SELF-EXTENSIONS FOR THE FINITE SYMPLECTIC GROUPS VIA RELATED EXTENSIONS FOR THE FROBENIUS KERNELS

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Abstract. For large primes it was shown in [BNP1, BNP3] that a finite group of Lie Type does not admit self-extensions, i.e. non-trivial extensions of a simple module with itself, unless the group is one of the symplectic groups $Sp_{2n}(F_p)$, $n \geq 1$. In this paper it is shown that self-extensions indeed exist for these groups in odd characteristic. In addition, it is shown that these self-extensions are closely related to certain extensions between simple modules for the Frobenius kernel that are exceptional for type $C_n$.

1 Introduction

1.1 Let $G$ be a connected simply connected almost simple algebraic group defined and split over the field $F_p$ with $p$ elements, and $k$ be the algebraic closure of $F_p$. Let $G(F_q)$ be the finite Chevalley group consisting of $F_q$-rational points of $G$ where $q = p^r$ for a positive integer $r$. Moreover, let $G_r$ be the $r$th Frobenius kernel.

In 1984 H.H. Andersen [And] showed that Frobenius kernels do not admit self-extensions of simple modules, i.e. non-trivial extensions of a simple module with itself, unless the underlying root system is of type $C_n$ ($n \geq 1$) and the prime is two. In [BNP2] the following generalization of Andersen’s result was given: Given a pair of $p$-restricted weights $\lambda$ and $\mu$ that are “close”, i.e. $\langle \lambda - \mu, \alpha^\vee \rangle < p/3$ for any root $\alpha$, then the $G$ and $G_1$-extensions between two simple modules affording these highest weights coincide, unless $G$ is of type $C_n$.

The purpose of this paper is to show that such pairs of simple $G$-modules whose highest weights are “close” but whose $G$ and $G_1$-extensions differ indeed exist in type $C_n$. Moreover, it is shown that the constructions of these extensions give rise to self-extensions for the finite group $Sp_{2n}(F_p)$ for odd $p$ and arbitrary rank.

It is well-known that self-extensions of simple modules for the finite Chevalley groups $SL_2(F_p)$ exist for arbitrary primes $p$. In [Hum1] J.E. Humphreys constructed examples of self-extensions between simple modules for the symplectic groups $Sp_4(F_p)$ with $p$ odd. In the same paper Humphreys conjectured that the root systems of type $C_n$ might be exceptional for the existence of self-extensions. For large primes it was was proved in [BNP1, BNP3] that self-extensions can only exist for type $C_n$, thus confirming Humphreys’ conjecture.

Here we construct self-extensions for the finite symplectic group of arbitrary rank. One family of self-extensions described in the paper (Proposition 4.3) has been discovered independently by Tiep and Zalesskiı in [TZ] using quite different methods. The ideas used...
in our set up involve extensions between certain pairs of simple $G$-modules, one restricted and one non-restricted, whose restrictions to the finite group $G(\mathbb{F}_p)$ contain the desired self-extensions as submodules and whose restriction to the Frobenius kernel produce the aforementioned exceptional extensions. Such pairs of weights were also used in the construction in [Hum1].

Unfortunately, the methods used do not allow for a complete classification of self-extensions for finite Chevalley groups. But at least for large primes one obtains additional examples besides the ones in [TZ]. In [BNP1] necessary conditions on the highest weights of potential candidates for self-extensions for the groups $Sp_{2n}(\mathbb{F}_p)$ were given. The highest weight $\lambda$ of a simple module $L(\lambda)$ has to be “close” to the hyperplane defined via $\langle x + \rho, \alpha_n^\vee \rangle = \frac{p}{2}$. Here $\alpha_n$ denotes the unique long simple root. One might conjecture that any pair of $p$-restricted weights $\lambda$ and $\lambda - \frac{1}{2} \alpha_n$ with $\langle \lambda, \alpha_n^\vee \rangle = \frac{p-1}{2}$ should yield a pair of self-extensions for $Sp_{2n}(\mathbb{F}_p)$.

1.2 Notation: $G$ will always denote a connected simply connected almost simple algebraic group that is defined and split over the field $\mathbb{F}_p$ with $p$ elements. $k$ denotes the algebraic closure of $\mathbb{F}_p$ and $G_1$ is the first Frobenius kernel of $G$. The conventions in the paper will follow the ones used in [Jan1]. Let $T$ be a maximal torus in $G$ and $\Phi$ the associated root system. The positive roots are denoted by $\Phi^+$ and the negative roots by $\Phi^-$. Let $B$ be a Borel subgroup containing $T$ and corresponding to the negative roots. $X(T)$ denotes the weight lattice, $X(T)_+$ the dominant weights, and $X_1(T)$ the $p$-restricted weights. For a weight $\gamma \in X(T)_+, H^0(\gamma), V(\gamma)$, and $L(\gamma)$ denote the induced module, the Weyl module, and the simple module, respectively.

Starting with Section 2.2, we will assume in addition that $G$ is of type $C_n$. We follow [Bou, p.254] and denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i < n$, the short simple roots, while $\alpha_n = 2\epsilon_n$ is the unique long simple root. The fundamental weights are $\omega_i = \sum_{k=1}^i \epsilon_k$, with $\omega_1$ being the unique minuscule weight. For convenience we will frequently switch between the $\epsilon_i$, $\omega_i$, and $\alpha_i$ as a basis. The highest short root is $\alpha_0 = \epsilon_1 + \epsilon_2$ and the longest element of the Weyl group $W$ is $-1$. The simple modules are therefore self-dual and $H^0(\gamma)$ and $V(\gamma)$ are dual to each other.

2 $G_1$-Extensions between Weyl modules and induced modules

It is well-known that $\text{Ext}^1_{G_1}(V(\mu), H^0(\lambda)) = 0$ for any pair of weights $\lambda, \mu$. In [And] Andersen proved that $\text{Ext}^1_{G_1}(V(\lambda), H^0(\lambda)) = 0$ unless $G$ is of type $C_n$ and $p = 2$. In [BNP2, 5.2, Prop.(a)] a generalization of Andersen’s result for odd primes was found. It was shown that $\text{Ext}^1_{G_1}(V(\mu), H^0(\lambda))$ vanishes for a pair of restricted weights $\lambda$ and $\mu$ that are “close” (i.e. $\langle \mu - \lambda, \alpha_i^\vee \rangle < p/3$, for any root $\alpha$), unless $G$ is of type $C_n$ and the weights are reflections of each other across the hyperplane $\langle \gamma + \rho, \alpha_i^\vee \rangle = \frac{p}{2}$. In this section we will show that extensions for such “close” pairs of weights indeed exist for type $C_n$.

2.1 The following Lemma is well-known. It is included for the benefit of the reader. We will make repeated use of it in later arguments.
Lemma. Let $i > 0$ be an integer, $\alpha$ be a simple root, $\mu \in X(T)_+$, and $\gamma \in X(T)$ with $-p \leq \langle \gamma, \alpha^\vee \rangle \leq -1$ then

$$\text{Ext}^i_B(V(\mu), \gamma) \cong \begin{cases} 0 & \text{if } \langle \gamma, \alpha^\vee \rangle = -1 \\ \text{Ext}^{i-1}(V(\mu), s_\alpha \cdot \gamma) & \text{else.} \end{cases}$$

Proof. We apply the spectral sequence [Jan1, I.4.5]

$$\text{Ext}^i_{P(a)}(V(\mu), R^j \text{ind}^{(a)}_B \gamma) \Rightarrow \text{Ext}^{i+j}_B(V(\mu), \gamma).$$

If $\langle \gamma, \alpha^\vee \rangle = -1$ then $R^j \text{ind}^{(a)}_B \gamma = 0$ for all $j \geq 0$ [Jan1, II.5.2(b)], which forces $\text{Ext}^i_B(V(\mu), \gamma) = 0$ for all $i > 0$.

Otherwise it follows from [Jan1, II.5.2(d)] that

$$\text{Ext}^i_B(V(\mu), \gamma) \cong \text{Ext}^{i-1}(V(\mu), R^1 \text{ind}^{(a)}_B \gamma).$$

Now $-p \leq \langle \gamma, \alpha^\vee \rangle \leq -2$ implies that $0 \leq \langle \gamma, \alpha^\vee \rangle \leq p - 2$. It follows from [Jan1, II.5.3(b)] that $R^j \text{ind}^{(a)}_B \gamma \cong \text{ind}^{(a)}_B(s_\alpha \cdot \gamma)$. Finally, [Jan1, II.4.7(1)] yields

$$\text{Ext}^i_B(V(\mu), \gamma) \cong \text{Ext}^{i-1}_B(V(\mu), \text{ind}^{(a)}_B(s_\alpha \cdot \gamma)) \cong \text{Ext}^{i-1}_B(V(\mu), s_\alpha \cdot \gamma).$$

□

2.2 The G-module $L(\omega_1) \cong H^0(\omega_1)$ is multiplicity free with dimension $2n$. The weight spaces are expressed most conveniently in the form $\pm \epsilon_i$ with $i = 1, \ldots, n$.

Lemma. Let $G$ be of type $C_n$, $p$ odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_i^\vee \rangle = (p - 1)/2$ then

(a) for any weight $\sigma$ of $L(\omega_1)$,

$$\text{Ext}^1_B(V(\lambda - \frac{1}{2} \alpha_n), \lambda + p\sigma) \cong \begin{cases} k & \text{if } \sigma = -\epsilon_n \\ 0 & \text{else,} \end{cases}$$

(b) for $i < n$

$$\text{Ext}^2_B(V(\lambda - \frac{1}{2} \alpha_n), \lambda - p\epsilon_i) = 0.$$

Proof. (a) If $\sigma = \epsilon_1 = \omega_1$ the weight $\lambda + p\omega_1$ is dominant and the claim follows from [Jan1, II.4.13].

For $i > 1$ and $\sigma = \epsilon_i$ it follows from the fact that $\lambda$ is restricted that $-p \leq \langle \lambda + p\epsilon_i, \alpha_i^\vee \rangle \leq -1$. We apply Lemma 2.1 to obtain

$$\text{Ext}^1_B(V(\lambda - \frac{1}{2} \alpha_n), \lambda + p\epsilon_i) \cong \begin{cases} 0 & \text{if } \langle \lambda, \alpha_i^\vee \rangle = p - 1 \\ \text{Hom}_B(V(\lambda - \frac{1}{2} \alpha_n), s_{\alpha_{i-1}} \cdot (\lambda + p\epsilon_i)) & \text{else.} \end{cases}$$

Now $\text{Hom}_B(V(\lambda - \frac{1}{2} \alpha_n), s_{\alpha_{i-1}} \cdot (\lambda + p\epsilon_i)) = 0$ unless

$$\lambda - \frac{1}{2} \alpha_n = s_{\alpha_{i-1}} \cdot (\lambda + p\epsilon_i) = \lambda + p\epsilon_i + (p - 1 - \langle \lambda, \alpha_{i-1}^\vee \rangle)\alpha_{i-1} \text{ and } \langle \lambda, \alpha_{i-1}^\vee \rangle \leq p - 2.$$
The later implies that
\[ 0 = p\epsilon_i + \frac{1}{2} \alpha_n + (p - 1 - \langle \lambda, \alpha_{i-1}^\vee \rangle)\alpha_{i-1} = (p - 1 - \langle \lambda, \alpha_{i-1}^\vee \rangle)\alpha_{i-1} + p\left(\frac{1}{2} \sum_{k=i}^{n-1} \alpha_k\right) + \frac{p + 1}{2} \alpha_n \geq \alpha_{i-1}, \]
which is absurd.

Similarly, we can argue for \( \sigma = -\epsilon_i \) that \(-p \leq \langle \lambda - p\epsilon_i, \alpha_i^\vee \rangle \leq -1 \) and
\[
\text{Ext}_B^1(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, \lambda - p\epsilon_i) \cong \begin{cases} 
0 & \text{if } \langle \lambda, \alpha_i^\vee \rangle = p - 1 \\
\text{Hom}_B(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, s_{\alpha_i} \cdot (\lambda - p\epsilon_i)) & \text{else.}
\end{cases}
\]
Clearly,
\[
\text{Hom}_B(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, s_{\alpha_i} \cdot (\lambda - p\epsilon_i)) \cong \begin{cases} 
k & \text{if } \lambda - \frac{1}{2} \alpha_n = s_{\alpha_i} \cdot (\lambda - p\epsilon_i) \\
0 & \text{else.}
\end{cases}
\]
The equation \( \lambda - \frac{1}{2} \alpha_n = s_{\alpha_i} \cdot (\lambda - p\epsilon_i) = \lambda - p\epsilon_i + (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_i \) holds, if and only if
\[
p\epsilon_i - \frac{1}{2} \alpha_n = p\left(\frac{1}{2} \sum_{k=i}^{n-1} \alpha_k\right) + \frac{p - 1}{2} \alpha_n = (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_i,
\]
which is true only in the case \( i = n \).

(b) We apply Lemma 2.1 once to obtain
\[
\text{Ext}_B^2(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, \lambda - p\epsilon_i) \cong \begin{cases} 
0 & \text{if } \langle \lambda, \alpha_i^\vee \rangle = p - 1 \\
\text{Ext}_B^1(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, s_{\alpha_i} \cdot (\lambda - p\epsilon_i)) & \text{else.}
\end{cases}
\]
Now \( \langle s_{\alpha_i} \cdot (\lambda - p\epsilon_i), \alpha_j^\vee \rangle = \langle \lambda - p\epsilon_i + (p - 1 - \langle \lambda, \alpha_j^\vee \rangle)\alpha_i, \alpha_j^\vee \rangle \geq 0 \) for all \( j \neq i + 1 \), while
\[
\langle s_{\alpha_i} \cdot (\lambda - p\epsilon_i), \alpha_i^\vee \rangle = \langle \lambda - p\epsilon_i + (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_i, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle + (\lambda, \alpha_{i+1}^\vee) + 1 - p.
\]
If \( p - 1 \leq \langle \lambda, \alpha_i^\vee \rangle + (\lambda, \alpha_{i+1}^\vee) \) then \( s_{\alpha_i} \cdot (\lambda - p\epsilon_i) \) is dominant and \( \text{Ext}_B^1(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, s_{\alpha_i} \cdot (\lambda - p\epsilon_i)) \) vanishes [Jan1, II.4.13].

Otherwise \( 1 - p \leq \langle s_{\alpha_i} \cdot (\lambda - p\epsilon_i), \alpha_i^\vee \rangle \leq -1 \). In this case Lemma 2.1 yields
\[
\text{Ext}_B^2(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, \lambda - p\epsilon_i) \cong \text{Hom}_B(V\langle \lambda - \frac{1}{2} \alpha_n \rangle, s_{\alpha_i + 1} s_{\alpha_i} \cdot (\lambda - p\epsilon_i)) \cong \begin{cases} 
k & \text{if } \lambda - \frac{1}{2} \alpha_n = s_{\alpha_i + 1} s_{\alpha_i} \cdot (\lambda - p\epsilon_i) \\
0 & \text{else.}
\end{cases}
\]
If \( \lambda - \frac{1}{2} \alpha_n = s_{\alpha_i + 1} s_{\alpha_i} \cdot (\lambda - p\epsilon_i) \) then \( s_{\alpha_i} \cdot (\lambda - \frac{1}{2} \alpha_n) = s_{\alpha_i} \cdot (\lambda - p\epsilon_i) \). This implies that
\[
\lambda - \frac{1}{2} \alpha_n - (\langle \lambda - \frac{1}{2} \alpha_n, \alpha_{i+1}^\vee \rangle + 1)\alpha_{i+1} = \lambda - p\epsilon_i + (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_i,
\]
which forces
\[
(p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_i + (\langle \lambda - \frac{1}{2} \alpha_n, \alpha_{i+1}^\vee \rangle + 1)\alpha_{i+1} = p\epsilon_i - \frac{1}{2} \alpha_n = p\left(\frac{1}{2} \sum_{k=i}^{n-1} \alpha_k\right) + \frac{p - 1}{2} \alpha_n.
\]
Comparing the coefficients for \( \alpha_i \) shows that this is impossible.

\[\square\]

2.3 The following Proposition is the first step in the construction of self-extensions for the finite symplectic groups.
Proposition. Let $G$ be of type $C_n$, $p$ odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n \rangle = (p-1)/2$, then
\[ \text{Ext}^1_G(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)) \approx k. \]

Proof. The $G$-module $L(\omega_1)$ is multiplicity free with weight spaces $\pm \epsilon_i$, $i = 1, \ldots, n$. Let $S$ denote the $B$-submodule consisting of the weight spaces $-\epsilon_1, \ldots, -\epsilon_n$, $T$ be the $B$-submodule consisting of the weight spaces $-\epsilon_1, \ldots, -\epsilon_{n-1}$, and $Q$ be the $B$-quotient $L(\omega_1)/S$. The weight spaces of $Q$ are $\epsilon_1, \ldots, \epsilon_n$.

The short exact sequence $0 \rightarrow S \rightarrow L(\omega_1) \rightarrow Q \rightarrow 0$ yields the exact sequence
\[ \text{Hom}_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes Q^{(1)}) \rightarrow \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes S^{(1)}) \]
\[ \rightarrow \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes (L(\omega_1)) \rightarrow \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes Q^{(1)}). \]

The first term in this sequence is zero because $\lambda - \frac{1}{2}\alpha_n$ is not a weight of $\lambda \otimes Q^{(1)}$. The last term vanishes by Lemma 2.2(a). Therefore
\[ \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes (L(\omega_1)) \approx \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes S^{(1)}). \]

Next we use the short exact sequence $0 \rightarrow T \rightarrow S \rightarrow -\epsilon_n \rightarrow 0$ to obtain
\[ \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes T^{(1)}) \rightarrow \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes S^{(1)}) \]
\[ \rightarrow \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes \epsilon_n) \rightarrow \text{Ext}^2_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes T^{(1)}) \]

Clearly the first term is zero and the last term vanishes by Lemma 2.2(b). One concludes from Lemma 2.2(a) that
\[ \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda \otimes (L(\omega_1)) \approx \text{Ext}^1_B(V(\lambda - \frac{1}{2}\alpha_n), \lambda - p\epsilon_n) \approx k. \]

\[ \square \]

Corollary. Let $G$ be of type $C_n$, $p$ odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n \rangle = (p-1)/2$, then
\[ \text{Ext}^1_{G_1}(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda)) \neq 0. \]

Proof. Consider the Lyndon-Hochschild-Serre spectral sequence
\[ E_2^{ij} = \text{Ext}^i_{G/G_1}(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda), L(\omega_1)) \Rightarrow \text{Ext}^{i+j}_{G_1}(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)). \]

Since $\text{Hom}_{G_1}(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda)) = 0$, we have $E_2^{1,0} = E_2^{2,0} = 0$ and from the corresponding five-term sequence $E_2^1 \approx E_2^{0,1}$. \[ \square \]

3 $G_1$-extensions for pairs of simple modules whose highest weights are “close”

Throughout this section $G$ is of type $C_n$. We establish the existence of certain non-trivial $G$-extensions between simple $G$-modules, one with restricted highest weight and the other non-restricted. These will yield $G_1$-extensions for pairs of simple modules whose highest weights are “close” as well as self-extensions for the finite symplectic groups.
3.1 First needed are some vanishing results for extensions.

**Lemma.** Let $G$ be of type $C_n$, $p$ odd, $\mu, \gamma \in X(T)_+$ and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$. Moreover, assume that $\langle \lambda, \alpha_n^\vee \rangle < (p-1)/2$ for $i < n$. Then the following hold

(a) If $\mu \leq \lambda - \frac{1}{2}\alpha_n$, then $\Ext_G^1(V(\mu), H^0(\lambda) \otimes L(\omega_1)(1)) = 0$ unless $\mu = \lambda - \frac{1}{2}\alpha_n$.

(b) If $\gamma \leq \lambda$, then $\Ext_G^1(V(\gamma), H^0(\lambda - \frac{1}{2}\alpha_n) \otimes L(\omega_1)(1)) = 0$ unless $\gamma = \lambda$.

**Proof.** (a) Assume that $\Ext_G^1(V(\mu), H^0(\lambda) \otimes L(\omega_1)(1)) \cong \Ext_B^1(V(\mu), \lambda \otimes L(\omega_1)(1)) \neq 0$. Then there exists a weight $\sigma$ of $L(\omega_1)$ such that $\Ext_B^1(V(\mu), \lambda + p\sigma) \neq 0$. It follows from [Jan1, II.4.13] that $\sigma \neq \epsilon_i$. If $\sigma = \epsilon_i$ with $i > 1$ we apply Lemma 2.1 and conclude as in the proof of Lemma 2.2 that

$$\mu = s_{\alpha_i} \cdot (\lambda + p\epsilon_i) = \lambda + p\epsilon_i + (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_{i-1}$$

$$= \lambda - \frac{1}{2}\alpha_n + p \sum_{k=i}^{n-1} \alpha_k + \frac{p+1}{2}\alpha_n + (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_{i-1} > \lambda - \frac{1}{2}\alpha_n.$$

This contradicts the assumption that $\mu \leq \lambda - \frac{1}{2}\alpha_n$.

If $\sigma = -\epsilon_i$, Lemma 2.1 and the argument in the proof of Lemma 2.2 show that

$$\mu = s_{\alpha_i} \cdot (\lambda - p\epsilon_i) = \lambda - p\epsilon_i + (p - 1 - \langle \lambda, \alpha_i^\vee \rangle)\alpha_i.$$ 

If $i < n$ the inner product with $\alpha_{i+1}$ yields $\langle \mu, \alpha_{i+1}^\vee \rangle = \langle \lambda, \alpha_{i+1}^\vee \rangle + \langle \lambda, \gamma^\vee \rangle - (p - 1) < 0$.

This contradicts the fact that $\mu$ is dominant.

That leaves the case $i = n$. Here one obtains $\mu = \lambda - p\epsilon_n + (p - 1 - \frac{p-1}{2})\alpha_n = \lambda - \epsilon_n = \lambda - \frac{1}{2}\alpha_n$, as claimed.

Part (b) is left to the reader. \hfill $\Box$

3.2 The following Proposition is believed to hold even without conditions (i) and (ii).

**Proposition.** Let $G$ be of type $C_n$, $p$ odd, and $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_n^\vee \rangle = (p-1)/2$. Moreover, assume that

(i) $\langle \lambda, \alpha_i^\vee \rangle < (p-1)/2$ for $i < n$,

(ii) $H^0(\lambda)$ and $H^0(\lambda - \frac{1}{2}\alpha_n)$ have only $p$-restricted composition factors,

then $\Ext_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda) \otimes L(\omega_1)(1)) \cong k$.

**Proof.** By Proposition 2.3 $\Ext_G^1(V(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)(1)) \cong k$. Therefore there exists a weight $\mu \leq \lambda - \frac{1}{2}\alpha_n$ such that $\Ext_G^1(L(\mu), H^0(\lambda) \otimes L(\omega_1)(1)) \neq 0$. Define the $G$-module $S$ via the short exact sequence $0 \rightarrow S \rightarrow V(\mu) \rightarrow L(\mu) \rightarrow 0$ and obtain the exact sequence $\Hom(S, H^0(\lambda) \otimes L(\omega_1)(1)) \rightarrow \Ext_G^1(L(\mu), H^0(\lambda) \otimes L(\omega_1)(1)) \rightarrow \Ext_G^1(V(\mu), H^0(\lambda) \otimes L(\omega_1)(1))$. All the weights in $S$ are less than $\lambda + p\omega_1$ and therefore

$$0 \neq \Ext_G^1(L(\mu), H^0(\lambda) \otimes L(\omega_1)(1)) \rightarrow \Ext_G^1(V(\mu), H^0(\lambda) \otimes L(\omega_1)(1)).$$

Now Lemma 3.1 forces $\mu = \lambda - \frac{1}{2}\alpha_n$ and $\Ext_G^1(L(\lambda - \frac{1}{2}\alpha_n), H^0(\lambda) \otimes L(\omega_1)(1)) \neq 0$.

From condition (ii) we conclude that there exists a $p$-restricted weight $\gamma \leq \lambda$ such that $\Ext_G^1(L(\lambda - \frac{1}{2}\alpha_n), L(\gamma) \otimes L(\omega_1)(1)) \neq 0$. We make use of the short exact sequence $0 \rightarrow T \rightarrow
V(\lambda - \frac{1}{2} \alpha_n) \to L(\lambda - \frac{1}{2} \alpha_n) \to 0 \text{ to obtain the exact sequence}

\text{Hom}_G(T, L(\gamma) \otimes L(\omega_1)) \to \text{Ext}_G^1(L(\lambda - \frac{1}{2} \alpha_n), L(\gamma) \otimes L(\omega_1))

\to \text{Ext}_G^1(V(\lambda - \frac{1}{2} \alpha_n), L(\gamma) \otimes L(\omega_1)).

The first term vanishes because \gamma and the composition factors of T are restricted. Therefore

\text{Ext}_G^1(L(\gamma), H^0(\lambda - \frac{1}{2} \alpha_n) \otimes L(\omega_1)) \cong \text{Ext}_G^1(V(\lambda - \frac{1}{2} \alpha_n), L(\gamma) \otimes L(\omega_1)) \neq 0.

Repeating the earlier argument yields Ext^1_G(V(\gamma), H^0(\lambda-\frac{1}{2} \alpha_n) \otimes L(\omega_1)) \neq 0. Now Lemma 3.1 forces \gamma = \lambda and the claim follows. \hfill \Box

In [BNP2, 5.3, Thm.(A)] it was shown that Ext^1_G(L(\mu), L(\lambda)) \cong Ext^1_G(L(\mu), L(\lambda)) for any pair of restricted weights \lambda and \mu that are "close" (i.e. \langle \mu - \lambda, \alpha \rangle < p/3, for any root \alpha), unless G is of type C_n and the weights are reflections of each other across the hyperplane \langle \gamma + \rho, \alpha \rangle = \frac{p}{2}. For type C_n the Corollary gives examples of such "close" pairs of simple modules where Ext^1_G and Ext^1_{G_1} differ. Notice that \lambda = \frac{p-1}{2} \omega_n satisfies conditions (i) and (ii).

Corollary. Let G be of type C_n, p odd, and \lambda \in X_1(T) with \langle \lambda, \alpha_n^\vee \rangle = (p-1)/2. Moreover, assume that

(i) \langle \lambda, \alpha_i^\vee \rangle < (p-1)/2 for i < n,
(ii) \text{H}^0(\lambda) \text{ and } \text{H}^0(\lambda - \frac{1}{2} \alpha_n) \text{ have only p-restricted composition factors},

then Ext^1_G(L(\lambda - \frac{1}{2} \alpha_n), L(\lambda)) \neq Ext^1_G(L(\lambda - \frac{1}{2} \alpha_n), L(\lambda)) = 0.

Proof. Obviously, Ext^1_G(L(\lambda - \frac{1}{2} \alpha_n), L(\lambda)) = 0. On the other hand, [And, Thm 5.2(a)] yields an isomorphism

\text{Hom}_{G/G_1}(\text{Ext}^1_{G_1}(L(\lambda - \frac{1}{2} \alpha_n), L(\lambda)), L(\omega_1)) \cong \text{Ext}^1_G(L(\lambda - \frac{1}{2} \alpha_n), L(\lambda) \otimes L(\omega_1)).

The assertion follows from the previous Proposition. \hfill \Box

It was shown in [BNP2, 5.4, Cor.(A)] that root systems other than type C_n (n \geq 1) do not admit non-trivial G-extensions between simple modules whose highest weights are inside the same alcove. Notice that for p > 2n the weights \frac{p-1}{2} \omega_n - \frac{1}{2} \alpha_n and \frac{p-1}{2} \omega_n are contained in the same alcove. Again the root systems of type C_n are exceptional for the existence of such G_1-extensions.

4 Self-extensions for Sp_{2n}(F_p)

Let G be of type C_n. By Proposition 3.2 there exist non-trivial G-extensions E,

0 \to L(\frac{p-1}{2} \omega_n) \to E \to L(\frac{p-1}{2} \omega_n - \frac{1}{2} \alpha_n) \otimes L(\omega_1) \to 0.

We will show that E contains a self-extension of L(\frac{p-1}{2} \omega_n) as a G(F_p)-submodule.

4.1 In order to describe the G(F_p)-structure of such non-trivial G-extensions, we embed them in tensor products with the Steinberg module.
Lemma. Let $G$ be of type $C_n$, $p$ odd, and $\lambda \in X_1(T)$. Assume that there exist non-trivial $G$-extensions

$$0 \to L(\lambda) \to E \to L(\lambda + \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)} \to 0,$$

then $E$ is a $G$-submodule of $St \otimes L((p-1)\rho - \lambda)$.

Proof. The short exact sequence defining $E$ yields the exact sequence

$$\Hom_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}, St \otimes L((p-1)\rho - \lambda))$$

$$\to \Hom_G(E, St \otimes L((p-1)\rho - \lambda))$$

$$\to \Hom_G(L(\lambda), St \otimes L((p-1)\rho - \lambda))$$

$$\to \Ext^1_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}, St \otimes L((p-1)\rho - \lambda)).$$

We will show that the first and last term in this sequence vanish. Recall that $St \otimes L(\omega_1)^{(1)}$ is a simple $G$-module. Comparing highest weights shows that

$$\Hom_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}, St \otimes L((p-1)\rho - \lambda)) = 0.$$

Similar weight comparisons show that

$$\Hom_G(L(\lambda + \frac{1}{2}\alpha_n), St \otimes L((p-1)\rho - \lambda)) \cong \Hom_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L((p-1)\rho - \lambda), St)$$

$$\cong \Hom_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L((p-1)\rho - \lambda), St).$$

Clearly, $\Hom_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L((p-1)\rho - \lambda), St) = 0$, because not both the weights $(p-1)\rho - \lambda + \frac{1}{2}\alpha_n$ and $\lambda$ are contained in the root lattice.

The Steinberg module is injective as a $G_1$-module. It follows from the five-term-exact sequence of the Lyndon-Hochschild-Serre spectral sequence and the above that

$$\Ext^1_G(L(\lambda + \frac{1}{2}\alpha_n) \otimes L(\omega_1)^{(1)}, St \otimes L((p-1)\rho - \lambda)) = 0.$$

From [Jan1, II.10.15] one obtains $\Hom_G(L(\lambda), St \otimes L((p-1)\rho - \lambda)) \cong k$. This forces $\Hom_G(E, St \otimes L((p-1)\rho - \lambda + \frac{1}{2}\alpha_n)) \cong k$. The assumption on $E$ to be a non-trivial extension makes the homomorphism an embedding. \hfill \Box

4.2 Here we show that $L(\frac{p-1}{2}\omega_n)$ appears in the $G(\mathbb{F}_p)$-socle of $L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes L(\omega_1)$.

Lemma. Let $G$ be of type $C_n$, $\alpha_n$ be the unique long simple root and $\omega_n$ the corresponding fundamental weight. If $p$ odd, then $\Hom_G(L(\frac{p-1}{2}\omega_n), L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes L(\omega_1)) \cong k$.

Proof. Recall that $H^0(\omega_1) \cong L(\omega_1)$. The tensor product $H^0(\frac{p-1}{2}\omega_n) \otimes H^0(\omega_1)$ has a good filtration [Jan1, II.4.19]. Direct computation shows that the factors of the filtration are $H^0(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n)$ and $H^0(\frac{p-1}{2}\omega_n + \omega_1)$ with $\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n$ being the smaller weight. Therefore
Hom\(_G(L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n), H^0(\frac{p-1}{2}\omega_n) \otimes L(\omega_1))\)
\[= \text{Hom}_G(V(\frac{p-1}{2}\omega_n), L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes L(\omega_1)) \cong k.\]

Assume that \(\text{Hom}_G(L(\frac{p-1}{2}\omega_n), L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes L(\omega_1)) = 0.\) Then there exists a composition factor \(L(\gamma)\) of \(V(\frac{p-1}{2}\omega_n)\) whose highest weight satisfies \(\gamma \uparrow \frac{p-1}{2}\omega_n\) \([\text{Jan}1, \text{II}.6.13]\) such that \(0 \neq \text{Hom}_G(L(\gamma), L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes H^0(\omega_1)) \hookrightarrow \text{Hom}_G(L(\gamma), H^0(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes H^0(\omega_1)).\)

The tensor product \(H^0(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes H^0(\omega_1)\) has a filtration with factors \(H^0(\frac{p-1}{2}\omega_n - \alpha_n), H^0(\frac{p-1}{2}\omega_n - \alpha_{n-1} - \alpha_n), H^0(\frac{p-1}{2}\omega_n),\) and \(H^0(\frac{p-1}{2}\omega_n + \alpha_1 + \ldots + \alpha_{n-1}).\) This forces \(\gamma = \frac{p-1}{2}\omega_n - \alpha_n\) or \(\gamma = \frac{p-1}{2}\omega_n - \alpha_{n-1} - \alpha_n.\) We will proceed to show that neither weight is strongly linked to \(\frac{p-1}{2}\omega_n.\)

The only weights that lie between \(\frac{p-1}{2}\omega_n - \alpha_n\) and \(\frac{p-1}{2}\omega_n\) are \(\frac{p-1}{2}\omega_n - \alpha_{n-1} = \omega \cdot \frac{p-1}{2}\omega_n\) and \(\frac{p-1}{2}\omega_n - \alpha_n.\) By \([\text{Jan}1, \text{II}.6.4(1)]\) it is sufficient to show that \(\frac{p-1}{2}\omega_n - \alpha_n\) and \(\frac{p-1}{2}\omega_n - \alpha_{n-1} - \alpha_n\) are not reflections of \(\frac{p-1}{2}\omega_n\) and that \(\frac{p-1}{2}\omega_n - \alpha_{n-1} - \alpha_n\) is not a reflection of \(\frac{p-1}{2}\omega_n - \alpha_{n-1}.\) Set \(\alpha\) equal to \(\alpha_n\) or \(\alpha_n + \alpha_{n-1}\) and assume that there is a root \(\beta\) and an integer \(m\) with \(s_{\beta, mp} \cdot \frac{p-1}{2}\omega_n = \frac{p-1}{2}\omega_n - ((\frac{p-1}{2}\omega_n + \rho, \beta') - mp)\beta = \frac{p-1}{2}\omega_n - \alpha.\) This forces \(\beta = \alpha\) and \(\frac{p-1}{2} + 1 = (\frac{p-1}{2}, \omega_n + \rho, \alpha') = mp + 1.\) Similarly, one obtains from \(s_{\beta, mp} \cdot \frac{p-1}{2}\omega_n - \alpha_{n-1} = \frac{p-1}{2}\omega_n - \alpha_{n-1} + \alpha_n\) that \(\beta = \alpha\) and \(\frac{p+1}{2} + 1 = (\frac{p-1}{2}\omega_n - \alpha_{n-1} + \rho, \alpha') = mp + 1.\) Either case leads via \(p \pm 1 = 2mp\) to a contradiction.

\[\square\]

4.3 Self-extensions for odd primes

**Proposition.** Let \(\alpha_n\) denote the unique long simple root in the root system of a symplectic algebraic group \(Sp_2n(k)\) and \(\omega_n\) the corresponding fundamental weight. If \(p\) is odd then

(i) \(\text{Ext}^1_{Sp_2n}(\mathbb{F}_p)(L(\frac{p-1}{2}\omega_n), L(\frac{p-1}{2}\omega_n)) \neq 0\) and

(ii) \(\text{Ext}^1_{Sp_2n}(\mathbb{F}_p)(L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n), L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n)) \neq 0.\)

**Proof.** (i) The weight \(\frac{p-1}{2}\omega_n\) satisfies the conditions of Proposition 3.2 because \((\frac{p-1}{2}\omega_n, \alpha_0') = p - 1.\) Therefore

\[k \cong \text{Ext}^1_G(L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n), L(\frac{p-1}{2}\omega_n) \otimes L(\omega_1)) \cong \text{Ext}^1_G(L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes L(\omega_1)(1), L(\frac{p-1}{2}\omega_n))\]

and there exists a non-trivial \(G\)-extensions

\[0 \to L(\frac{p-1}{2}\omega_n) \to E \to L(\frac{p-1}{2}\omega_n - \frac{1}{2}\alpha_n) \otimes L(\omega_1)(1) \to 0.\]

(4.3.1)

The \(G\)-module \(L(\frac{p-1}{2}\omega_n) \otimes L(\omega_1)(1)\) is isomorphic to \(L(\frac{p-1}{2}\omega_n) \otimes L(\omega_1)\) as a \(G(\mathbb{F}_p)\)-module. From the exact sequence (4.3.1) we obtain the following exact sequence of \(G(\mathbb{F}_p)\)-modules

\[0 \to \text{Hom}_G(\mathbb{F}_p)(L(\frac{p-1}{2}\omega_n), L(\frac{p-1}{2}\omega_n)) \to \text{Hom}_G(\mathbb{F}_p)(L(\frac{p-1}{2}\omega_n), E)\]
Notice that the module \( L^{(p-1)/2} \omega_n \) is simple as a \( G(F_p) \)-module and that the \( G \) and \( G(F_p) \)-socle of \( L^{(p-1)/2} \omega_n \circledast L(\omega_1) \) coincide. We apply Lemma 4.2 to the above sequence to obtain the exact sequence

\[
0 \rightarrow k \rightarrow \text{Hom}_{G(F_p)}(L^{(p-1)/2} \omega_n, E) \rightarrow k \rightarrow \text{Ext}_{G(F_p)}^1(L^{(p-1)/2} \omega_n, L^{(p-1)/2} \omega_n).
\]

It is therefore sufficient for our claim to show that \( \text{Hom}_{G(F_p)}(L^{(p-1)/2} \omega_n, E) \cong k \). We will show that \( L^{(p-1)/2} \omega_n \) appears only once in the \( G(F_p) \)-socle of \( E \). It follows from Lemma 4.1 that the \( G(F_p) \)-socle of \( E \) is contained in the \( G(F_p) \)-socle of the \( G \)-module \( St \circledast L((p-1)\rho - \frac{p-1}{2} \omega_n) \). Our claim follows from [Jan2, Satz 1.5] and comparison of highest weights via

\[
\dim \text{Hom}_{G(F_p)}(L^{(p-1)/2} \omega_n, St \circledast L((p-1)\rho - \frac{p-1}{2} \omega_n))
= [L^{(p-1)/2} \omega_n] \circledast L((p-1)\rho - \frac{p-1}{2} \omega_n) : St|_{G(F_p)}
= \sum_{\nu \in \chi(T)_+} [L^{(p-1)/2} \omega_n] \circledast L((p-1)\rho - \frac{p-1}{2} \omega_n) \circledast L(\nu) : St \circledast L(\nu)^{(1)}|_G
= [L^{(p-1)/2} \omega_n] \circledast L((p-1)\rho - \frac{p-1}{2} \omega_n) : St|_G = 1.
\]

The proof of (ii) is left to the reader. \( \square \)

These self-extensions have also been discovered by Tiep and Zalesskiı (see [TZ, 3.18]). Their method of proof is quite different. They show that the existence of self-extensions is a necessary condition for certain irreducible \( p \)-modular representations to have a lift to characteristic zero. A result due to Zalesskiı and Suprunenko [ZS] says that \( L^{(p-1)/2} \omega_n - \frac{1}{2} \alpha_n \) and \( L^{(p-1)/2} \omega_n \) are reduction modulo \( p \) of irreducible complex Weil representations for \( Sp_{2n}(F_p) \). Our methods have the advantage that they produce more families of self-extensions, at least for large primes.

### 4.4 More examples of self-extensions for large primes

Let \( G \) be of type \( C_n, p > 2n \), and \( \lambda \in X_1(T) \) with \( \langle \lambda, \alpha^\vee_i \rangle = (p-1)/2 \). Moreover, assume that \( \lambda \) is a \( p \)-regular weight. It follows from the translation principle [Jan1, II.7] that the module \( L(\lambda) \circledast L(\omega_1) \) is semi-simple both as a \( G \)-modules as well as a \( G(F_p) \)-modules (see [Hum2, p. 350]). A straightforward calculation shows that the weight \( \lambda - \epsilon_n = \lambda - \frac{1}{2} \alpha_n \) is also \( p \)-regular and contained in the same alcove. Hence \( \text{Hom}(L(\lambda - \frac{1}{2} \alpha_n), L(\lambda) \circledast L(\omega_1)) \cong k \). Using the same arguments as in the proof of Proposition 1.1 one obtains:

**Proposition.** Let \( G \) be of type \( C_n, p > 2n \), and \( \lambda \in X_1(T) \) with \( \langle \lambda, \alpha^\vee_n \rangle = (p-1)/2 \). Moreover, assume that

(i) \( \langle \lambda, \alpha^\vee_i \rangle < (p-1)/2 \) for \( i < n \),

(ii) \( H^0(\lambda) \) has only \( p \)-restricted composition factors,
then $\text{Ext}^1_{G(F_p)}(L(\lambda), L(\lambda))$ and $\text{Ext}^1_{G(F_p)}(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n))$ do not vanish.

Similarly, it follows from the translation principle that any $p$-regular weight $\lambda$ with $\langle \lambda, \alpha_n^\vee \rangle = (p - 1)/2$ that shares an alcove with $\frac{p-1}{2}\omega_n$ satisfies $\text{Ext}^1_{G_1}(L(\lambda - \frac{1}{2}\alpha_n)), L(\lambda) \otimes L(\omega_1)^{(1)} \cong k$. We can argue as above and conclude that $\text{Ext}^1_{G(F_p)}(L(\lambda), L(\lambda))$ and $\text{Ext}^1_{G(F_p)}(L(\lambda - \frac{1}{2}\alpha_n), L(\lambda - \frac{1}{2}\alpha_n))$ do not vanish in these cases.

**References**


