Worksheet (13.1 / 11.6): Simplified Quadric Surfaces

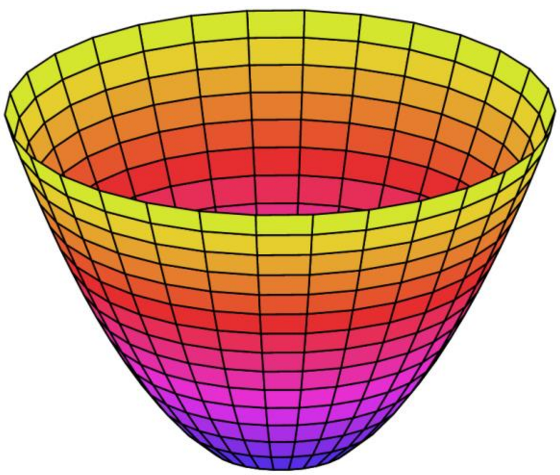
A quadric surface is a surface that can be expressed as a second degree polynomial in x, y and z. In this worksheet, we'll look at simplified quadric surfaces in which all the level curves of a certain height k ($z = k$) are circles.

A Paraboloid

$$f(x, y) = x^2 + y^2$$

$$z = x^2 + y^2$$

1. On the *x,y*-plane, draw several level curves $f(x, y) = x^2 + y^2 = k$ for different values of k.



2. Consider the intersection of the surface with the *yz*-plane by setting $x = 0$:

3. Consider the intersection of the surface with the *xz*-plane by setting $y = 0$:

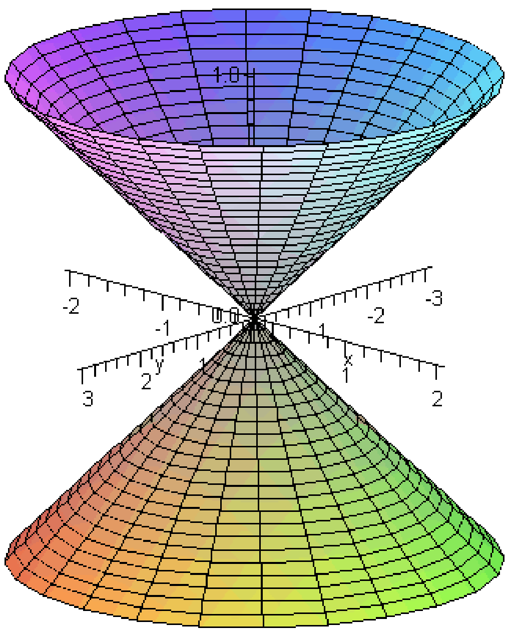
4. Combine these data in a single *x,y,z* plot:

A Cone

$$f(x, y) = \pm\sqrt{x^2 + y^2}$$

$$z^2 = x^2 + y^2$$

1. On the *x*,*y*-plane, draw several level curves $f(x, y) = \pm\sqrt{x^2 + y^2} = k$ for different values of *k*.



2. Consider the intersection of the surface with the *yz*-plane by setting $x = 0$:

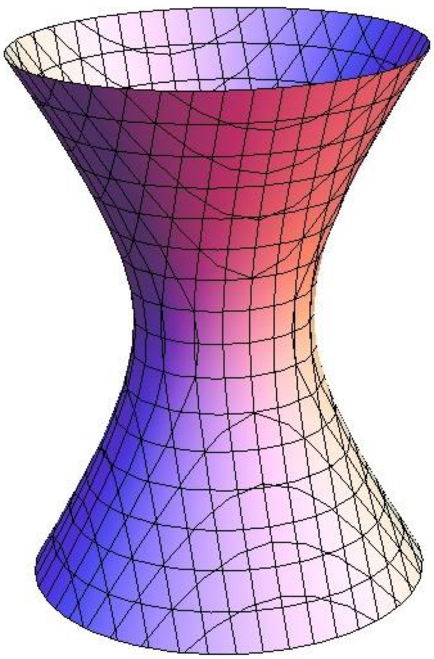
3. Consider the intersection of the surface with the *xz*-plane by setting $y = 0$:

4. Combine these data in a single *x*,*y*,*z* plot:

A Hyperboloid of One Sheet

$$f(x, y) = \pm\sqrt{x^2 + y^2 - 1}$$

$$z^2 = x^2 + y^2 - 1$$



1. Notice that since z^2 is non-negative, this imposes the constraint that $x^2 + y^2 - 1 \geq 0$. Sketch the subset of the *xy*-plane that satisfies this condition. (Compare the shape of the hyperboloid near the origin with that of the cone.)

2. On the *x*,*y*-plane, draw several level curves $f(x, y) = \pm\sqrt{x^2 + y^2 - 1} = k$ for different values of *k*.

3. Consider the intersection of the surface with the *yz*-plane by setting $x = 0$. (This shape is called a hyperbola.)

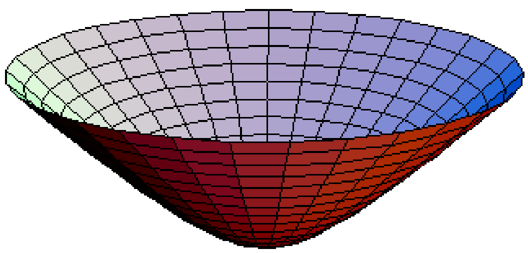
4. Combine these data in a single *x*,*y*,*z* plot:

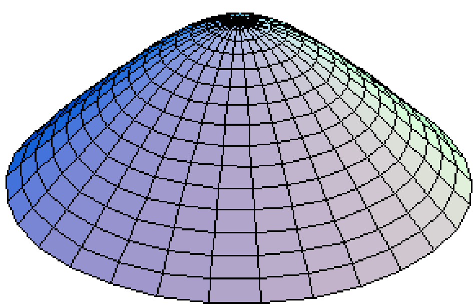
A Hyperboloid of Two Sheets

$$f(x,y) = \pm\sqrt{x^2 + y^2 + 1}$$

$$z^2 = x^2 + y^2 + 1$$

$$z^2 - 1 = x^2 + y^2$$





1. Notice that since $x^2 + y^2$ is non-negative, this imposes the constraint that $z^2 - 1 \geq 0$. Consider the intersection of the surface with the *yz*-plane by setting $x = 0$ and notice how the constraint on *z* results in two hyperbolas.

2. On the *x*,*y*-plane, draw several level curves $f(x,y) = \pm\sqrt{x^2 + y^2 + 1} = k$ for different values of *k* to confirm that the level curves are circles.

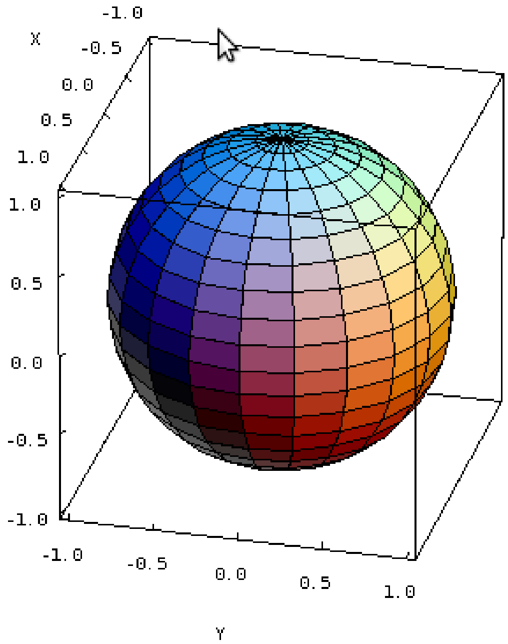
3. Combine these data in a single *x*,*y*,*z* plot:

A Sphere

$$f(x, y) = \pm\sqrt{1 - x^2 - y^2}$$

$$z^2 = 1 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 1$$



1. It is not surprising that this last quadric surface is a sphere when we consider the expression $x^2 + y^2 + z^2 = 1$. Using the expression $f(x, y) = \pm\sqrt{1 - x^2 - y^2}$, confirm that the level curves *z* = *k* are circles on the *xy*-plane. (Note that *k* must take values less than 1.)

2. Since z^2 is non-negative, the second equation highlights the constraint that $1 - x^2 - y^2 > 0$. How do you understand the effect of this constraint, possibly in connection with the hyperboloids?