definite and indefinite integrals, the Fundamental Theorem of Calculus.
1. (10 pts) For the function \( f(x) = -x^2 \) evaluate the limit of \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \). Show all the details.

2. (20 pts) Find the derivatives of the functions. Show all your work.
   (a) \( f(x) = \frac{x}{1 + x^2} \)
   (b) \( f(x) = (x + 1)e^{-x^2} \)
   (c) \( f(x) = \frac{\tan x}{1 + \tan x} \)
   (d) \( f(x) = \sin \left( \frac{x}{\sqrt{x + 1}} \right) \)

3. (10 pts) Find an equation of the tangent line of the curve \( y^2 + x^2 = y^4 - 2x \) at the point \((-2, 1)\).

4. (10 pts) Find the derivatives of the functions. Show all your work.
   (a) \( f(x) = \ln(\sec x + \tan x) \)
   \[ f'(x) = \frac{1}{\sec x + \tan x} \cdot (\sec x + \tan x)' \]
   \[ = \frac{1}{\sec x + \tan x} \cdot \sec x \cdot (\sec x + \tan x) = \sec x \]
   (b) \( f(x) = \tan^{-1} \left( \sqrt{x^2 + 1} \right) \)

5. (10 pts) Find the absolute maximum and the absolute minimum value of the function \( f(x) = \frac{1}{x} + \ln x \) on the interval \([\frac{1}{2}, 10]\).
   \[ f'(x) = -\frac{1}{x^2} + \frac{1}{x} \]
   We find critical points by setting \( f' \) equal to zero. Hence \( -\frac{1}{x^2} + \frac{1}{x} = 0 \). Multiplying by \( x^2 \) yields \(-1 + x = 0\). Only critical point at \( x = 1 \). Now \( f(1/2) = 2 - \ln 2 \), \( f(1) = 1 \), and \( f(10) = 1/10 + \ln 10 \).
   The maximum happens at \( x = 10 \) and the minimum at \( x = 1 \). (Note that \( \ln 2 \) is less than 1 and \( \ln 10 \) is larger than 1.

6. (10 pts) Find the limits. Show all your work and justify your answer.
   (a) \( \lim_{x \to 0} \frac{\cos x - 1}{8x^2} \)
   (b) \( \lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x} \)

7. (15 pts) For the function \( f(x) = x^4 - 2x^2 + 1 \) find
   (a) all critical points,
   (b) the intervals on which the function is decreasing or increasing,
   (c) all local maxima and minima,
   (d) the intervals on which the function is concave up or concave down,
   (e) all inflection points,
   (f) all asymptotes.
   (g) Use the above information to sketch the graph of \( f(x) \). Show all your work.

8. (10 pts) A square-based, open-top, rectangular cardboard box is to be made to have a volume of 4000 cm\(^3\). Find the dimensions that will minimize the amount of cardboard needed to build the box. Draw a picture, label and name your variables, show all your work.

9. (10 pts) Find antiderivatives of each of the following:
   (a) \( f(x) = \frac{1}{x^2} - x\sqrt{x} \)
   (b) \( f(x) = e^{x/2} \)
10. (5 pts) Use the Fundamental Theorem of Calculus to evaluate the definite integral \( \int_1^4 \frac{1}{2\sqrt{x}} \, dx \).