1 Introduction

1.1 Let $G$ be a simple algebraic group over an algebraically closed field $k$ of characteristic $p > 0$. For $r \geq 1$, let $G_r$ be the $r$th Frobenius kernel of $G$. It is well-known that the representations for $G_1$ are equivalent to the restricted representations for Lie $G$. Historically, the cohomology for Frobenius kernels has been best understood for large primes. Friedlander and Parshall [FP] first computed the cohomology ring $H^\bullet(G_1, k)$ for $p \geq 3(h - 1)$ where $h$ is the Coxeter number of the underlying root system. They proved that the cohomology ring can be identified with the coordinate algebra of the nullcone. Andersen and Jantzen [AJ] later verified this fact for $p \geq h$. Furthermore, they generalized this calculation by looking at $H^\bullet(G_1, H^0(\lambda))$ where $H^0(\lambda) = \text{ind}_G^B \lambda$ for $p \geq h$. Their results had some restrictions on the type of root system involved. Kumar, Lauritzen and Thomsen [KLT] removed the restrictions on the root systems through the use of Frobenius splittings.

The cohomology ring $H^\bullet(G_1, k)$ modulo nilpotents can be identified in general with the coordinate algebra of the restricted nullcone $N_1 = \{ x \in \text{Lie}(G) \mid x[p] = 0 \}$. For good primes Nakano, Parshall and Vella [NPV] proved that this variety is irreducible and can be identified with the closure of some Richardson orbit. Recently, Carlson, Lin, Nakano and Parshall [CLNP] have given an explicit description of $N_1$. These recent results provide some indication that one can systematically study extensions of Frobenius kernels for small primes by using general formulas which exhibit generic behavior for large primes.

1.2 This paper will first focus on the first extension groups (i.e. $\text{Ext}^1$) in the category of $G_r$-modules for arbitrary primes. The authors have shown that such computations are highly desirable because they can be used to provide vital information about extensions in the category of modules for the finite Chevalley group $G(F_{p^r})$ (see [BNP1, BNP2, BNP3]).

The paper is outlined as follows. In [Jan2], Jantzen provides extensive computations of the first cohomology groups of the first Frobenius kernel $G_1$ of $G$. In particular, he studies both $H^1(G_1, L(\lambda))$ and $H^1(G_1, H^0(\lambda))$ where $L(\lambda)$ is the simple $G$-module with highest weight $\lambda$. The goal in the first part of the paper is to extend these results to higher Frobenius kernels $G_r$. In Section 2, we recall Jantzen’s computations of $H^1(B_1, \lambda)$ and then use those results to compute $H^1(B_r, \lambda)$. In Section 3 the $B_r$-cohomology results are used to compute $H^1(G_r, H^0(\lambda))$. In particular, we determine all fundamental dominant weights $\omega$ for which...
$H^1(G_r, H^0(\omega))$ is non-zero. In Section 4, we present some results on the cohomology of simple modules.

In Section 5 we make use of the earlier computations (in Section 3) to prove a general formula for extensions between two simple $G_r$-modules for arbitrary primes (see Theorem 5.3). More specifically, we can relate extensions (i.e. $\text{Ext}^r_{G_r}L$) between simple $G_r$-modules with the extensions between certain $G$-modules. In particular for $m = 1$ and $\lambda, \mu \in X_r(T)$, we construct the following isomorphism (as vector spaces)

\begin{equation}
\text{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_h} \text{Ext}^1_{G}(L(\lambda), I_h(\nu) \otimes L(\mu)) \otimes L(\nu),
\end{equation}

where $I_h(\nu)$ is the injective hull of $L(\nu)$ in the bounded category $C_h$. Here $\pi_h = \{ \nu \in X(T) : \langle \nu, \alpha_0 \rangle < h \}$ and $C_h$ is the full subcategory of all $G$-modules whose composition factors $L(\lambda)$ have highest weights in $\pi_h$. For $p > h$, we can apply the explicit description of $H^*(G_1, H^0(\lambda))$ given in [KLT], to provide sharper results on the necessary bounds for our truncated categories. From this formula above, we can deduce that for $p \geq 2h - 1$,

\begin{equation}
\text{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_h} \text{Ext}^1_{G}(L(\lambda), L(\nu) \otimes L(\mu)) \otimes L(\nu).
\end{equation}

The preceding formulas significantly improve earlier results by the authors [BNP1] and Andersen [And1, Prop. 5.5].

1.3 Notation: Throughout this paper let $G$ be a simple simply connected algebraic group scheme defined and split over the finite field $\mathbb{F}_p$ with $p$ elements. The field $k$ is the algebraic closure of $\mathbb{F}_p$. For $r \geq 1$, let $G_r$ be the $r$th Frobenius kernel of $G$. The basic definitions and notation can be found in [Jan1].

Let $T$ be a maximal split torus and $\Phi$ be the root system associated to $(G, T)$. Moreover, let $\Phi^+$ (resp. $\Phi^-$) be positive (resp. negative) roots and $\Delta$ be a base consisting of simple roots. For a given root system of rank $n$, the simple roots will be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\alpha^\vee = 2\alpha / (\alpha, \alpha)$ be the coroot corresponding to $\alpha \in \Phi$. In this case, the fundamental weights (basis dual to $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee$) will be denoted by $\omega_1, \omega_2, \ldots, \omega_n$. We use the same ordering of roots as given in [Jan2] (following Bourbaki). In particular, for type $B_n$, $\alpha_n$ denotes the unique short simple root and for type $C_n$, $\alpha_n$ denotes the unique long simple root. For a generic simple root $\alpha$, $\omega_\alpha$ will denote the corresponding fundamental weight. Let $B$ be a Borel subgroup containing $T$ corresponding to the negative roots and $U$ be the unipotent radical of $B$.

Let $E$ be the Euclidean space associated with $\Phi$, and the inner product on $E$ will be denoted by $\langle \ , \ \rangle$. Moreover, let $X(T)$ be the integral weight lattice obtained from $\Phi$. The set $X(T)$ has a partial ordering defined as follows: if $\lambda, \mu \in X(T)$ then $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} N\alpha$. Set $\alpha_0$ to be the highest short root. Moreover, let $\rho$ be the half sum of positive roots and $w_0$ denote the long element of the Weyl group. The Coxeter number associated to $\Phi$ is $h = \langle \rho, \alpha_0^\vee \rangle + 1$. The set of dominant integral weights is defined by

$$X(T)_+ = \{ \lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta \},$$

and the set of $p^r$-restricted weights is

$$X_r(T) = \{ \lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in \Delta \}.$$
The simple modules for $G$ are indexed by the set $X(T)_+$ and denoted by $L(\lambda)$, $\lambda \in X(T)_+$ with $L(\lambda) = \text{soc}_G H^0(\lambda)$ where $H^0(\lambda) = \text{ind}_B^G \lambda$. A complete set of non-isomorphic simple $G_r$-modules are easily obtained by taking $\{ L(\lambda) : \lambda \in X_r(T) \}$. For $\lambda \in X(T)$, we will often use the notation $k_\lambda := \lambda$ to be the one-dimensional $B$-module obtained by taking the one-dimensional $T$-module $\lambda$ and extending it to $U$-trivially.

2 $B_r$-cohomology

2.1 This section is concerned with computing $B_r$-cohomology. Specifically, we compute $H^1(B_r, \lambda)$ for all $\lambda \in X(T)$. For an arbitrary $\lambda \in X(T)$, we may write $\lambda = \lambda_0 + p^r\lambda_1$ for a unique weight $\lambda_0 \in X_r(T)$. Furthermore,

$$H^1(B_r, \lambda) = H^1(B_r, \lambda_0 + p^r\lambda_1) \cong H^1(B_r, \lambda_0) \otimes p^r\lambda_1.$$ 

Hence, it suffices to compute the cohomology for weights $\lambda \in X_r(T)$.

2.2 Special cohomology modules: We define certain cohomology modules which will be used throughout the rest of the paper. Jantzen’s computations of the cohomology groups $H^1(B_1, \lambda)$ in [Jan2] begins with a computation of $H^1(U_1, k)$ as a $B$-module. In most cases, $H^1(U_1, k)$ decomposes as a direct sum of simple modules but not in all cases. In particular, certain indecomposable modules arise when the prime is small. This leads to the presence of certain indecomposable $B$-modules in the identification of $H^1(B_1, \lambda)$. We list these module here with our notation.

- Type $B_n$, $n \geq 3$, $p = 2$: Let $M_{B_n}$ denote the 2-dimensional indecomposable $B$-module from [Jan2] having a filtration with factors $k_{\omega_n}$ on the top and $k_{\omega_n - 1 - \omega_n}$ on the bottom.
- Type $C_n$, $n \geq 2$, $p = 2$: Let $M_{C_n}$ denote the $n$-dimensional indecomposable $B$-module from [Jan2] having a filtration with factors $k_{\omega_1}$, $k_{\omega_2 - \omega_1}$, $k_{\omega_3 - \omega_2}$, ..., and $k_{\omega_n - \omega_{n-1}}$ from top to bottom.
- Type $F_4$, $p = 2$: Let $M_{F_4}$ denote the 3-dimensional indecomposable $B$-module from [Jan2] having a filtration with factors $k_{\omega_4}$, $k_{\omega_3 - \omega_4}$, and $k_{\omega_2 - \omega_3}$ from top to bottom.
- Type $G_2$, $p = 2, 3$: Let $M_{G_2}$ denote the 2-dimensional indecomposable $B$-module from [Jan2] having a filtration with factors $k_{\omega_1}$ on the top and $k_{\omega_2 - \omega_1}$ on the bottom. Note that there are properly two modules here, one for each prime. As the prime will be clear in context, we abusively use the same notation for both.

2.3 For a simple root $\alpha$, whether or not the weight $p\omega_\alpha - \alpha$ is $p$-restricted affects the $B_1$-cohomology. For higher $r$, the question becomes whether $p^r\omega_\alpha - p^i\alpha$ is $p^r$-restricted for $0 \leq i \leq r - 1$. More generally, if $\omega$ is a weight and $p^r\omega - p^i\alpha$ is $p^r$-restricted, one would like to know what conditions $\omega$ must satisfy. It is not hard to see that in fact $\omega$ is usually $\omega_\alpha$ and in general is uniquely determined by $p$, $r$, $\alpha$, and $i$.

Lemma. Let $\lambda \in X_r(T)$. If $\lambda = p^r\omega - p^i\alpha \in X_r(T)$ for some $\omega \in X(T)$, $\alpha \in \Delta$, and $0 \leq i \leq r - 1$, then $\omega = \omega_\alpha$ EXCEPT in the following cases:
Theorem (B) \hspace{1cm}
\begin{align*}
&\text{Theorem (A)}
\end{align*}

Hence we have
\begin{align*}
\langle \alpha, \beta \rangle = &\, p^r \langle \omega, \alpha \rangle - p^j \langle \alpha, \alpha \rangle = p^r n_\alpha - 2p^j.
\end{align*}

\text{Hence } n_\alpha = 1. \text{ If the underlying root system is of type } A_1, \text{ we are done. Now, let } \beta \neq \alpha \text{ be another simple root. Then we have}
\begin{align*}
\langle \lambda, \beta \rangle = &\, p^r \langle \omega, \beta \rangle - p^j \langle \alpha, \beta \rangle = p^r n_\beta - p^j \langle \alpha, \beta \rangle.
\end{align*}

Now \( \langle \alpha, \beta \rangle = 0, -1, -2, \text{ or } -3. \) Hence \( n_\beta \) must be zero unless \( p = 2 \text{ or } 3, i = r - 1, \text{ and } \langle \alpha, \beta \rangle = -2 \text{ or } -3. \) Checking the various root systems one obtains the above list of “exceptional” cases. 

\textbf{2.4} Jantzen computed the cohomology groups \( H^1(B_1, \lambda) \) in [Jan2, §3] for all \( \lambda \in X(T). \) For the reader’s convenience, we recall these results. For small primes, the answer depends on the type of the root system and involves certain indecomposable \( B \)-modules which are identified in Section 2.2. Note that there is a “generic” answer for \( p > 3. \)

\textbf{Theorem (A).} \textit{Let } \( p > 3 \text{ and } \lambda \in X_1(T). \) \textit{Then}
\begin{align*}
H^1(B_1, \lambda) &\cong \begin{cases} k^{(1)}_{\omega_{\alpha}} & \text{if } \lambda = p\omega_{\alpha} - \alpha \text{ for } \alpha \in \Delta \\ 0 & \text{else.} \end{cases}
\end{align*}

\textbf{Theorem (B).} \textit{Let } \( p = 3 \text{ and } \lambda \in X_1(T). \)
\begin{align*}
(\text{a}) \text{ Assume that the underlying root system of } G \text{ is not of type } A_2 \text{ or } G_2. \text{ Then}
H^1(B_1, \lambda) &\cong \begin{cases} k^{(1)}_{\omega_{\alpha}} & \text{if } \lambda = p\omega_{\alpha} - \alpha \text{ for } \alpha \in \Delta \\ 0 & \text{else.} \end{cases}
\end{align*}

(\text{b}) \text{ Assume that the underlying root system of } G \text{ is of type } A_2. \text{ Then}
\begin{align*}
H^1(B_1, \lambda) &\cong \begin{cases} k^{(1)}_{\omega_{1}} \oplus k^{(1)}_{\omega_{2}} & \text{if } \lambda = \omega_{1} + \omega_{2} = 3\omega_{1} - \alpha_{1} = 3\omega_{2} - \alpha_{2} \\ 0 & \text{else.} \end{cases}
\end{align*}

(\text{c}) \text{ Assume that the underlying root system of } G \text{ is of type } G_2. \text{ Then}
\begin{align*}
H^1(B_1, \lambda) &\cong \begin{cases} k^{(1)}_{\omega_{1}} & \text{if } \lambda = \omega_{1} + \omega_{2} = 3\omega_{1} - \alpha_{1} \\ M^{(1)}_{G_2} & \text{if } \lambda = 3(\omega_{2} - \omega_{1}) - \alpha_{2} \\ 0 & \text{else.} \end{cases}
\end{align*}
We remark that in this proposition, one sees two phenomena which lead to a non-generic answer. When \( p \omega_i - \alpha_j = p \omega_j - \alpha_j \) for distinct \( i, j \), there is a “doubling” of the cohomology (in the sense of a direct sum of modules). The second phenomenon involves the question of whether the weight \( p \omega_j - \alpha_j \) is \( p \)-restricted. Notice that in type \( G_2 \) when \( p = 3 \), the weight \( 3 \omega_2 - \alpha_2 = 3 \omega_1 \) is not \( p \)-restricted. And it gets “replaced” by the \( p \)-restricted weight \( 3(\omega_2 - \omega_1) - \alpha_2 \). Furthermore, the cohomology involves a non-simple indecomposable \( B \)-module. We refer the reader to Lemma 2.3 which considers the question of whether \( p \omega_j - \alpha_j \) is \( p \)-restricted and note that one sees the same phenomena for \( p = 2 \).

**Theorem**  (C). Let \( p = 2 \) and \( \lambda \in X_1(T) \).

(a) Assume that the underlying root system of \( G \) is of type \( A_n \) with \( n \neq 3, E_6, E_7, \) or \( E_8 \). Then

\[
H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_3} & \text{if } \lambda = p \omega_i - \alpha \text{ for } \alpha \in \Delta \\
  0 & \text{else}
\end{cases}
\]

(b) Assume that the underlying root system of \( G \) is of type \( A_3 \). Then

\[
H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_2} + k^{(1)}_{\alpha_2} & \text{if } \lambda = \omega_3 = 2 \omega_1 - \alpha_1 = 2 \omega_3 - \alpha_3 \\
  k^{(1)}_{\omega_2} & \text{if } \lambda = \omega_1 + \omega_2 = 2 \omega_2 - \alpha_2 \\
  0 & \text{else}
\end{cases}
\]

(c) Assume that the underlying root system of \( G \) is of type \( B_3 \). Then

\[
H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_2} + k^{(1)}_{\alpha_2} & \text{if } \lambda = \omega_3 = 2 \omega_1 - \alpha_1 = 2 \omega_3 - \alpha_3 \\
  \langle M^1_{B_3} \rangle & \text{if } \lambda = \omega_2 = 2(\omega_3 - \alpha_2) \\
  0 & \text{else}
\end{cases}
\]

(d) Assume that the underlying root system of \( G \) is of type \( B_4 \). Then

\[
H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_1} + \langle M^1_{B_4} \rangle & \text{if } \lambda = \omega_3 = 2 \omega_1 - \alpha_1 = 2(\omega_3 - \omega_4) - \alpha_3 \\
  k^{(1)}_{\omega_1} + \langle M^1_{B_4} \rangle & \text{if } \lambda = 2 \omega_j - \alpha_j \text{ for } j \in \{2, 4\} \\
  0 & \text{else}
\end{cases}
\]

(e) Assume that the underlying root system of \( G \) is of type \( B_n, n \geq 5 \). Then

\[
H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_j} & \text{if } \lambda = 2 \omega_j - \alpha_j \text{ for } j \in \{1, 2, \ldots, n-2, n\} \\
  \langle M^1_{B_n} \rangle & \text{if } \lambda = \omega_{n-2} = 2(\omega_{n-1} - \omega_n) - \alpha_{n-1} \\
  0 & \text{else}
\end{cases}
\]

(f) Assume that the underlying root system of \( G \) is of type \( C_n, n \geq 2 \). Then

\[
H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_j} & \text{if } \lambda = 2 \omega_j - \alpha_j \text{ for } j \in \{1, 2, \ldots, n-1\} \\
  \langle M^1_{C_n} \rangle & \text{if } \lambda = 0 = 2(\omega_n - \omega_{n-1}) - \alpha_n \\
  0 & \text{else}
\end{cases}
\]
(g) Assume that the underlying root system of $G$ is of type $D_4$. Then
\[ H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_1} \oplus k^{(1)}_{\omega_3} \oplus k^{(1)}_{\omega_4} & \text{if } \lambda = \omega_2 = 2\omega_1 - \alpha_1 = 2\omega_3 - \alpha_3 = 2\omega_4 - \alpha_4 \\
  k^{(1)}_{\omega_2} & \text{if } \lambda = \omega_1 + \omega_3 = 2\omega_2 - \alpha_2 \\
  0 & \text{else}. 
\end{cases} \]

(h) Assume that the underlying root system of $G$ is of type $D_n$, $n \geq 5$. Then
\[ H^1(B_1, \lambda) \cong \begin{cases} 
  k^{(1)}_{\omega_j} & \text{if } \lambda = 2\omega_j - \alpha_j \text{ for } j \in \{1, 2, \ldots, n-2\} \\
  k^{(1)}_{\omega_{n-1}} \oplus k^{(1)}_{\omega_n} & \text{if } \lambda = \omega_{n-2} = 2\omega_{n-1} - \alpha_{n-1} = 2\omega_n - \alpha_n \\
  0 & \text{else}. 
\end{cases} \]

(i) Assume that the underlying root system of $G$ is of type $F_4$. Then
\[ H^1(B_1, \lambda) \cong \begin{cases} 
  M^{(1)}_{F_4} & \text{if } \lambda = 2(\omega_2 - \omega_3) - \alpha_2 \\
  k^{(1)}_{\omega_1} & \text{if } \lambda = 2\omega_{j} - \alpha_{j} \text{ for } j \in \{1, 3, 4\} \\
  k^{(1)}_{\omega_{n-1}} \oplus k^{(1)}_{\omega_n} & \text{if } \lambda = \omega_{n-2} = 2\omega_{n-1} - \alpha_{n-1} = 2\omega_n - \alpha_n \\
  0 & \text{else}. 
\end{cases} \]

(j) Assume that the underlying root system of $G$ is of type $G_2$. Then
\[ H^1(B_1, \lambda) \cong \begin{cases} 
  M^{(1)}_{G_2} & \text{if } \lambda = 2(\omega_2 - \omega_1) - \alpha_2 \\
  k^{(1)}_{\omega_1} & \text{if } \lambda = 2\omega_1 - \alpha_1 \\
  k^{(1)}_{\omega_{n-1}} \oplus k^{(1)}_{\omega_n} & \text{if } \lambda = \omega_{n-2} = 2\omega_{n-1} - \alpha_{n-1} = 2\omega_n - \alpha_n \\
  0 & \text{else}. 
\end{cases} \]

Using these propositions, one can compute $H^1(B_1, \lambda)$ for all weights $\lambda$ by writing $\lambda = \lambda_0 + p\lambda_1$ and using $H^1(B_1, \lambda) \cong H^1(B_1, \lambda_0) \otimes p\lambda_1$.

2.5 With the aid of the Lyndon-Hochschild-Serre spectral sequence, we now extend the results of Section 2.4 for $B_1$ to $B_r$ for all $r$. When $p > 3$, the answer fits a “generic” form that does not depend on the root system. We consider this case first.

**Theorem.** Suppose $p > 3$ and $\lambda \in X_r(T)$. Then
\[ H^1(B_r, \lambda) \cong \begin{cases} 
  k^{(r)}_{\omega_\alpha} & \text{if } \lambda = p^i\omega_\alpha - p^i\alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r-1 \\
  0 & \text{else}. 
\end{cases} \]

**Proof.** We proceed by induction on $r$ with the $r = 1$ case being Proposition 2.4(A). To proceed inductively, consider the LHS spectral sequence
\[ E_{2}^{i,j} = H^i(B_r/B_{r-1}, H^j(B_{r-1}, \lambda)) \Rightarrow H^{i+j}(B_r, \lambda) \]
and the corresponding five-term exact sequence
\[ 0 \to E^{1,0} \to E^1 \to E^{0,1} \to E^{2,0} \to E^2. \]
Write \( \lambda = \lambda_0 + p^{r-1} \lambda_1 \). By induction, we have
\[
E^{0,1} = \text{Hom}_{B_r/B_{r-1}}(k, \mathcal{H}^1(B_{r-1}, \lambda)) \\
\cong \text{Hom}_{B_r/B_{r-1}}(k, \mathcal{H}^1(B_{r-1}, \lambda_0) \otimes p^{r-1} \lambda_1) \\
\cong \begin{cases} 
\text{Hom}_{B_r/B_{r-1}}(k, k_{\omega}^{(r)} \otimes k_{\lambda}^{(r-1)}) & \text{if } \lambda_0 = p^{r-1} \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r-2 \\
0 & \text{else} 
\end{cases} \\
\cong \begin{cases} 
\text{Hom}_{B_1}(k, k_{\omega_\alpha + \lambda_1})^{(r-1)} & \text{if } \lambda_0 = p^{r-1} \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r-2 \\
0 & \text{else} 
\end{cases} \\
\cong \begin{cases} 
(k_{\omega}^{(1)})^{(r-1)} & \text{if } \lambda_0 = p^{r-1} \omega_\alpha - p^i \alpha \text{ as above and } \omega_\alpha + \lambda_1 = p \omega \text{ for } \omega \in X(T) \\
0 & \text{else} 
\end{cases} \\
\cong \begin{cases} 
k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^{r-1} \omega_\alpha + \lambda_1 - p^i \alpha = p^r \omega - p^i \alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r-2 \\
0 & \text{else} 
\end{cases}.
\]

Since \( \lambda \in X_r(T) \), applying Lemma 2.3, we get
\[
E^{0,1} \cong \begin{cases} 
k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r-2 \\
0 & \text{else} \end{cases}
\]

On the other hand, we have
\[
E^{1,0} = \mathcal{H}^1(B_r/B_{r-1}, \text{Hom}_{B_{r-1}}(k, \lambda)) \\
\cong \begin{cases} 
\mathcal{H}^1(B_r/B_{r-1}, k_{\lambda}^{(r-1)}) & \text{if } \lambda = p^{r-1} \lambda' \text{ for } \lambda' \in X(T) \\
0 & \text{else} 
\end{cases} \\
\cong \begin{cases} 
\mathcal{H}^1(B_1, k_{\lambda'})^{(r-1)} & \text{if } \lambda = p^{r-1} \lambda' \text{ for } \lambda' \in X(T) \\
0 & \text{else} 
\end{cases} 
\]

Note that since \( \lambda \in X_r(T) \), the weight \( \lambda' \) must lie in \( X_1(T) \). And so by induction (or simply Proposition 2.4(A)), we have
\[
E^{1,0} \cong \begin{cases} 
(k_{\omega_\alpha}^{(1)})^{(r-1)} & \text{if } \lambda = p^{r-1} \lambda' \text{ as above and } \lambda' = p \omega_\alpha - \alpha \text{ for } \alpha \in \Delta \\
0 & \text{else} 
\end{cases} \\
\cong \begin{cases} 
k_{\omega_\alpha}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^{r-1} \alpha \text{ for } \alpha \in \Delta \\
0 & \text{else} \end{cases}.
\]

If \( E^{0,1} = 0 \), then \( E^1 \cong E^{1,0} \) and the above computations confirm the claim. On the other hand, if \( E^{0,1} \neq 0 \), we must have \( \lambda = p^r \omega_\alpha - p^i \alpha \) for some \( \alpha \in \Delta \) and \( 0 \leq i \leq r-2 \). This implies that \( \lambda \) is not divisible by \( p^{r-1} \) and so \( \text{Hom}_{B_{r-1}}(k, \lambda) = 0 \). Hence \( E^{1,0} = 0 = E^{2,0} \) and then \( E^1 \cong E^{0,1} \) and the result follows. \( \square \)
2.6 For $p = 3$, one has to deal with the fact that $H^1(B_1, \lambda)$ may not be of the form $k^{(1)}_\omega$ (when it is not zero). However, the same basic inductive argument still works.

**Theorem.** Let $p = 3$ and $\lambda \in X_r(T)$.

(a) Assume that the underlying root system of $G$ is not of type $A_2$ or $G_2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_\omega & \text{if } \lambda = p^r \omega_i - p^r \alpha_i \text{ for } \alpha_i \in \Delta, 0 \leq i \leq r - 1 \\ 0 & \text{else.} \end{cases}$$

(b) Assume that the underlying root system of $G$ is of type $A_2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_\omega & \text{if } \lambda = p^r (\omega_1 + \omega_2) = p^r \omega_1 - p^r \alpha_1 = p^r \omega_2 - p^r \alpha_2 \\ k^{(r)}_\omega & \text{if } \lambda = p^r \omega_j - p^r \alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

(c) Assume that the underlying root system of $G$ is of type $G_2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_\omega & \text{if } \lambda = p^r (\omega_1 + \omega_2) = p^r \omega_1 - p^r \alpha_1 \\ M_{G_2}^{(r)} & \text{if } \lambda = p^r \omega_2 - \omega_1 = p^r \omega_2 - p^r \alpha_2 \\ k^{(r)}_\omega & \text{if } \lambda = p^r \omega_j - p^r \alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

**Proof.** The proof for part (a) is identical to the proof of Theorem 2.5. For part (b), we follow the same inductive argument. Let $\lambda = \lambda_0 + p^{r-1} \lambda_1$. In this case, we get

$$E^{0,1} \cong \begin{cases} \text{Hom}_{B_r/B_{r-1}}(\lambda_0, k^{(r-1)}_\omega \oplus k^{(r-1)}_\omega) \otimes k^{(r-1)}_{\lambda_1} & \text{if } \lambda_0 = p^{r-2} (\omega_1 + \omega_2) \\ \text{Hom}_{B_r/B_{r-1}}(\lambda_0, k^{(r-1)}_\omega) \otimes k^{(r-1)}_{\lambda_1} & \text{if } \lambda_0 = p^{r-2} \omega_j - p^r \alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 3 \\ 0 & \text{else.} \end{cases}$$

Note that $\text{Hom}_{B_1}(\lambda_0, k^{(r-1)}_\omega \oplus k^{(r-1)}_\omega)$ is at most one-dimensional since it is not possible to have both $\omega_1 + \lambda_1 = p \omega$ and $\omega_2 + \lambda_1 = p \omega'$ for weights $\omega, \omega' \in X(T)$. Hence, we have

$$E^{0,1} \cong \begin{cases} k^{(r)}_\omega & \text{if } \lambda_0 = p^{r-2} (\omega_1 + \omega_2) \text{ and } \omega_j + \lambda_1 = p \omega \text{ for } j \in \{1, 2\}, \omega \in X(T) \\ k^{(r)}_\omega & \text{if } \lambda_0 = p^{r-2} \omega_j - p^r \alpha_j \text{ and } \omega_j + \lambda_1 = p \omega \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 3, \omega \in X(T) \\ 0 & \text{else.} \end{cases}$$

Since $p^{r-2} (\omega_1 + \omega_2) = p^{r-2} \omega_1 - p^{r-2} \alpha_1 = p^{r-2} \omega_2 - p^{r-2} \alpha_2$, the non-vanishing conditions above can be combined. The requirement is that $\lambda = \lambda_0 + p^{r-1} \lambda_1 = p^{r-1} \omega_j - p^r \alpha_j + p^{r-1} \lambda_1 = p^{r-1} (\omega_j + \lambda_1) - p^r \alpha_j = p^r \omega_j - p^r \alpha_j$ for $j = 1$ or $2, 0 \leq i \leq r - 2$ and some $\omega \in X(T)$. Applying Lemma 2.3, $\omega$ must be $\omega_j$. Hence we get

$$E^{0,1} \cong \begin{cases} k^{(r)}_\omega & \text{if } \lambda = p^r \omega_j - p^r \alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$
Next we compute $E^{1,0}$. Again, we get

$$E^{1,0} \cong \begin{cases} \text{H}^1(B_1, k_{\lambda'})^{(r-1)} & \text{if } \lambda = p^{r-1}\lambda' \text{ for } \lambda' \in X(T) \\ 0 & \text{else.} \end{cases}$$

And by induction,

$$E^{1,0} \cong \begin{cases} k_{\omega_1}^{(r)} \oplus k_{\omega_2}^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1 + \omega_2) = p^r\omega_1 - p^{r-1}\alpha_1 = p^r\omega_2 - p^{r-1}\alpha_2 \\ 0 & \text{else.} \end{cases}$$

As in Theorem 2.5, $H^1(B_r, \lambda)$ may be identified with either $E^{1,0}$ or $E^{0,1}$ and the claim follows.

For part (c), we proceed analogously. With $\lambda = \lambda_0 + p^r\lambda_1$, we get

$$E^{0,1} \cong \begin{cases} \text{Hom}_{B_r/B_{r-1}}(k, k_{\omega_1}^{(r-1)} \otimes k_{\lambda_1}^{(r-1)}) & \text{if } \lambda_0 = p^{r-1}\omega_1 - p^{r-2}\alpha_1 \\ \text{Hom}_{B_r/B_{r-1}}(k, M_{G_2}^{(r-1)} \otimes k_{\lambda_1}^{(r-1)}) & \text{if } \lambda_0 = p^{r-2}\omega_2 = p^{r-1}(\omega_2 - \omega_1) - p^{r-2}\alpha_2 \\ \text{Hom}_{B_r/B_{r-1}}(k, k_{\omega_2}^{(r-2)} \otimes k_{\lambda_1}^{(r-2)}) & \text{if } \lambda_0 = p^{r-1}\omega_j - p^i\alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 3 \\ 0 & \text{else.} \end{cases}$$

The condition in the second case above arises because the module $M_{G_2} \otimes k_{\lambda_1}$ is a two-dimensional indecomposable module with bottom factor being $k_{\omega_2 - \omega_1} \otimes k_{\lambda_1} \cong k_{\omega_2 - \omega_1 + \lambda_1}$.

Applying Lemma 2.3, we further get

$$E^{0,1} \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^r\omega_1 - p^{r-2}\alpha_1 \\ k_{\omega_2}^{(r)} & \text{if } \lambda = p^r\omega_2 - p^{r-2}\alpha_2 \\ k_{\omega_j}^{(r)} & \text{if } \lambda = p^r\omega_j - p^i\alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 3 \\ 0 & \text{else.} \end{cases}$$

Next we compute $E^{1,0}$. Again, we get

$$E^{1,0} \cong \begin{cases} \text{H}^1(B_1, k_{\lambda'})^{(r-1)} & \text{if } \lambda = p^{r-1}\lambda' \text{ for } \lambda' \in X(T) \\ 0 & \text{else.} \end{cases}$$

And by induction,

$$E^{1,0} \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1 + \omega_2) = p^r\omega_1 - p^{r-1}\alpha_1 \\ M_{G_2}^{(r)} & \text{if } \lambda = p^{r-1}\omega_2 = p^r(\omega_2 - \omega_1) - p^{r-1}\alpha_2 \\ 0 & \text{else.} \end{cases}$$

As before, $H^1(B_r, \lambda)$ may be identified with either $E^{1,0}$ or $E^{0,1}$ and the claim follows. □
2.7 For $p = 2$, the computation of $H^1(B_r, \lambda)$ involves even more special cases. As for $p = 3$, one must deal with the presence of direct sums and non-simple indecomposable modules. The arguments are similar to those for $p = 3$ and left to the interested reader.

**Theorem.** Let $p = 2$ and $\lambda \in X_r(T)$.

(a) Assume that the underlying root system of $G$ is of type $A_n$ with $n \neq 3$, $E_6$, $E_7$, or $E_8$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k_{\omega_i}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 1 \\ 0 & \text{else.} \end{cases}$$

(b) Assume that the underlying root system of $G$ is of type $A_3$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k_{\omega_1}^{(r)} \oplus k_{\omega_3}^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r \omega_3 - p^{r-1} \alpha_3 \\ k_{\omega_2}^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1 + \omega_3) = p^r \omega_2 - p^{r-1} \alpha_2 \\ k_{\omega_3}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

(c) Assume that the underlying root system of $G$ is of type $B_3$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k_{\omega_1}^{(r)} \oplus k_{\omega_3}^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r \omega_3 - p^{r-1} \alpha_3 \\ M_{B_3}^{(r)} & \text{if } \lambda = p^{r-1} \omega_1 = p^r (\omega_2 - \omega_3) - p^{r-1} \alpha_2 \\ k_{\omega_3}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

(d) Assume that the underlying root system of $G$ is of type $B_4$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k_{\omega_1}^{(r)} \oplus M_{B_4}^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r (\omega_3 - \omega_4) - p^{r-1} \alpha_3 \\ k_{\omega_1}^{(r)} & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{2, 4\} \\ k_{\omega_3}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

(e) Assume that the underlying root system of $G$ is of type $B_n$, $n \geq 5$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{1, 2, \ldots, n - 2, n\} \\ M_{B_n}^{(r)} & \text{if } \lambda = p^{r-1} \omega_{n-2} = p^r (\omega_{n-1} - \omega_n) - p^{r-1} \alpha_{n-1} \\ k_{\omega_n}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

(f) Assume that the underlying root system of $G$ is of type $C_n$, $n \geq 2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k_{\omega_1}^{(r)} & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{1, 2, \ldots, n - 1\} \\ M_{C_n}^{(r)} & \text{if } \lambda = 0 = p^r (\omega_n - \omega_{n-1}) - p^{r-1} \alpha_n \\ k_{\omega_n}^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$
2.8 With the computations of $H^1(B_r, \lambda)$ for all $\lambda \in X_r(T)$ above, one can readily compute $H^1(B_r, \lambda)$ for arbitrary $\lambda \in X(T)$. First, we make the following observation.

Corollary. Let $\lambda \in X(T)$. Then $H^1(B_r, \lambda) \neq 0$ if and only if $\lambda = p^r \omega - p^i \alpha$ for some $\omega \in X(T), \alpha \in \Delta$, and $0 \leq i \leq r - 1$. Moreover, if $\lambda \in X_r(T)$, then the weight $\omega$ is the weight determined by $p$, $r$, $\alpha$, and $i$ in Lemma 2.3.

Proof. Given $\lambda \in X(T)$, we first show that $\lambda$ must have the desired form. Write $\lambda = \lambda_0 + p^r \lambda_1$ for (unique) $\lambda_0 \in X_r(T)$ and $\lambda_1 \in X(T)$. Then $H^1(B_r, \lambda) \cong H^1(B_r, \lambda_0) \otimes p^r \lambda_1$ and the answer depends on $\lambda_0$. From Theorems 2.5 through 2.7, $H^1(B_r, \lambda_0) \neq 0$ if and only if $\lambda_0 = p^r \omega' - p^i \alpha$ for some $\omega \in \Delta$ where $\omega'$ is the weight (corresponding to $p$, $r$, $\alpha$, and $i$) determined in Lemma 2.3. Thus $\lambda = \lambda_0 + p^r \lambda_1 = p^r \omega' - p^i \alpha + p^r \lambda_1 = p^r (\omega' + \lambda_1) - p^i \alpha$ has the form $p^r \omega - p^i \alpha$.

Conversely, given any weight $\lambda = p^r \omega - p^i \alpha$, one can always express $\omega$ as $\omega = \omega' + \lambda_1$ for the required weight $\omega'$ and some weight $\lambda_1 \in X(T)$. And since non-vanishing is independent of $\lambda_1$, $H^1(B_r, \lambda)$ will be non-zero for all such $\lambda$. Finally, the “moreover” part follows immediately from Lemma 2.3.
Given $\lambda = p^i\omega - p^j\alpha$, to compute $H^1(B_r, \lambda)$, write $\lambda = (p^i\omega' - p^j\alpha) + p^k\lambda_1$ for the specific weight $\omega'$ (from Lemma 2.3) and some weight $\lambda_1 \in X(T)$. In terms of the given weight $\omega$, $\lambda_1 = \omega - \omega'$. Thus we get

$$H^1(B_r, \lambda) \cong H^1(B_r, \lambda_1) \otimes k^{(r)}_{\lambda_1} = H^1(B_r, p^i\omega' - p^j\alpha) \otimes k^{(r)}_{\lambda_1} \cong H^1(B_r, p^i\omega' - p^j\alpha) \otimes k^{(r)}_{\omega - \omega'}$$

and one simply substitutes the answers from Theorem 2.5 through 2.7. For example, in the generic case, $\omega' = \omega_\alpha$ and $H^1(B_r, p^i\omega_\alpha - p^j\alpha) \cong k^{(r)}_{\omega_\alpha}$ so that $H^1(B_r, p^i\omega - p^j\alpha) \cong k^{(r)}_{\omega_\alpha} \otimes k^{(r)}_{\omega - \omega_\alpha} \cong k^{(r)}_{\omega}$. For completeness, we include the answers here omitting the straightforward details in the non-generic cases.

**Theorem (A).** Let $p > 3$ and $\lambda \in X(T)$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_{\omega} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 1 \\ k^{(r)}_{\omega} & \text{else.} \end{cases}$$

**Theorem (B).** Let $p = 3$ and $\lambda \in X(T)$.

(a) Assume that the underlying root system of $G$ is not of type $A_2$ or $G_2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_{\omega} \oplus k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), j, \ell \in \{1,2\}, j \neq \ell \\ k^{(r)}_{\omega} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$

(b) Assume that the underlying root system of $G$ is of type $A_2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_{\omega} \oplus k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), j, \ell \in \{1,2\}, j \neq \ell \\ M^{(r)}_{G_2} \otimes k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), j \in \{1,2\}, 0 \leq i \leq r - 2 \\ k^{(r)}_{\omega} & \text{else.} \end{cases}$$

(c) Assume that the underlying root system of $G$ is of type $G_2$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_{\omega} \oplus k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), j, \ell \in \{1,2\}, j \neq \ell \\ M^{(r)}_{G_2} \otimes k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), j \in \{1,2\}, 0 \leq i \leq r - 2 \\ k^{(r)}_{\omega} & \text{else.} \end{cases}$$

**Theorem (C).** Let $p = 2$ and $\lambda \in X(T)$.

(a) Assume that the underlying root system of $G$ is of type $A_n$ with $n \neq 3$, $E_6$, $E_7$, or $E_8$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_{\omega} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 1 \\ k^{(r)}_{\omega} & \text{else.} \end{cases}$$

(b) Assume that the underlying root system of $G$ is of type $A_3$. Then

$$H^1(B_r, \lambda) \cong \begin{cases} k^{(r)}_{\omega} \oplus k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), j, \ell \in \{1,3\}, j \neq \ell \\ k^{(r)}_{\omega} \oplus k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T) \\ k^{(r)}_{\omega} & \text{if } \lambda = p^i\omega - p^j\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\ 0 & \text{else.} \end{cases}$$
(c) Assume that the underlying root system of $G$ is of type $B_3$. Then

$$\begin{align*}
H^1(B_r, \lambda) &\cong \begin{cases} 
    k^{(r)}_\omega & \text{if } \lambda = p^r\omega - p^r-1\alpha_j \text{ for } \omega \in X(T), j, \ell \in \{1, 3\}, j \neq \ell \\
    M_{B_3}^{(r)} \otimes k^{(r)}_{\omega + \omega_1 - \omega_j} & \text{if } \lambda = p^r\omega - p^r-1\alpha_2 \text{ for } \omega \in X(T) \\
    k^{(r)}_{\omega + \omega_2 - \omega_j} & \text{if } \lambda = p^r\omega - p^r\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
    0 & \text{else.}
\end{cases}
\end{align*}$$

(d) Assume that the underlying root system of $G$ is of type $B_4$. Then

$$\begin{align*}
H^1(B_r, \lambda) &\cong \begin{cases} 
    k^{(r)}_\omega & \text{if } \lambda = p^r\omega - p^r-1\alpha_1 \text{ for } \omega \in X(T) \\
    (M_{B_4}^{(r)} \otimes k^{(r)}_{\omega - \alpha_1}) & \text{if } \lambda = p^r\omega - p^r-1\alpha_{1, j} \text{ for } \omega \in X(T), j \in \{2, 4\} \\
    k^{(r)}_{\omega + \omega_1 + \omega_3 - \omega_2} & \text{if } \lambda = p^r\omega - p^r\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
    0 & \text{else.}
\end{cases}
\end{align*}$$

(e) Assume that the underlying root system of $G$ is of type $B_n, n \geq 5$. Then

$$\begin{align*}
H^1(B_r, \lambda) &\cong \begin{cases} 
    k^{(r)}_\omega & \text{if } \lambda = p^r\omega - p^r-1\alpha_1 \text{ for } \omega \in X(T), j \in \{1, 2, \ldots, n - 2, n\} \\
    M_{B_n}^{(r)} \otimes k^{(r)}_{\omega - \omega_{n-1}} & \text{if } \lambda = p^r\omega - p^r-1\alpha_{n-1} \text{ for } \omega \in X(T) \\
    k^{(r)}_{\omega + \omega_n - \omega_{n-1}} & \text{if } \lambda = p^r\omega - p^r\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
    0 & \text{else.}
\end{cases}
\end{align*}$$

(f) Assume that the underlying root system of $G$ is of type $C_n, n \geq 2$. Then

$$\begin{align*}
H^1(B_r, \lambda) &\cong \begin{cases} 
    k^{(r)}_\omega & \text{if } \lambda = p^r\omega - p^r-1\alpha_1 \text{ for } \omega \in X(T), j \in \{1, 2, \ldots, n - 1\} \\
    M_{C_n}^{(r)} \otimes k^{(r)}_{\omega + \omega_{n-1} - \omega_n} & \text{if } \lambda = p^r\omega - p^r-1\alpha_n \text{ for } \omega \in X(T) \\
    k^{(r)}_{\omega - \omega_n - \omega_{n-1}} & \text{if } \lambda = p^r\omega - p^r\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
    0 & \text{else.}
\end{cases}
\end{align*}$$

(g) Assume that the underlying root system of $G$ is of type $D_4$. Then

$$\begin{align*}
H^1(B_r, \lambda) &\cong \begin{cases} 
    k^{(r)}_\omega & \text{if } \lambda = p^r\omega - p^r-1\alpha_j \text{ for } \omega \in X(T), j, \ell, s \in \{1, 3, 4\}, j \neq \ell, j \neq s, \ell \neq s \\
    k^{(r)}_{\omega + \omega_e - \omega_j} & \text{if } \lambda = p^r\omega - p^r-1\alpha_2 \text{ for } \omega \in X(T) \\
    k^{(r)}_{\omega_a - \omega_4} & \text{if } \lambda = p^r\omega - p^r\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
    0 & \text{else.}
\end{cases}
\end{align*}$$

(h) Assume that the underlying root system of $G$ is of type $D_n, n \geq 5$. Then

$$\begin{align*}
H^1(B_r, \lambda) &\cong \begin{cases} 
    k^{(r)}_\omega & \text{if } \lambda = p^r\omega - p^r-1\alpha_j \text{ for } \omega \in X(T), j \in \{1, 2, \ldots, n - 2\} \\
    k^{(r)}_{\omega + \omega_j - \omega_e} & \text{if } \lambda = p^r\omega - p^r-1\alpha_j \text{ for } \omega \in X(T), j, \ell \in \{n - 1, n\}, j \neq \ell \\
    k^{(r)}_{\omega - \omega_n} & \text{if } \lambda = p^r\omega - p^r\alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
    0 & \text{else.}
\end{cases}
\end{align*}$$
(i) Assume that the underlying root system of \(G\) is of type \(F_4\). Then
\[
H^1(B_r, \lambda) \cong \begin{cases} 
 k^{(r)}_\omega & \text{if } \lambda = p^r \omega - p^{r-1} \alpha_j \text{ for } \omega \in X(T), \ j \in \{1, 3, 4\} \\
 M_{F_4}(r) \otimes k^{(r)}_{\omega + \omega_3 - \omega_2} & \text{if } \lambda = p^r \omega - p^{r-1} \alpha_2 \text{ for } \omega \in X(T) \\
k^{(r)}_\omega & \text{if } \lambda = p^r \omega - p^i \alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]

(j) Assume that the underlying root system of \(G\) is of type \(G_2\). Then
\[
H^1(B_r, \lambda) \cong \begin{cases} 
 k^{(r)}_\omega & \text{if } \lambda = p^r \omega - p^{r-1} \alpha_1 \text{ for } \omega \in X(T) \\
 M^{(r)}_{G_2} \otimes k^{(r)}_{\omega + \omega_1 - \omega_2} & \text{if } \lambda = p^r \omega - p^{r-1} \alpha_2 \text{ for } \omega \in X(T) \\
k^{(r)}_\omega & \text{if } \lambda = p^r \omega - p^i \alpha \text{ for } \omega \in X(T), \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]

We have chosen to present the results for \(\lambda \in X_r(T)\) first and then those for general \(\lambda \in X(T)\). If one prefers, this can be done in the opposite order: one can inductively obtain the results for arbitrary \(\lambda\) and then use Lemma 2.3 to deduce the results in Sections 2.5 through 2.7 for \(p^r\)-restricted weights.

3 \textbf{G}_r\text{-cohomology of induced modules}

3.1 According to Kempf’s vanishing theorem, \(H^0(\lambda) = \text{ind}^G_B \lambda\) is zero unless \(\lambda \in X(T)_+\). For dominant weights \(\lambda\), the preceding computations of \(B_r\)-cohomology can now be used to compute \(H^1(G_r, H^0(\lambda))\) thanks to the isomorphism
\[
H^1(G_r, H^0(\lambda))^{(-r)} \cong \text{ind}^G_B(H^1(B_r, \lambda)^{(-r)})
\]
(cf. [Jan1, II.12.2]). Indeed, in the “generic” case, we simply have \(\text{ind}^G_B(H^1(B_r, \lambda)^{(-r)}) \cong \text{ind}^G_B(k_\omega) = H^0(\omega)\). In general, for \(p^r\)-restricted weights, the computations follow readily from Theorems 2.5 through 2.7. However, some work is required when the \(B_r\)-cohomology involves a non-simple indecomposable module. For \(p \geq 3(h - 1)\), the following theorem (in conjunction with Lemma 2.3) is stated on p. 392 of [And2].

**Theorem (A).** Let \(p > 3\) and \(\lambda \in X_r(T)\). Then
\[
H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
 H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 1 \\
0 & \text{else.}
\end{cases}
\]

**Theorem (B).** Let \(p = 3\) and \(\lambda \in X_r(T)\).

(a) Assume that the underlying root system of \(G\) is not of type \(A_2\) or \(G_2\). Then
\[
H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
 H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 1 \\
0 & \text{else.}
\end{cases}
\]
Assume that the underlying root system of $G$ is of type $A_2$. Then
\[
H^1(G_r, H_0^{(0)}(\lambda)) \cong \begin{cases} 
H^0(\omega_1)^{(r)} \oplus H^0(\omega_2)^{(r)} & \text{if } \lambda = p^{r-1}(\omega_1 + \omega_2) = p^r \omega_1 - p^{r-1} \alpha_1 = p^r \omega_2 - p^{r-1} \alpha_2 \\
H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r \omega_j - p^i \alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]

Assume that the underlying root system of $G$ is of type $G_2$. Then
\[
H^1(G_r, H_0^{(0)}(\lambda)) \cong \begin{cases} 
H^0(\omega_1)^{(r)} & \text{if } \lambda = p^r \omega_1 - p^{r-1} \alpha_1 \\
H^0(\omega_1)^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r (\omega_2 - \omega_1) - p^{r-1} \alpha_2 \\
H^0(\omega_j)^{(r)} & \text{if } \lambda = p^r \omega_j - p^i \alpha_j \text{ for } j \in \{1, 2\}, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]

Proof. There is one case where the computation involves inducing a non-simple indecomposable $B$-module. That is in part (c) when $\lambda = p^{r-1} \omega_2$ and $H^1(B_r, \lambda) \cong M^{(r)}_{G_2}$. However, Jantzen shows in [Jan2, Prop. 5.2] that $\text{ind}_{G_2}(M_{G_2}) \cong H^0(\omega_1)$ which gives the claim. \hfill \Box

Theorem (C). Let $p = 2$ and $\lambda \in X_r(T)$.

(a) Assume that the underlying root system is of type $A_n$ with $n \neq 3$, $E_6$, $E_7$, or $E_8$. Then
\[
H^1(G_r, H_0^{(0)}(\lambda)) \cong \begin{cases} 
H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 1 \\
0 & \text{else.}
\end{cases}
\]

(b) Assume that the underlying root system of $G$ is of type $A_3$. Then
\[
H^1(G_r, H_0^{(0)}(\lambda)) \cong \begin{cases} 
H^0(\omega_1)^{(r)} \oplus H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r \omega_3 - p^{r-1} \alpha_3 \\
H^0(\omega_2)^{(r)} & \text{if } \lambda = p^{r-1} (\omega_1 + \omega_3) = p^r \omega_2 - p^{r-1} \alpha_2 \\
H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]

(c) Assume that the underlying root system of $G$ is of type $B_3$. Then
\[
H^1(G_r, H_0^{(0)}(\lambda)) \cong \begin{cases} 
H^0(\omega_1)^{(r)} \oplus H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r \omega_3 - p^{r-1} \alpha_3 \\
H^0(\omega_3)^{(r)} & \text{if } \lambda = p^{r-1} \omega_1 = p^r (\omega_2 - \omega_3) - p^{r-1} \alpha_2 \\
H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]

(d) Assume that the underlying root system of $G$ is of type $B_4$. Then
\[
H^1(G_r, H_0^{(0)}(\lambda)) \cong \begin{cases} 
H^0(\omega_1)^{(r)} \oplus H^0(\omega_4)^{(r)} & \text{if } \lambda = p^{r-1} \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r (\omega_3 - \omega_4) - p^{r-1} \alpha_3 \\
H^0(\omega_4)^{(r)} & \text{if } \lambda = p^r \omega_j - p^i \alpha_j \text{ for } j = 2, 4 \\
H^0(\omega_\alpha)^{(r)} & \text{if } \lambda = p^r \omega_\alpha - p^i \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}
\]
(e) Assume that the underlying root system of $G$ is of type $B_n$, $n \geq 5$. Then

$$H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
H^0(\omega_j)(r) & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{1, 2, \ldots, n - 2, n\} \\
H^0(\omega_n)(r) & \text{if } \lambda = p^r \omega_{n-2} - p^r(\omega_{n-1} - \omega_n) - p^{r-1} \alpha_{n-1} \\
H^0(\omega_\alpha)(r) & \text{if } \lambda = p^r \omega_\alpha - p^r \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}$$

(f) Assume that the underlying root system of $G$ is of type $C_n$, $n \geq 2$. Then

$$H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
H^0(\omega_j)(r) & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{1, 2, \ldots, n - 1\} \\
H^0(\omega_1)(r) & \text{if } \lambda = 0 = p^r(\omega_1 - \omega_{n-1}) - p^{r-1} \alpha_n \\
H^0(\omega_\alpha)(r) & \text{if } \lambda = p^r \omega_\alpha - p^r \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}$$

(g) Assume that the underlying root system is of type $D_4$. Then

$$H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
H^0(\omega_1)(r) \oplus H^0(\omega_3)(r) \oplus H^0(\omega_4)(r) & \text{if } \lambda = p^r \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 = p^r \omega_3 - p^{r-1} \alpha_3 = p^r \omega_4 - p^{r-1} \alpha_4 \\
H^0(\omega_2)(r) & \text{if } \lambda = p^r(\omega_1 + \omega_3) = p^r \omega_2 - p^{r-1} \alpha_2 \\
H^0(\omega_\alpha)(r) & \text{if } \lambda = p^r \omega_\alpha - p^r \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}$$

(h) Assume that the underlying root system of $G$ is of type $D_n$, $n \geq 5$. Then

$$H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
H^0(\omega_j)(r) & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{1, 2, \ldots, n - 2\} \\
H^0(\omega_{n-1})(r) \oplus H^0(\omega_n)(r) & \text{if } \lambda = p^r \omega_{n-2} = p^r \omega_{n-1} - p^{r-1} \alpha_{n-1} = p^r \omega_n - p^{r-1} \alpha_n \\
H^0(\omega_\alpha)(r) & \text{if } \lambda = p^r \omega_\alpha - p^r \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}$$

(i) Assume that the underlying root system of $G$ is of type $F_4$. Then

$$H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
H^0(\omega_j)(r) & \text{if } \lambda = p^r \omega_j - p^{r-1} \alpha_j \text{ for } j \in \{1, 3, 4\} \\
H^0(\omega_4)(r) & \text{if } \lambda = p^r \omega_1 = p^r(\omega_2 - \omega_3) - p^{r-1} \alpha_2 \\
H^0(\omega_\alpha)(r) & \text{if } \lambda = p^r \omega_\alpha - p^r \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}$$

(j) Assume that the underlying root system of $G$ is of type $G_2$. Then

$$H^1(G_r, H^0(\lambda)) \cong \begin{cases} 
H^0(\omega_1)(r) & \text{if } \lambda = p^r \omega_2 = p^r \omega_1 - p^{r-1} \alpha_1 \\
H^0(\omega_1)(r) & \text{if } \lambda = p^r \omega_1 = p^r(\omega_2 - \omega_1) - p^{r-1} \alpha_2 \\
H^0(\omega_\alpha)(r) & \text{if } \lambda = p^r \omega_\alpha - p^r \alpha \text{ for } \alpha \in \Delta, 0 \leq i \leq r - 2 \\
0 & \text{else.}
\end{cases}$$
Proof. As in the previous proposition, the only difficulty arises in computing the induced module for the non-simple indecomposable modules. From [Jan2, 5.1, 5.2], we have

\[
\begin{align*}
\text{ind}_B^G(M_{B_n}) &\cong H^0(\omega_n), \\
\text{ind}_B^G(M_{C_n}) &\cong H^0(\omega_1), \\
\text{ind}_B^G(M_{F_4}) &\cong H^0(\omega_1), \\
\text{ind}_B^G(M_{G_2}) &\cong H^0(\omega_1).
\end{align*}
\]

\[\square\]

3.2 From Corollary 2.8 and Theorem 2.5, one immediately gets the following. Parts (a) and (b) lower the condition on \( p \) found in [And2, p. 392].

**Corollary.** Suppose \( \lambda \in X(T)_+ \).

(a) \( H^1(G_r, H^0(\lambda)) \neq 0 \) if and only if \( \lambda = p^r\omega - p^i\alpha \) for some \( w \in X(T) \), \( \alpha \in \Delta \), and \( 0 \leq i \leq r - 1 \).

(b) If \( p > 3 \) and \( \lambda = p^r\omega - p^i\alpha \), then \( H^1(G_r, H^0(\lambda)) = H^0(\omega)(r) \).

(c) If \( \lambda \in X_r(T) \), then the weight \( \omega \) is the weight determined by \( p \), \( r \), \( \alpha \), and \( i \) inLemma 2.3.

As done in Section 3.1 for \( p^r \)-restricted weights, one can use Theorems 2.8(A-C) to compute \( H^1(G_r, H^0(\lambda)) \) in terms of induced modules for all \( \lambda \in X(T)_+ \). As most cases simply involve inducing simple \( B \)-modules, for brevity, we do not include these results. However, some of the answers involve a module of the form \( \text{ind}_B^G(M_{X_n} \otimes k_\sigma) \). Specifically, the computation of \( H^1(G_r, H^0(\lambda)) \) for the following dominant weights \( \lambda \) involves inducing the given module:

- \( p = 3 \), type \( G_2 \), \( \lambda = p^r\omega - p^{r-1}\alpha_2 \): \( M_{G_2} \otimes k_{\omega+\omega_1-\omega_2} \)
- \( p = 2 \), type \( B_n \), \( n \geq 3 \), \( \lambda = p^r\omega - p^{r-1}\alpha_{n-1} \): \( M_{B_n} \otimes k_{\omega+\omega_n-\omega_{n-1}} \)
- \( p = 2 \), type \( B_4 \), \( \lambda = p^r\omega - p^{r-1}\alpha_1 \): \( M_{B_4} \otimes k_{\omega-\omega_2} \)
- \( p = 2 \), type \( C_n \), \( n \geq 2 \), \( \lambda = p^r\omega - p^{r-1}\alpha_2 \): \( M_{C_n} \otimes k_{\omega+\omega_n-\omega_2} \)
- \( p = 2 \), type \( F_4 \), \( \lambda = p^r - p^{r-1}\omega_2 \): \( M_{F_4} \otimes k_{\omega+\omega_2} \)
- \( p = 2 \), type \( G_2 \), \( \lambda = p^r\omega - p^{r-1}\alpha_2 \): \( M_{G_2} \otimes k_{\omega+\omega_1-\omega_2} \)

3.3 The following homological algebra fact will allow us to identify a filtration of the modules listed in Section 2.2 by \( H^0(\gamma) \)s. This strategy is based on the proof of [Jan2, Prop. 5.1].

**Lemma.** Let \( M \) be a finite-dimensional \( B \)-module with a filtration (from top to bottom) by \( k_{\sigma_1}, k_{\sigma_2}, \ldots, k_{\sigma_n} \). Assume further that \( R^i \text{ind}_B^G(\sigma_j) = 0 \) for all \( i \geq 1 \) and all \( j \). Then \( R^i \text{ind}_B^G(M) = 0 \) for all \( i \geq 1 \) and \( \text{ind}_B^G(M) \) has a filtration by factors (from top to bottom):

\[ H^0(\sigma_1), H^0(\sigma_2), \ldots, H^0(\sigma_n) \]

where any weights \( \sigma_j \) that are not dominant are omitted.
Proof. We argue by induction on \( n \) and are trivially done if \( n = 1 \). For \( n > 1 \), there is a short exact sequence

\[
0 \rightarrow k_{\sigma_1} \rightarrow M \rightarrow N \rightarrow 0
\]

for some module \( N \). Associated to such a short exact sequence is a long exact sequence

\[
0 \rightarrow \text{ind}_B^G(\sigma_1) \rightarrow \text{ind}_B^G(M) \rightarrow \text{ind}_B^G(N) \rightarrow R^1 \text{ind}_B^G(\sigma_1) \rightarrow R^1 \text{ind}_B^G(M) \rightarrow R^1 \text{ind}_B^G(N) \rightarrow \ldots .
\]

By the hypothesis and induction, \( R^i \text{ind}_B^G(\sigma_1) = 0 \) for all \( i \geq 1 \) and there is a short exact sequence

\[
0 \rightarrow \text{ind}_B^G(\sigma_1) \rightarrow \text{ind}_B^G(M) \rightarrow \text{ind}_B^G(N) \rightarrow 0.
\]

Of course, \( \text{ind}_B^G(\sigma_1) = 0 \) if \( \sigma_1 \) is not dominant. And by induction \( \text{ind}_B^G(N) \) has a filtration by \( H^0(\sigma_2), H^0(\sigma_3), \ldots, H^0(\sigma_n) \) for those \( \sigma_j \) which are dominant. Therefore, we get the desired filtration of \( M \).

\[\square\]

3.4 Given a \( B \)-module \( M \) and a weight \( \sigma \) of \( M \), we note that there are two easy conditions under which \( R^i \text{ind}_B^G(\sigma) = 0 \) for all \( i \geq 1 \). First, this holds if \( \sigma \) is dominant (cf. \cite[I.4.5]). Secondly, if \( \langle \sigma, \alpha \rangle = -1 \) for some simple root \( \alpha \), then \( R^i \text{ind}_B^G(\sigma) = 0 \) for all \( i \geq 0 \) by \cite[I.5.4(a)]. (Of course, for the latter \( \sigma, H^0(\sigma) = 0 \) and will not appear in the answer.) For the modules of interest to us, we will see that all weights satisfy one of these two conditions.

**Proposition** (A). Suppose that \( p = 2 \) or \( 3 \) and the underlying root system of \( G \) is of type \( G_2 \). Let \( \omega \in X(T) \) be such that \( p^r \omega - p^{r-1} \alpha_2 \) lies in \( X(T)_+ \). Then \( \langle \omega, \alpha_1^{(2)} \rangle \geq -1 \) and \( \langle \omega, \alpha_2^{(2)} \rangle \geq 1 \). Furthermore,

(a) if \( \langle \omega, \alpha_1^{(2)} \rangle \geq 0 \), then \( \text{ind}_B^G(M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}) \) has a filtration with factors \( H^0(\omega + 2\omega_1 - \omega_2) \) on the top and \( H^0(\omega) \) on the bottom.

(b) Whereas, if \( \langle \omega, \alpha_1^{(2)} \rangle = -1 \), then \( \text{ind}_B^G(M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}) \cong H^0(\omega + 2\omega_1 - \omega_2) \).

**Proof.** Let \( \omega = m_1 \omega_1 + m_2 \omega_2 \) for integers \( m_1, m_2 \). In order for the weight \( p^r \omega - p^{r-1} \alpha_2 = (p^r m_1 + 3 p^{r-1}) \omega_1 + (p^r m_2 - 2 p^{r-1}) \omega_2 \) to be dominant, it is necessary that \( \langle \omega, \alpha_1^{(2)} \rangle = m_1 \geq -1 \) and \( \langle \omega, \alpha_2^{(2)} \rangle = m_2 \geq 1 \) as claimed. Now, consider the \( B \)-module \( M = M_{G_2} \otimes k_{\omega+\omega_1-\omega_2} \). This is a two-dimensional indecomposable \( B \)-module with a filtration having factors \( k_{\omega_1+\omega_1-\omega_2} = k_{\omega_2+2\omega_1-\omega_2} \) on the top and \( k_{\omega_2-\omega_1+\omega_1-\omega_2} = k_{\omega} \) on the bottom. From the conditions on \( \omega \), the weight \( 2\omega_1 - \omega_2 \) will be dominant but \( \omega \) will be dominant only if \( m_1 \geq 0 \). On the other hand, if \( m_1 = \langle \omega, \alpha_1^{(2)} \rangle = -1 \), we are in the case of \cite[I.5.4(a)]{Jan1} mentioned above and have \( R^i \text{ind}_B^G(\omega) = 0 \) for all \( i \). And so the claims follow from Lemma 3.3. \[\square\]

**Proposition** (B). Suppose \( p = 2 \).

(a) Assume the underlying root system of \( G \) is of type \( B_n \) with \( n \geq 3 \). Let \( \omega \in X(T) \) be such that \( p^r \omega - p^{r-1} \alpha_{n-1} \) lies in \( X(T)_+ \). Then \( \langle \omega, \alpha_j^{(n)} \rangle \geq 0 \) for \( 1 \leq j \leq n - 2 \), \( \langle \omega, \alpha_{n-1}^{(n)} \rangle \geq 1 \), and \( \langle \omega, \alpha_n^{(n)} \rangle \geq -1 \). Further

(i) if \( \langle \omega, \alpha_n^{(n)} \rangle \geq 0 \), then \( \text{ind}_B^G(M_{B_n} \otimes k_{\omega+\omega_n-\omega_{n-1}}) \) has a filtration with factors \( H^0(\omega + 2\omega_n - \omega_{n-1}) \) on the top and \( H^0(\omega) \) on the bottom.

(ii) Whereas, if \( \langle \omega, \alpha_n^{(n)} \rangle = -1 \), then \( \text{ind}_B^G(M_{B_n} \otimes k_{\omega+\omega_n-\omega_{n-1}}) \cong H^0(\omega + 2\omega_n - \omega_{n-1}) \).
(b) Assume that the underlying root system of $G$ is of type $B_4$. Let $\omega \in X(T)$ be such that $p^r \omega - p^{r-1} \alpha_1$ lies in $X(T)_+$. Then $\langle \omega, \alpha_j^\vee \rangle \geq 1$ and $\langle \omega, \alpha_j^\vee \rangle \geq 0$ for $j = 2, 3, 4$. Further

(i) if $\langle \omega, \alpha_j^\vee \rangle \geq 1$, then $\text{ind}_B^G(M_{B_4} \otimes k_{\omega-\omega_1})$ has a filtration with factors $H^0(\omega + \omega_4 - \omega_1)$ on the top and $H^0(\omega + \omega_3 - \omega_1 - \omega_4)$ on the bottom.

(ii) Whereas, if $\langle \omega, \alpha_j^\vee \rangle = 0$, then $\text{ind}_B^G(M_{B_4} \otimes k_{\omega-\omega_1}) \cong H^0(\omega + \omega_1 - \omega_4)$.

(c) Assume that the underlying root system of $G$ is of type $C_n$, $n \geq 2$. Let $\omega \in X(T)$ be such that $p^r \omega - p^{r-1} \alpha_n$ lies in $X(T)_+$. Then $\langle \omega, \alpha_j^\vee \rangle \geq 0$ for $1 \leq j \leq n - 2$, $\langle \omega, \alpha_{n-1}^\vee \rangle \geq -1$, and $\langle \omega, \alpha_n^\vee \rangle \geq 1$. Furthermore, the module $M_{C_n} \otimes k_{\omega+\omega_n-\omega_n}$ has a filtration by factors (from top to bottom):

- $H^0(\omega + \omega_1 + \omega_{n-1} - \omega_n)$,
- $H^0(\omega + \omega_3 - \omega_1 + \omega_{n-3} - \omega_n)$,
- $H^0(\omega + \omega_{n-1} - \omega_{n-2} + \omega_n - \omega_n)$,
- $H^0(\omega + \omega_n - \omega_{n-1} + \omega_n - \omega_n) = H^0(\omega)$

with not necessarily all factors present. Specifically,

- $H^0(\omega + \omega_1 + \omega_{n-1} - \omega_n)$ is always present.
- For $n \geq 3$ and $1 \leq j \leq n - 2$, $H^0(\omega + \omega_{j+1} - \omega_j + \omega_{n-1} - \omega_n)$ is present if $\langle \omega, \alpha_j^\vee \rangle \geq 1$ and not present if $\langle \omega, \alpha_j^\vee \rangle = 0$.
- $H^0(\omega)$ is present if $\langle \omega, \alpha_{n-1}^\vee \rangle \geq 0$ and is not present if $\langle \omega, \alpha_{n-2}^\vee \rangle = -1$.

(d) Assume that the underlying root system of $G$ is of type $F_4$. Let $\omega \in X(T)$ be such that $p^r \omega - p^{r-1} \alpha_2$ lies in $X(T)_+$. Then $\langle \omega, \alpha_j^\vee \rangle \geq 0$, $\langle \omega, \alpha_3^\vee \rangle \geq 1$, $\langle \omega, \alpha_3^\vee \rangle \geq -1$, and $\langle \omega, \alpha_4^\vee \rangle \geq 0$. Further $M_{F_4} \otimes k_{\omega+3-\omega_2}$ has a filtration by factors (from top to bottom):

- $H^0(\omega + \omega_3 + \omega_4 - \omega_2)$,
- $H^0(\omega + 2\omega_3 - \omega_2 - \omega_4)$,
- $H^0(\omega)$

with not necessarily all factors present. Specifically,

- $H^0(\omega + \omega_3 + \omega_4 - \omega_2)$ is always present.
- $H^0(\omega + 2\omega_3 - \omega_2 - \omega_4)$ is present if $\langle \omega, \alpha_3^\vee \rangle \geq 1$ and not present if $\langle \omega, \alpha_3^\vee \rangle = 0$.
- $H^0(\omega)$ is present if $\langle \omega, \alpha_4^\vee \rangle \geq 0$ and is not present if $\langle \omega, \alpha_4^\vee \rangle = -1$.

Proof. As in the preceding proof, we simply compute the conditions on $\omega$ and then apply Lemma 3.3. For part (a), write $\omega = \sum_{i=1}^n m_i \omega_i$. In order for $2^r \omega - 2^{r-1} \alpha_{n-1} = 2^r m_1 \omega_1 + 2^r m_2 \omega_2 + \cdots + 2^r m_{n-3} \omega_{n-3} + (2^r m_{n-2} + 2^{r-1}) \omega_{n-2}$

$+ (2^r m_{n-1} - 2^r) \omega_{n-1} + (2^r m_n + 2^r) \omega_n$

to be dominant, we must have $m_i \geq 0$ for $1 \leq i \leq n - 2$, $m_{n-1} \geq 1$, and $m_n \geq -1$ as claimed. The module $M_{B_n} \otimes k_{\omega+\omega_n-\omega_1}$ has a filtration with $k_{\omega+2\omega_n-\omega_{n-1}}$ on the top and $k_\omega$ on the bottom. From above, the weight $\omega + 2\omega_n - \omega_n$ is necessarily dominant. If $m_n \geq 0$, then $\omega$ is also dominant. Alternatively, we must have $\langle \omega, \alpha_n^\vee \rangle = m_n = -1$. In either case, the filtrations follow from Lemma 3.3.

For part (b), let $\omega = \sum_{i=1}^4 m_i \omega_i$. In order for the weight $2^r \omega - 2^{r-1} \alpha_1 = 2^r \omega - 2^r \omega_1 + 2^{r-1} \omega_2$ to be dominant, we must have $m_1 \geq 1$ and $m_i \geq 0$ for $i = 2, 3, 4$ as claimed. The module $M_{B_4} \otimes k_{\omega-\omega_1}$ has a filtration with $k_{\omega+\omega_1-\omega_1}$ on the top and $k_{\omega+\omega_1-\omega_1}$ on the
bottom. The weight \( \omega + \omega_4 - \omega_1 \) is dominant. However, the weight \( \omega + \omega_3 - \omega_1 - \omega_4 \) will be dominant only if \( m_4 = \langle \omega, \omega \rangle \geq 1 \). Alternatively, when \( m_4 = 0 \), we have \( \langle \omega + \omega_3 - \omega_1 - \omega_4, \omega \rangle = -1 \) which suffices.

For part (c), let \( \omega = \sum_{i=1}^{n} m_i \omega_i \). In order for the weight \( 2^r \omega - 2^{r-1} \alpha_n = 2^r \omega + 2^r \omega_n - 2^2 \omega_n \) to be dominant, we must have \( m_i \geq 0 \) for \( 1 \leq i \leq n - 2 \), \( m_{n-1} \geq -1 \), and \( m_n \geq 1 \) as claimed. The module \( M_{G_n} \otimes k_{\omega_{n-1} - \omega_n} \) has a filtration with factors (from top to bottom):

\[
\begin{align*}
k_{\omega + \omega_{n-1} - \omega_n}, & \\ k_{\omega + \omega_2 - \omega_1 + \omega_{n-1} - \omega_n}, & \\ k_{\omega + \omega_3 - \omega_2 + \omega_2 - \omega_1 + \omega_{n-1} - \omega_n}, & \\ \ldots, & \\ k_{\omega + \omega_{n-2} - \omega_{n-1} + \omega_{n-1} - \omega_n}, & \\ k_{\omega + \omega_n - \omega_{n-1} + \omega_{n-1} - \omega_n} = k_\omega.
\end{align*}
\]

The weight \( \omega + \omega_{n-1} - \omega_n \) is always dominant. On the other hand, the remaining weights need not be. Indeed, for \( 1 \leq j \leq n - 2 \), the weight \( \sigma_j = \omega + \omega_{j+1} - \omega_j + \omega_{n-1} - \omega_n \) is dominant if and only if \( m_j = \langle \omega, \alpha_j \rangle \geq 1 \). Moreover, if \( m_j = 0 \), then \( \langle \sigma_j, \alpha_j \rangle = -1 \) as needed. Similarly, the weight \( \omega \) is dominant if and only if \( \langle \omega, \alpha_{n-1} \rangle \geq 0 \). Alternatively, we have \( \langle \omega, \alpha_{n-1} \rangle = -1 \).

For part (d), let \( \omega = \sum_{j=1}^{4} m_j \omega_j \). In order for the weight \( 2^r \omega - 2^{r-1} \alpha_2 = 2^r \omega + 2^{r-1} \omega_1 - 2^2 \omega_2 + 2^2 \omega_3 \) to be dominant, we must have \( m_1 \geq 0 \), \( m_2 \geq 1 \), \( m_3 \geq -1 \), and \( m_4 \geq 0 \) as claimed. The module \( M_{E_4} \otimes k_{\omega + \omega_3 - \omega_2} \) has a filtration by factors (from top to bottom):

\[
\begin{align*}
k_{\omega + \omega_3 - \omega_2}, & \\ k_{\omega + \omega_4 - \omega_3 - \omega_2}, & \\ k_{\omega + \omega_2 - \omega_2 + \omega_2 - \omega_1 - \omega_3}, & \\ k_\omega.
\end{align*}
\]

The weight \( \omega + \omega_3 - \omega_2 \) is dominant only if \( m_1 \geq 1 \) whereas if \( m_4 = 0 \), then \( \langle \sigma, \alpha_4 \rangle = -1 \). Lastly, if \( m_3 \geq 0 \), then \( \omega \) is dominant. Alternatively, we have \( \langle \omega, \alpha_4 \rangle = -1 \).

### 3.5

In this section, we apply our results to provide a complete determination of when \( H^1(G_r, H^0(\omega_\beta)) \) (or equivalently \( H^1(B_r, \omega_\beta) \)) is non-zero for a fundamental dominant weight \( \omega_\beta \). For a given fundamental weight \( \omega_\beta \), by Corollary 3.2, \( H^1(G_r, H^0(\omega_\beta)) \neq 0 \) if and only if \( \omega_\beta = p^r \omega - p^i \alpha \) for some simple root \( \alpha \) and \( 0 \leq i \leq r - 1 \) where \( \omega \) is the corresponding weight from Lemma 2.3. Suppose \( \omega_\beta = p^r \omega - p^i \alpha \). Then

\[
1 = \langle \omega_\beta, \beta^\vee \rangle = p^r \langle \omega, \beta^\vee \rangle - p^i \langle \alpha, \beta^\vee \rangle.
\]

Observe that the right hand side is divisible by \( p \) unless \( i = 0 \). This reduces the assumption to \( \omega_\beta = p^r \omega - \alpha \). By checking Lemma 2.3, notice that we always have \( \langle \omega, \alpha^\vee \rangle = 1 \). So

\[
\langle \omega_\beta, \alpha^\vee \rangle = p^r \langle \omega, \alpha^\vee \rangle - \langle \alpha, \alpha^\vee \rangle = p^r - 2.
\]

As the left hand side equals 0 or 1, we must have \( r = 1 \) and \( p = 2 \) or 3. Therefore, we have the following result.

**Proposition**  (A). If \( r > 1 \) or \( p > 3 \), then

\[
H^1(B_r, \omega_\beta) = 0 = H^1(G_r, H^0(\omega_\beta))
\]

for all fundamental dominant weights \( \omega_\beta \).

The reader will have already observed that there are fundamental weights \( \omega_\beta \) of the form \( p \omega - \alpha \) when \( p = 2 \) or 3. We proceed to precisely identify these.

**Case 1**: Assume first that the weight \( \omega \) equals \( \omega_\alpha \). That is, we lie in the generic case. So, we are assuming \( \omega_\beta = p \omega_\alpha - \alpha \). Then we have

\[
\langle \omega_\beta, \alpha^\vee \rangle = p \langle \omega_\alpha, \alpha^\vee \rangle - \langle \alpha, \alpha^\vee \rangle = p - 2.
\]
The left hand side equals 1 if \( \beta = \alpha \) and 0 if \( \beta \neq \alpha \).

**Case 1.1:** Suppose \( \beta = \alpha \). Then the left hand side of (3.5.1) is 1 and we must have \( p = 3 \). In type \( A_1 \), one does indeed have \( \omega_\alpha = 3 \omega_\alpha - \alpha \). On the other hand, in any other type, we cannot have \( \omega_\alpha = 3 \omega_\alpha - \alpha \) for there exists a simple root \( \sigma \neq \alpha \) with \( \langle \omega_\alpha, \sigma^\vee \rangle \neq 0 \) while \( \langle \omega_\alpha, \sigma^\vee \rangle = 0 \).

**Case 1.2:** Suppose \( \beta \neq \alpha \). Then the left hand side of (3.5.1) is zero and we must have \( p = 2 \). So \( \omega_\beta = 2 \omega_\alpha - \alpha \). In other words \( \alpha = 2 \omega_\alpha - \omega_\beta \). One can readily identify all simple roots which have this form:

- **Type** \( A_n \), \( n \geq 2 \): \( \alpha_1 = 2 \omega_1 - \omega_2 \) so \( \omega_2 = 2 \omega_1 - \alpha_1 \); \( \alpha_n = 2 \omega_n - \omega_{n-1} \) so \( \omega_{n-1} = 2 \omega_n - \alpha_n \)
- **Type** \( B_n \), \( n \geq 3 \): \( \alpha_1 = 2 \omega_1 - \omega_2 \) so \( \omega_2 = 2 \omega_1 - \alpha_1 \); \( \alpha_n = 2 \omega_n - \omega_{n-1} \) so \( \omega_{n-1} = 2 \omega_n - \alpha_n \)
- **Type** \( C_n \), \( n \geq 2 \): \( \alpha_1 = 2 \omega_1 - \omega_2 \) so \( \omega_2 = 2 \omega_1 - \alpha_1 \)
- **Type** \( D_n \), \( n \geq 4 \): \( \alpha_1 = 2 \omega_1 - \omega_2 \) so \( \omega_2 = 2 \omega_1 - \alpha_1 \); \( \alpha_{n-1} = 2 \omega_{n-1} - \omega_n - 2 \) and \( \alpha_n = 2 \omega_n - \omega_{n-2} \) so \( \omega_{n-2} = 2 \omega_{n-1} - \alpha_{n-1} = 2 \omega_n - \alpha_n \)
- **Type** \( E_n \), \( n = 6, 7, 8 \): \( \alpha_1 = 2 \omega_1 - 3 \omega_3 \) so \( \omega_3 = 2 \omega_1 - \alpha_1 \); \( \alpha_2 = 2 \omega_2 - \omega_4 \) so \( \omega_4 = 2 \omega_2 - \alpha_2 \); \( \alpha_n = 2 \omega_n - \omega_{n-1} \) so \( \omega_{n-1} = 2 \omega_n - \alpha_n \)
- **Type** \( F_4 \): \( \alpha_1 = 2 \omega_1 - \omega_2 \) so \( \omega_2 = 2 \omega_1 - \alpha_1 \); \( \alpha_4 = 2 \omega_4 - \omega_3 \) so \( \omega_3 = 2 \omega_4 - \alpha_4 \)
- **Type** \( G_2 \): \( \alpha_1 = 2 \omega_1 - \omega_2 \) so \( \omega_2 = 2 \omega_1 - \alpha_1 \)

**Case 2:** Suppose that \( \omega_\beta = p \omega - \alpha \) and \( \omega \) from Lemma 2.3 has non-generic form. For the list of “exceptional” weights \( \omega \), we simply check whether \( p \omega - \alpha \) is a fundamental weight:

- **Type** \( B_n \), \( n \geq 3 \), \( p = 2 \): \( 2(\omega_{n-1} - \omega_n) - \alpha_{n-1} = \omega_{n-2} \)
- **Type** \( C_n \), \( n \geq 2 \), \( p = 2 \): \( 2(\omega_n - \omega_{n-1}) - \alpha_n = 0 \) (not fundamental)
- **Type** \( F_4 \), \( p = 2 \): \( 2(\omega_2 - \omega_3) - \alpha_2 = \omega_1 \)
- **Type** \( G_2 \), \( p = 2 \): \( 2(\omega_2 - \omega_1) - \alpha_2 = \omega_1 \)
- **Type** \( G_2 \), \( p = 3 \): \( 3(\omega_2 - \omega_1) - \alpha_2 = \omega_2 \)

Thus we have all the fundamental dominant weights with non-vanishing cohomology.

**Proposition (B).** Let \( p = 2 \) or \( 3 \) and \( \omega_\beta \) is a fundamental dominant weight. Then

\[
H^1(B_1, \omega_\beta) = 0 = H^1(G_1, H^0(\omega_\beta))
\]

**EXCEPT for the following weights:**

(a) Assume \( p = 2 \).

- **Type** \( A_2 \): \( H^1(G_1, \omega_1) \cong H^0(\omega_2)^{(1)}, H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \)
- **Type** \( A_3 \): \( H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_3)^{(1)} \)
- **Type** \( A_n \), \( n \geq 4 \): \( H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \)
- **Type** \( B_3 \): \( H^1(G_1, \omega_1) \cong H^0(\omega_3)^{(1)}, H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_3)^{(1)} \)
- **Type** \( B_4 \): \( H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_4)^{(1)} \), \( H^1(G_1, \omega_3) \cong H^0(\omega_1)^{(1)} \)
- **Type** \( B_n \), \( n \geq 5 \): \( H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)}, H^1(G_1, \omega_{n-2}) \cong H^0(\omega_n)^{(1)} \), \( H^1(G_1, \omega_{n-1}) \cong H^0(\omega_{n-1})^{(1)} \)
- **Type** \( C_n \), \( n \geq 2 \): \( H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \)
- **Type** \( D_4 \): \( H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)} \oplus H^0(\omega_3)^{(1)} \oplus H^0(\omega_4)^{(1)} \)
• Type $D_n$, $n \geq 5$: $H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)}$, $H^1(G_1, \omega_{n-2}) \cong H^0(\omega_{n-1})^{(1)}$ ⊕ $H^0(\omega_n)^{(1)}$,
• Type $E_n$, $n = 6, 7, 8$: $H^1(G_1, \omega_3) \cong H^0(\omega_1)^{(1)}$, $H^1(G_1, \omega_4) \cong H^0(\omega_2)^{(1)}$, $H^1(G_1, \omega_{n-1}) \cong H^0(\omega_n)^{(1)}$
• Type $F_4$: $H^1(G_1, \omega_1) \cong H^0(\omega_4)^{(1)}$, $H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)}$, $H^1(G_1, \omega_3) \cong H^0(\omega_4)^{(1)}$
• Type $G_2$: $H^1(G_1, \omega_1) \cong H^0(\omega_1)^{(1)}$, $H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)}$

(b) Assume $p = 3$.
• Type $A_1$: $H^1(G_1, \omega_1) \cong H^0(\omega_1)^{(1)}$
• Type $G_2$: $H^1(G_1, \omega_2) \cong H^0(\omega_1)^{(1)}$

4 Simple $G_r$-modules

4.1 The computation of the cohomology groups $H^1(G_r, L(\lambda))$ for $\lambda \in X(T)_+$ is not as straightforward. Following [Jan2, 4.2, 4.3], consider the short exact sequence

$$0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0$$

and the long exact sequence in cohomology

$$0 \to (L(\lambda))^{G_r} \to H^0(\lambda)^{G_r} \to (H^0(\lambda)/L(\lambda))^{G_r}$$

$$\to H^1(G_r, L(\lambda)) \to H^1(G_r, H^0(\lambda)) \to H^1(G_r, H^0(\lambda)/L(\lambda)) \to \cdots.$$ 

If $\lambda, \mu \in X_r(T)$ then $\text{Hom}_{G_r}(L(\mu), H^0(\lambda))$ is zero if $\lambda \neq \mu$ and $k$ otherwise. It follows that if $\lambda \in X_r(T)$ and $\lambda \neq 0$ then $H^0(\lambda)^{G_r} = 0$, and otherwise it is $k$. Consequently, for any $\lambda \in X_r(T)$, there is an exact sequence

$$0 \to (H^0(\lambda)/L(\lambda))^{G_r} \to H^1(G_r, L(\lambda)) \to H^1(G_r, H^0(\lambda)) \to H^1(G_r, H^0(\lambda)/L(\lambda)).$$

The following is now immediate.

**Proposition.** If $\lambda \in X_r(T)$ and $H^1(G_r, H^0(\lambda)) = 0$, then $H^1(G_r, L(\lambda)) \cong (H^0(\lambda)/L(\lambda))^{G_r}$.

4.2 By combining the previous results with Corollary 3.2, we obtain an identification of $H^1(G_r, L(\lambda))$ for most weights (up to an understanding of the module $H^0(\lambda)$).

**Corollary.** Suppose $\lambda \in X_r(T)$. If $\lambda \neq p^i \omega - p^i \alpha$ for $\alpha \in \Delta$, and $0 \leq i \leq r - 1$ where $\omega$ is determined by $p$, $r$, $\alpha$, and $i$ from Lemma 2.3, then

$$H^1(G_r, L(\lambda)) \cong (H^0(\lambda)/L(\lambda))^{G_r}.$$ 

We remark that it is still an open problem in terms of what happens to $H^1(G_r, L(\lambda))$ when $H^1(G_r, H^0(\lambda)) \neq 0$.

5 Ext$^1$-formula between simple modules
5.1 Let \( k[G] \) be the coordinate algebra of \( G \). For each \( \nu \in X(T)_+ \), let \( I(\nu) \) be the injective hull of the simple \( G \)-module \( L(\nu) \). As a \( G \)-module

\[
(5.1.1) \quad k[G] \cong \bigoplus_{\nu \in X(T)_+} I(\nu)^{\dim_k L(\nu)}.
\]

Here \( I(\nu)^{\dim_k L(\nu)} = \bigoplus_{i=1}^m I(\nu) \) where \( m = \dim_k L(\nu) \). Therefore,

\[
\text{ind}_{G, r}^G k \cong k[G/G_r] \cong k[G]^{(r)} \cong \bigoplus_{\nu \in X(T)_+} (I(\nu)^{(r)})^{\dim_k L(\nu)}.
\]

Now let \( \lambda, \mu \in X_r(T) \). By Frobenius reciprocity and the preceding isomorphism, we have for \( m \geq 0, \)

\[
\text{Ext}^m_{G, r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in X(T)_+} \text{Ext}^m_{G, r}(L(\lambda), L(\mu) \otimes \text{ind}_{G, r}^G k)
\]

\[
\cong \bigoplus_{\nu \in X(T)_+} \text{Ext}^m_{G, r}(L(\lambda), L(\mu) \otimes I(\nu)^{(r)})^{\dim_k L(\nu)}
\]

\[
\cong \bigoplus_{\nu \in X(T)_+} \text{Ext}^m_{G, r}(L(\lambda), L(\mu) \otimes I(\nu)^{(r)}) \otimes L(\nu)^{(r)}.
\]

Note that the last isomorphism is in general only an isomorphism of vector spaces.

5.2 Let \( \pi_s = \{ \nu \in X(T)_+ : \langle \nu, \alpha_0^\vee \rangle < s \} \) and \( C_s \) be the full subcategory of all \( G \)-modules whose composition factors \( L(\nu) \) have highest weights lying in \( \pi_s \). For \( L(\nu) \) in \( C_s \), let \( I_s(\nu) \) be the injective hull of \( L(\nu) \) in the category \( C_s \). We remark that \( C_s \) is a highest weight category as defined in [CPS]. The category \( C_s \) is equivalent to the module category for a finite-dimensional quasi-hereditary algebra. Moreover, the injective module \( I_s(\nu) \) is a finite-dimensional \( G \)-module.

**Proposition.** Let \( \lambda, \mu \in X_r(T) \) and \( p \) be an arbitrary prime. Then \( \text{Ext}^m_{G, r}(L(\lambda), L(\mu))^{(-r)} \) is a \( G \)-module in \( C_{s(m)} \) where

\[
s(m) = \begin{cases} 
1 & \text{if } m = 0, \\
h & \text{if } m = 1, \\
(m-1)(2h-3) + 3 & \text{if } m > 1.
\end{cases}
\]

**Proof.** For \( m = 0 \) the statement is clear. The proof for the case \( m = 1 \) is inspired from the ideas in [And2, Lem. 2.3]. Set \( \lambda^s = -w_0\lambda \). Consider the short exact sequence

\[
0 \to L(\lambda^s) \otimes L(\mu) \to H^0(\lambda^s) \otimes H^0(\mu) \to N \to 0.
\]

Since \( \lambda, \mu \in X_r(T) \), we have \( \text{Hom}_{G, r}(L(\lambda), L(\mu)) \cong \text{Hom}_{G, r}(V(\lambda), H^0(\mu)) \). Therefore, from the short exact sequence above and the associated long exact sequence in cohomology, we obtain the following exact sequence

\[
0 \to N^{G, r} \to \text{Ext}^1_{G, r}(L(\lambda), L(\mu)) \to H^1(G_r, H^0(\lambda^s) \otimes H^0(\mu)).
\]

We first show that if \( \nu \) is a weight of \( \text{Ext}^1_{G, r}(L(\lambda), L(\mu))^{(-r)} \) then

\[
(5.2.1) \quad p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \lambda^s + \mu, \alpha_0^\vee \rangle + 3p^{r-1}.
\]
All the weights of $N$ are less than $\lambda^* + \mu$ so (5.2.1) is true for the weights of $(N^{G_r})^{(-r)}$. Consequently it suffices to prove that (5.2.1) holds for all weights of $H^1(G_r, H^0(\lambda^*) \otimes H^0(\mu))^{(-r)}$. Let $L(\sigma) = L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ be a composition factor of $H^0(\lambda^*) \otimes H^0(\mu)$ where $\sigma_0 \in X_r(T)$. Then

$$H^1(G_r, L(\sigma)) \cong H^1(G_r, L(\sigma_0)) \otimes L(\sigma_1)^{(r)}.$$ 

Consider the short exact sequence

$$0 \rightarrow L(\sigma_0) \rightarrow H^0(\sigma_0) \rightarrow Q \rightarrow 0.$$

As above, this exact sequence induces an exact sequence of the form

$$0 \rightarrow Q^{G_r} \rightarrow H^1(G_r, L(\sigma_0)) \rightarrow H^1(G_r, H^0(\sigma_0)).$$

Let $\mu$ be a weight of $H^1(G_r, H^0(\sigma_0))^{(-r)}$. Then one can directly verify using Theorem 3.1(A-C) that

$$p^r \langle \mu, \alpha_0^\vee \rangle \leq \langle \sigma_0, \alpha_0^\vee \rangle + 3p^{r-1}. \tag{5.2.2}$$

Indeed, in the generic case, $H^1(G_r, H^0(\sigma_0))$ is non-zero only when $\sigma_0 = p^r \omega_\alpha - p^i \alpha$ for a simple root $\alpha$ with $i < r$ in which case $H^1(G_r, H^0(\sigma_0))^{(-r)} \cong H^0(\omega_\alpha)$. Since $p^r \omega_\alpha = \sigma_0 + p^r \alpha$, (5.2.2) readily holds. For $p > 3$ only the generic case occurs. For $p = 2, 3$, verification of the non-generic cases is left to the interested reader. Since every weight of $Q^{G_r}$ is less than $p^r \sigma_0$, it follows that (5.2.1) holds for all weights of $H^1(G_r, L(\sigma_0))^{(-r)}$.

If $\nu$ is a weight of $H^1(G_r, H^0(\lambda^* \otimes H^0(\mu))^{(-r)}$, then $p^r \nu \leq p^r \mu + p^r \sigma_1$ where $L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ is a composition factor of $H^0(\lambda^*) \otimes H^0(\mu)$ and $\mu$ is a weight of $H^1(G_r, L^0(\sigma_0))^{(-r)}$. Using (5.2.2), it follows that

$$p^r \langle \nu, \alpha_0^\vee \rangle \leq p^r \langle \mu, \alpha_0^\vee \rangle + p^r \langle \sigma_1, \alpha_0^\vee \rangle \leq \langle \sigma_0, \alpha_0^\vee \rangle + 3p^{r-1} + p^r \langle \sigma_1, \alpha_0^\vee \rangle \leq \langle \lambda^* + \mu, \alpha_0^\vee \rangle + 3p^{r-1}$$

This verifies (5.2.1).

Consider the short exact sequence of $G$-modules

$$0 \rightarrow L(\mu) \rightarrow St_r \otimes L(\hat{\mu}) \rightarrow R \rightarrow 0$$

where $\hat{\mu} = (p^r - 1) \rho - \mu^*$. By applying the long exact sequence in cohomology along with the projectivity of $St_r$ over $G_r$, we see that for $m \geq 2$

$$\text{Ext}^m_{G_r}(L(\lambda), L(\mu)) \cong \text{Ext}^{m-1}_{G_r}(L(\lambda), R). \tag{5.2.3}$$

In fact, equation (5.2.3) also holds for $m = 1$. Since $\text{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \text{Ext}^1_{G_r}(L(\mu), L(\lambda))$, we may assume without loss of generality that $\lambda \nless \mu$. Then $\text{Hom}_{G_r}(L(\lambda), St_r \otimes L(\hat{\mu}))$ is trivial unless $\lambda = \mu$ in which case it is $k$. Hence the first map in the long exact sequence $\text{Hom}_{G_r}(L(\lambda), L(\mu)) \rightarrow \text{Hom}_{G_r}(L(\lambda), St_r \otimes L(\hat{\mu}))$ is an isomorphism and (5.2.3) also holds for $m = 1$.

The highest weight of $R$ is less than $2(p^r - 1) \rho - \mu^*$. Thus, any weight $\nu$ of $\text{Ext}^1_{G_r}(L(\lambda), L(\mu))^{(-r)} \cong \text{Hom}_{G_r}(L(\lambda), R)^{(-r)}$ must satisfy

$$p^r \nu \leq 2(p^r - 1) \rho - \mu^* - \lambda.$$
Applying the inner product with $\alpha_0^\vee$ we get
\begin{equation}
\langle p^r \nu, \alpha_0^\vee \rangle \leq 2(p^r - 1)(h - 1) - \langle \mu^* + \lambda, \alpha_0^\vee \rangle.
\end{equation}
Observe that for any weight $\xi$, $\langle \xi^*, \alpha_0^\vee \rangle = \langle \xi, \alpha_0^\vee \rangle$. Thus combining (5.2.1) and (5.2.4) together yields
\[ 2p^r \langle \nu, \alpha_0^\vee \rangle \leq 2(p^r - 1)(h - 1) + 3p^{r-1}. \]
This implies that
\[ \langle \nu, \alpha_0^\vee \rangle \leq (h - 1) + \frac{3}{2p} < h. \]
as required.

Next, we argue the case $m = 2$. By (5.2.3) we have
\[ \text{Ext}^2_{G_r}(L(\lambda), L(\mu))^{(-r)} \cong \text{Ext}^1_{G_r}(L(\lambda), R)^{(-r)}. \]
Let $L(\sigma) \cong L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ be a composition factor of $R$. Then $p^r \sigma_1 \leq 2(p^r - 1)p - \mu^* - \sigma_0$ and one obtains
\[ p^r (\sigma_1, \alpha_0^\vee) \leq 2(p^r - 1)(h - 1) - \langle \mu^* + \sigma_0, \alpha_0^\vee \rangle. \]
It follows from (5.2.1) that any weight of $\nu$ of $\text{Ext}^1_{G_r}(L(\lambda), L(\sigma_0))^{(-r)}$ satisfies
\[ p^r (\nu, \alpha_0^\vee) \leq \langle \lambda^* + \sigma_0, \alpha_0^\vee \rangle + 3p^{r-1}. \]
Thus any weight $\nu$ of $\text{Ext}^1_{G_r}(L(\lambda), L(\sigma))^{(-r)} \cong \text{Ext}^1_{G_r}(L(\lambda), L(\sigma_0))^{(-r)} \otimes L(\sigma_1)$ satisfies
\begin{equation}
\langle \nu, \alpha_0^\vee \rangle \leq 3p^{r-1} + 2(p^r - 1)(h - 1)
\end{equation}
which implies
\[ \langle \nu, \alpha_0^\vee \rangle < 2h. \]

Finally, we apply (5.2.3) to the case $m > 2$. The highest weight of $R$ is less than $2(p^r - 1)p$. Thus, any weight $\sigma = \sigma_0 + p^r \sigma_1$ of $R$ must satisfy $\langle \sigma, \alpha_0^\vee \rangle \leq 2(p^r - 1)(h - 1)$, which implies that $\langle \sigma_1, \alpha_0^\vee \rangle \leq 2(h - 1) - 1 = 2h - 3$.

Let $L(\sigma)$ be a composition factor of $R$ and $m > 1$. Then, as $G$-modules,
\[ \text{Ext}^{m-1}_{G_r}(L(\lambda), L(\sigma)) \cong \text{Ext}^{m-1}_{G_r}(L(\lambda), L(\sigma_0)) \otimes L(\sigma_1)^{(r)}. \]
Inductively we conclude that any weight $\nu$ of $\text{Ext}^1_{G_r}(L(\lambda), L(\mu))^{(-r)} \cong \text{Ext}^{m-1}_{G_r}(L(\lambda), R)^{(-r)}$ must satisfy
\[ \langle \nu, \alpha_0^\vee \rangle < s(m - 1) + 2h - 3. \]

\[ \square \]

5.3 The previous proposition can be refined in the case when $p > h$ by using the work in [KLT].
Proposition. Let $\lambda, \mu \in X_r(T)$ and $p > h$. Then $\text{Ext}^{m}_{G_r}(L(\lambda), L(\mu))^{(r)}$ is a $G$-module in $C_{s(m)}$ where

$$s(m) = \begin{cases} 
1 & \text{if } m = 0 \\
2 & \text{if } m = 1 \\
2(h - 1) + mk & \text{if } m > 1,
\end{cases}$$

where $\kappa = 3/2$ if $G$ is of type $G_2$ and $\kappa = 1$ otherwise.

Proof. We will first prove the following.

Step 1: Let $\sigma \in X(T)_+$, $p > h$ and $m \geq 0$, then any weight $p\nu$ of $H^m(G_1, H^0(\sigma))$ satisfies

$$p\langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p(m + 1)\kappa,$$

where $\kappa = 3/2$ if $G$ is of type $G_2$ and $\kappa = 1$ otherwise.

We use [KLT, Thm. 8] which says that

$$H^m(G_1, H^0(w \cdot 0 + p\lambda)) \cong \begin{cases} \text{ind}_G^B(S^{(m - l(w))/2}u^* \otimes \lambda)^{(1)} & \text{if } m = l(w) \text{ mod } 2 \\
0 & \text{else.}\end{cases}$$

Here $u = \text{Lie } U$. The weights of the $i$th symmetric powers $S^i u^*$ are just sums of $i$ positive roots. Therefore any weight $\gamma$ of $S^i u^*$ satisfies

$$\langle \gamma, \alpha_0^\vee \rangle \leq i \cdot \max \{ \langle \beta, \alpha_0^\vee \rangle \mid \beta \in \Phi^+ \} \leq 2ik.$$ 

Now let $\sigma = w \cdot 0 + p\lambda$ where $w \in W$ and $\lambda \in X(T)_+$. If $l(w) = 0$ then $\sigma = p\lambda$ and it follows that any weight $p\nu$ of $H^m(G_1, H^0(\sigma))$ satisfies $p\langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p\kappa$. If $l(w) > 0$ then $p\lambda \leq \sigma + 2p\rho$ and it follows that any weight $\nu$ of $H^m(G_1, H^0(\sigma))$ satisfies $p\langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + 2(h - 1) + p(m - 1)\kappa < \langle \sigma, \alpha_0^\vee \rangle + p(m + 1)\kappa$. This completes the proof of Step 1.

Consider the short exact sequence

$$0 \rightarrow L(\sigma) \rightarrow H^0(\sigma) \rightarrow Q \rightarrow 0,$$

which induces exact sequences of the form

$$H^{m-1}(G_1, Q) \rightarrow H^m(G_1, L(\sigma)) \rightarrow H^m(G_1, H^0(\sigma)).$$

Any highest weight of $Q$ is strictly less than $\sigma$ and by using induction on $m$ we get the following.

Step 2: For $\sigma \in X(T)_+$, $p > h$ and $m \geq 0$, any weight $p\nu$ of $H^m(G_1, L(\sigma))$ satisfies

$$p\langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p(m + 1)\kappa.$$
Next we will use induction on \( r \) to show

**Step 3:** For \( \sigma \in X(T)_+, p > h \) and \( m \geq 1 \), any weight \( p^r \nu \) of \( H^m(G_r, L(\sigma)) \) satisfies
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa.
\]

For \( r = 1 \) the statement was proved in Step 2. Assume that \( r > 1 \). We will use the Lyndon-Hochschild-Serre spectral sequence
\[
E_2^{i,j} = H^i(G_r/G_{r-1}, H^j(G_{r-1}, L(\sigma))) \Rightarrow H^{i+j}(G_r, L(\sigma)).
\]

The differentials in the spectral sequence are \( G \)-equivariant. Moreover, the cohomology is a subquotient of \( E_3^{i,j} \) where \( i + j = m \). Therefore, any composition factor of \( H^m(G_r, L(\sigma)) \) must be a composition factor of some \( H^i(G_r/G_{r-1}, H^j(G_{r-1}, L(\sigma))) \) with \( i + j = m \).

It is sufficient to show that any weight \( p^r \nu \) of \( H^i(G_r/G_{r-1}, H^j(G_{r-1}, L(\sigma))) \) with \( i + j = m \) satisfies
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa.
\]

We first discuss the case \( j = 0 \). \( \text{Hom}_{G_{r-1}}(k, L(\sigma)) \) has a composition series with simple modules \( L(\gamma)^{(r-1)} \). Clearly the highest weights \( \gamma \) of each factor satisfy
\[
(5.3.1) \quad p^{r-1} \langle \gamma, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle.
\]

The composition factors of \( H^m(G_r/G_{r-1}, \text{Hom}_{G_{r-1}}(k, L(\sigma))) \) are subquotients of some \( H^m(G_r/G_{r-1}, L(\gamma)^{(r-1)}) \). By Step 2 any weight \( p^\nu \) of \( H^m(G_r/G_{r-1}, L(\gamma)^{(r-1)}) \) satisfies
\[
p^\nu \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^\nu (m + 1) \kappa.
\]

Multiplying the inequality with \( p^{r-1} \) and using (5.3.1) yields
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq p^{r-1} \langle \gamma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa \leq \langle \sigma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa.
\]

We conclude that any weight \( p^r \nu \) of \( H^r(G_r/G_{r-1}, \text{Hom}_{G_{r-1}}(k, L(\sigma))) \) satisfies
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa.
\]

Next assume that \( j > 0 \). By the induction hypothesis we may assume that any weight \( p^{r-1} \gamma \) of \( H^i(G_{r-1}, L(\sigma)) \) satisfies
\[
(5.3.2) \quad p^{r-1} \langle \gamma, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^{r-1} (j + 1) \kappa.
\]

\( H^i(G_{r-1}, L(\sigma)) \) has a composition series with simple factors \( L(\gamma)^{(r-1)} \). The composition factors of \( H^i(G_r/G_{r-1}, H^j(G_{r-1}, L(\sigma))) \) are subquotients of some \( H^i(G_r/G_{r-1}, L(\gamma)^{(r-1)}) \).

By Step 2 any weight \( p^\nu \) of \( H^i(G_r/G_{r-1}, L(\gamma)^{(r-1)}) \) satisfies
\[
p^\nu \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^\nu (i + 1) \kappa.
\]

Multiplying the inequality with \( p^{r-1} \) and using (5.3.2) yields
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq p^{r-1} \langle \gamma, \alpha_0^\vee \rangle + p^r (i + 1) \kappa \leq \langle \sigma, \alpha_0^\vee \rangle + p^{r-1} (j + 1) \kappa + p^r (i + 1) \kappa.
\]

Now \( p^{r-1} (j + 1) \leq p^{r-1} 2j \leq p^r j \). Therefore
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^r (j + i + 1) \kappa = \langle \sigma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa.
\]

We conclude that any weight \( p^r \nu \) of \( H^i(G_r/G_{r-1}, H^j(G_{r-1}, L(\sigma))) \) satisfies
\[
p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^r (m + 1) \kappa.
\]
Finally, we now prove our claim concerning the size of the weights in $\text{Ext}^m_{G_r}(L(\lambda), L(\mu))$. Notice that the statement for $m \leq 1$ follows from Proposition 5.2. We assume that $m > 1$. One has the following sequence of isomorphisms

$$H^m(G_r, L(\lambda^*) \otimes L(\mu)) \cong \text{Ext}^m_{G_r}(L(\lambda), L(\mu)) \cong \text{Ext}^m_{G_r}(L(\mu), L(\lambda)) \cong H^m(G_r, L(\mu^*) \otimes L(\lambda)).$$

Without a loss of generality we may assume that $\langle \lambda, \alpha_0^\vee \rangle \leq \langle \mu, \alpha_0^\vee \rangle$.

Consider the short exact sequence of $G$-modules

$$0 \to L(\lambda^*) \otimes L(\mu) \to L(\lambda^*) \otimes \text{St}_r \otimes L(\hat{\mu}) \to R \to 0$$

where $\hat{\mu} = (p^r - 1)p - \mu^*$. By applying the long exact sequence in cohomology along with the projectivity of $\text{St}_r$ over $G_r$, we see that

$$\text{Ext}^m_{G_r}(L(\mu), L(\mu)) \cong H^m(G_r, L(\lambda^*) \otimes L(\mu)) \cong H^{m-1}(G_r, R).$$

Notice that any composition factor $L(\sigma)$ of $R$ satisfies $\langle \sigma, \alpha_0^\vee \rangle \leq 2(p^r - 1)(h - 1)$. It follows that any highest weight $p^r \nu$ of $H^{m-1}(G_r, R)$ satisfies

$$p^r \langle \nu, \alpha_0^\vee \rangle \leq \langle \sigma, \alpha_0^\vee \rangle + p^r m \kappa \leq 2(p^r - 1)(h - 1) + p^r m \kappa.$$

Dividing by $p^r$ yields

$$\langle \nu, \alpha_0^\vee \rangle \leq 2(h - 1) + m \kappa.$$

The assertion follows.

$\square$

5.4 We can now prove the following formula which relates extensions between simple modules in $G_r$ with certain $G$-modules. Note that the isomorphism is in general only an isomorphism of vector spaces not necessarily of $G$-modules.

**Theorem.** Let $\lambda, \mu \in X_r(T)$, and $p$ be an arbitrary prime. Then for $m \geq 0$,

$$\text{Ext}^m_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_s(m)} \text{Ext}_{G_r}^m(L(\lambda), I_{s(m)}(\nu) \otimes L(\mu)) \otimes L(\nu)^{(r)},$$

where

$$s(m) = \begin{cases} 1 & \text{if } m = 0, \\ h & \text{if } m = 1, \\ 2(h - 1) + m & \text{if } m > 1, p > h \text{ and } G \text{ is not of type } G_2, \\ 2(h - 1) + 3/2m & \text{if } m > 1, p > h \text{ and } G \text{ is of type } G_2, \\ (m - 1)(2h - 3) + 3 & \text{otherwise.} \end{cases}$$

**Proof.** Let $N$ be a $G$-module. First consider the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{i,j} = \text{Ext}_{G/G_r}^i(k, \text{Ext}_{G_r}^j(L(\lambda), L(\mu) \otimes N^{(r)})) \Rightarrow \text{Ext}_{G_r}^{i+j}(L(\lambda), L(\mu) \otimes N^{(r)}).$$

For $i > 0$ and $0 \leq j \leq m$, let us look at

$$E_2^{i,j} = \text{Ext}_{G/G_r}^i(k, \text{Ext}_{G_r}^j(L(\lambda), L(\mu)) \otimes N^{(r)}).$$
When \( N = I(\nu) \) we have \( E^{i,j}_2 = 0 \) for \( i > 0 \) because \( N^{(r)} \) is an injective \( G/G_r \)-module. On the other hand, if \( N = I_{s(m)}(\nu) \) then \( E^{i,j}_2 = 0 \) for \( i > 0 \) and \( 0 \leq j \leq m \) because \( M^{(-r)} \equiv (\text{Ext}^2_{C_r}(L(\lambda), L(\mu))^{(r)})^{(-r)} \) is a \( G \)-module in \( C_{s(m)} \) (by Propositions 5.2 and 5.3) and \( N \) is injective in \( C_{s(m)} \). It follows that if \( N = I(\nu) \) or \( I_{s(m)}(\nu) \) then \( E^{0,m}_2 \cong E^m \). Consequently, if \( \nu \notin \pi_{s(m)} \), then

\[
\text{Ext}^m_G(L(\lambda), L(\mu) \otimes I(\nu)^{(r)}) \cong \text{Hom}_G(M^{(-r)}, I(\nu)) = 0
\]

since the \( G \)-socle of \( I(\nu) \) is \( L(\nu) \) and \( M^{(-r)} \) lies in \( C_{s(m)} \). On the other hand, for \( \nu \in \pi_{s(m)} \), we have

\[
\text{Ext}^m_G(L(\lambda), L(\mu) \otimes I(\nu)^{(r)}) \cong \text{Hom}_{G/G_r}(M, I(\nu)^{(r)})
\]

\[
\cong \text{Hom}_{G/G_r}(M, I_{s(m)}(\nu)^{(r)})
\]

\[
\cong \text{Ext}^m_G(L(\lambda), L(\mu) \otimes I_{s(m)}(\nu)^{(r)}).
\]

Note that the isomorphism between the first and second lines follows by Propositions 5.2 and 5.3. The statement of the theorem now follows by the isomorphisms given in Section 5.1.

\( \Box \)

### 5.5  By specializing to the case when \( m = 1 \), we have the following corollary.

**Corollary (A).** Let \( \lambda, \mu \in X_r(T) \) and \( p \) be an arbitrary prime. Then

\[
\text{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_{s_1}} \text{Ext}^1_G(L(\lambda), I_h(\nu)^{(r)} \otimes L(\mu) \otimes L(\nu)^{(r)}).
\]

Corollary (A) takes on even a nicer formulation when \( p \geq 2(h - 1) \).

**Corollary (B).** Let \( \lambda, \mu \in X_r(T) \) and \( p \geq 2(h - 1) \). Then \( \text{Ext}^1_{G_r}(L(\lambda), L(\mu)) \) is a semisimple \( G \)-module and

\[
\text{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_{s_1}} \text{Ext}^1_G(L(\lambda), L(\nu)^{(r)} \otimes L(\mu) \otimes L(\nu)^{(r)}).
\]

For higher cohomologies we get the following.

**Corollary (C).** Let \( \lambda, \mu \in X_r(T) \), \( m \geq 2 \), and \( p \geq 3(h - 1) + mk - 1 \), where \( k = 3/2 \) if \( G \) is of type \( G_2 \) and \( k = 1 \) otherwise. Then \( \text{Ext}^m_{G_r}(L(\lambda), L(\mu)) \) is a semisimple \( G \)-module and

\[
\text{Ext}^m_{G_r}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \pi_{s(m)}} \text{Ext}^m_G(L(\lambda), L(\nu)^{(r)} \otimes L(\mu) \otimes L(\nu)^{(r)}),
\]

where \( s(m) = 2(h - 1) + mk \).

The following proves both Corollaries (B) and (C).

**Proof.** For \( m = 1 \), set \( s(m) = h \). Let \( \overline{C}_Z = \{ \lambda \in X(T) : 0 \leq \langle \lambda + \rho, \alpha_0^\vee \rangle \leq p \} \) denote the closure of the “bottom alcove” under the action of the affine Weyl group. By the Strong Linkage Principle, if \( \sigma_1, \sigma_2 \in \overline{C}_Z \), then \( \text{Ext}^1_G(L(\sigma_1), L(\sigma_2)) = 0 \). Let \( \nu \) be such that \( \langle \nu, \alpha_0^\vee \rangle < s(m) \). If \( p \geq s(m) + (h - 2) \), then

\[
\langle \nu + \rho, \alpha_0^\vee \rangle < s(m) + h - 1 \leq p + 1,
\]

then
which implies that $\nu \in \overline{C}_Z$. Consequently, the category $C_{s(m)}$ is semisimple and $I_{s(m)}(\nu) = L(\nu)$ for all $\nu \in C_{s(m)}$. The result now holds by Theorem 5.4. □

As noted, the isomorphism in Theorem 5.4 is only an isomorphism of vector spaces. However, one obtains the composition factors of $\text{Ext}^m_{Gr}(L(\lambda), L(\mu))$ via

$$[\text{Ext}^m_{C_{s(m)}}(L(\lambda), L(\mu)) : L(\nu)^{(r)}]_G = \dim \text{Ext}^m_{Gr}(L(\lambda), I_{s(m)}(\nu)^{(r)} \otimes L(\mu)).$$

For $p \geq s(m) + (h - 2)$ the category $C_{s(m)}$ is semisimple. Therefore the isomorphism in Corollaries (B) and (C) is actually an isomorphism of $G$-modules.

The preceding result improves results by the authors in [BNP1] and sharpens results by Andersen [And1] who proved this for $m = 1$ and $p \geq 3(h - 1)$.

**References**


Department of Mathematics, Statistics and Computer Science, University of Wisconsin-Stout, Menomonie, WI 54751, USA

E-mail address: bendelc@uwstout.edu

Department of Mathematics, University of Georgia, Athens, GA 30602, USA

E-mail address: nakano@math.uga.edu

Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA

E-mail address: pillen@jaguar1.usouthal.edu