The Role of Knots and Higher Dimensional Knots in Our Understanding of the Quantum World

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Abstract

According to the organizers of this event, this talk is meant to be speculative. The title alone satisfies that criterion! Yet, there seem to be at least two conflicting views of matter within our collective consciousness: the discrete and the continuous. Atoms, electrons, quarks, *etc.* are discrete particles while the meaning of particle in the first place has to do with a continuum — the collection of representations of a (product of) unitary groups.

Since the time of Taite and Thompson (Lord Kelvin), and perhaps earlier, matter was thought to be composed of knotted vortices in the aether. Such a model is notoriously wrong, yet some aspects persist. Knotted vortices can be created in the lab via the quantum hall effect. The space-time trace of a particle is a string, and indeed a 4-dimensional view of the universe suggests that we should examine the interactions of critical events.

So in this talk we start from toy models of particles that are created or annihilated, and that radiate and examine their critical interactions. Those aspects that you would like to consider "the same," I will consider as isomorphic. And I will then study the isomorphism among them. To continue the speculation, we'll successively project figures to planes and see if the interactions among critical events therein resemble the physical models of reality.

1 Introduction

These notes are to accompany my talk dated April 1, 2016 at the Mathematics Department, George Washington University. I write the notes for three reasons at least. Firstly, I am aware that fitting my talk into the allotted 50 minutes will be difficult. I am inspired by the scope of the venue to be speculative; yet I am humbled to give some details into my on-going work. The opportunity to present such a broad overview has given me the sense that I should attempt to fit this work into an over-arching structure. Secondly, the talk, as it appears on the slides, is primarily pictures. To better frame the talk, I need a libretto. Even here, I doubt that I will follow the script carefully. Still one of your former neighbors, an occupant of 1600 Pennsylvania Ave, has been criticized for not producing transcripts of her presentations for which her renumeration was considerably more than I can expect in my entire career. I think it is a good idea to have a record of that which is meant to be said; so I will release the transcripts early. Thirdly, the talk has given me a framework to write this material in a more cogent and coherence fashion. Most coherent and cogent endeavors start in a bit of disarray. As I write, I see that this talk is no exception. However, my hope is that this accompanying text will help you as you contemplate that which I said.

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2 Discrete or Continuous? What Is a Particle? Different Scales/Different Models

Is the universe discrete or continuous? I don't know. The reflection of the sun on top of the waves provides me with a pleasant metaphor for the quantum mechanical world. The light moves in blobs that may or may not coincide with the bulging water. The blobs of light merge and separate as the water moves. Each wave and each reflection of sunlight appears to be particulate but only for a blink of time. The water which appears as a continuum is filled with organisms of a variety of sizes and complexity. The creatures exist in a medium of hydrogen and hydroxide ions in which sodium, clorine, *etc.* combine and recombine.

At the turn of the last century, physicists were compelled, by experimental result, to disassociate the wave theory of light from a medium through which waves travel. Yet, general relativity also compels us to conceive of an underlying space-time that consists of matter which in turn affects its geometry.

I once asked Karen Uhlenbeck, "What is a particle?" She replied that it is a representation. A representation of what, where? Integral spin particles are representations of $SO(3)$ — the set of symmetries of the space-like slice of the space-time continuum. Half-integer particles are representations of $SU(2)$. Electrons take two trips before they return to their initial state. But the particle occupies space, and even though I conceive of it as a small bb that interacts with other particles in a billiard-like fashion, their behaviors are much more complicated. Perhaps, behaviors are complex. Let's believe that particles are discrete. But they are also representations of continuous groups. Clearly, I need different metaphors to conceptualize the world at scales that I can't directly experience.

2.1 Particles, Diagrams, and Geometry

The group of (2×2) unitary matrices with entries in the complex numbers is called $SU(2)$. As a topological space it coincides with the 3-dimensional sphere. The representation theory of $SU(2)$ considers those irreducible (complex) finite dimensional vector spaces upon which $SU(2)$ acts. These can be constructed as the image of the tensor product of the fundamental 2-dimensional vector space under a collection of projections that land in the space of symmetric polynomials. The representation theory is closely related to the so-called spin networks that were invented by Penrose. In such networks, trivalent vertices occur at junctions in which the out-going and in-coming spins coincide. Recombinations of spins correspond to the $6j$-symbols that measure the non-associativity of the tensor products of representations.

Both the spin networks and the identities that are satisfied by the $6j$-symbols are reflective of geometric properties. A $6j$-symbols may be a tetrahedron. Two tetrahedra glued along a face recombine into three that are joined at an edge. A little more than 25 years ago, our (primarily Russian) colleagues taught us an additional quantization in which the representations that were associated to knot theoretic invariants could be used to construct invariants of manifolds.

Category theory and diagrammatic calculi are central to all of these ideas. We think of $\cup$ as the birth of particle/anti-particle pairs. We think of $\cap$ as the annihilation of such. Similarly, X is an exchange of particles in a 1-dimensional space while I is a particle persisting in time and interacting with none. The diagrammatic calculus will be described further in a moment. For now, let us posit that there is an algebraic framework that mimics the diagrammatic depiction of particles as representations.

2.2 Dancing in a Large Room

A group of socially awkward dancers stands along a line with each 3 feet from her neighbor(s). Each has a phone in hand and a dance begins. All but two adjacent dancers remain stationary. These two move towards each other along that line, and before they collide, the each veer to their own left, and each returns to the line at the position of the other. This process continues with different (or the same) dancers switching position. Their phones, with highly sensitive GPS engaged, record the dance. After a finite amount of integral time, the phones map the set of motions of the dance. A braid appears.

We can ask, "When are two dances equivalent?" If the interchange between two dancers occurs with the dancer to the east moving to the south, turning around and then again turning off the line towards the south, then we could erase that movement and compress the time in which the motion occurred.

Virtual particles that are created, interact, and annihilate within time scales smaller than Planck's constant do not appear to contribute to a physical system. Perhaps, though, such infinitesimal interactions are precisely that which creates space and time.

3 Cartoon and Category

A *(small) category* consists of a set of objects such that between any two objects there is also a set or morphisms or arrows. Among the morphisms from an object to itself is an identity morphisms. A morphism (from A to B) is represented as an arrow $B \stackrel{f}{\longleftarrow} A$ whose source, $s(f)$, is A and whose target $t(f)$, is B. A pair of arrows $B \stackrel{f}{\longleftarrow} A$ and $C \stackrel{g}{\longleftarrow} B$ is said to be composable because the source $s(g)$ of the latter coincides with the target $t(f)$ of the former, and the composition is $C \stackrel{g \circ f}{\longleftarrow} A$. Composition of arrows is associative; that is, $(h \circ g) \circ f = h \circ (g \circ f)$. For any object A, there is an identity arrow $\|_A$. The identity arrows $|_A$ and $|_B$ when composed with f, leave f unchanged. That is, $|_B \circ f = f = f \circ |_A$.

There are a number of other adjectives that can be added to describe a category. We are interested in braided monoidal categories since these are all modeled by braids.

It is convenient for me to draw the arrows as either pointing to the left or pointing up.

3.1 Dots, Unary Notation, and Monoidal Categories

In the category of tangle diagrams, the objects are finite sequences of dots evenly spaced along line segments. Such a sequence was represented by the dancers above. The monoidal structure (associative binary operation with identity) on the objects is given by juxtaposition. A (possibly empty) sequence corresponds to a non-negative integer presented in unary notation. After all, $\mathbb{N} \cup \{0\}$ is the prototypical example of a monoid. Here the identity object $\{0\}$ is presented by the empty sequence. Juxtaposition is, by default, associative. The morphisms are generated by five kinds of basic morphisms: $\cup$, $\cap$, X, $\overline{\mathsf{X}}$, and I. The monoidal structure on morphisms is also given by horizontal juxtaposition. In the sequel, we will have cause to also include morphisms Y and $\curlywedge$. Clearly, the typography indicates that $(\bullet\bullet) \xleftarrow{\mathsf{Y}} (\bullet)$ while $(\bullet) \xleftarrow{\curlywedge} (\bullet\bullet)$.

In full generality, we'll assume that $F \in \{\cup, \cap, \mathsf{X}, \overline{\mathsf{X}}, \mathsf{Y}, \curlywedge\}$ and write $F(i,j)$ to indicate the tensor product

$$F(i,j) = \underbrace{(\mathsf{I} \otimes \cdots \otimes \mathsf{I})}_{i} \otimes F \otimes \underbrace{(\mathsf{I} \otimes \cdot \otimes \mathsf{I})}_{j}.$$

The existence of morphisms of the form $\cup$, $\cap$, $\curlywedge$, and Y indicates that the identity strings in the diagrams can move left or right as needed. We will allow any string that has no critical points to represent an identity.

We remark that at this point in the discussion no orientation is posited. More importantly, there are no identities assumed among the morphisms. However, like our dancers, I would like, for now, to assume that at most one non-identity morphism occurs at a given level. So that the tensor product $F \otimes G$ where $F, G \in \{\cup, \cap, \mathsf{X}, \overline{\mathsf{X}}, \mathsf{Y}, \curlywedge\}$ is represented by $(F \circ \|) \otimes (\| \circ G) = (\| \circ F) \otimes (G \circ \|)$ where $\|$ is shorthand for an appropriate number of identity strings. I'll read these morphisms as, "F over I to the left of I over G," or "I over F to the left of G over I."

3.2 Morphisms and Identities

Now in the category of tangles, certain identities between morphisms are presupposed. These include that X and $\overline{X}$ are inverse to each other, zig-zag identities for $\cap$ and $\cup$, and a Yang-Baxter relation which is stated as

$$(\mathsf{X} \otimes \mathsf{I}) \circ (\mathsf{I} \otimes \mathsf{X}) \circ (\mathsf{X} \otimes \mathsf{I}) = (\mathsf{I} \otimes \mathsf{X}) \circ (\mathsf{X} \otimes \mathsf{I}) \circ (\mathsf{I} \otimes \mathsf{X}).$$

Here I want to create a souped-up version of a category — the notion of a 2-category. And so, whenever a classical knot theorist — one who studies knotted and linked circles in 3-space— would suppose an identity, I want to suppose a type of natural isomorphism, and in addition I will suppose identities among these isomorphisms. More generally, when a lower dimensional knot theorist supposes that there is a knot move, I want to suppose instead that there is an isomorphism and the isomorphisms satisfy additional identities.

Even more generally still, there are a move to knots that change the topological type. For example in the classical, there are births

$$[(\bullet)_n \xleftarrow{(\cap\circ\cup)_{(i,j)}} (\bullet)_n] \overset{B(i,j)}{\Longleftarrow} [(\bullet)_n \xleftarrow{\mathsf{I}_{i+j}} (\bullet)_n],$$

deaths

$$[(\bullet)_n \overset{\mathsf{I}_{i+j}}{\longleftarrow} (\bullet)_n] \overset{D(i,j)}{\Longleftarrow} [(\bullet)_n \overset{(\cap \circ \cup)_{(i,j)}}{\longleftarrow} (\bullet)_n],$$

and saddles

$$[(\bullet)_n \overset{(\cup \circ \cap)_{(i,j)}}{\longleftarrow} (\bullet)_n] \overset{S(i,j)}{\Longleftarrow} [(\bullet)_n \overset{\mathsf{I}_n}{\longleftarrow} (\bullet)_n]$$

(where $(i+j) = n$), and these are to be treated as 2-morphisms.

Now the tricky part in this form of categorification is that the identities among isomorphisms are informed by singularity theory. So again, the algebraic framework is a kind of discretization, but the search for higher order relations always depends on analytical considerations. In some sense, this is why I believe that the theoretic underpinnings of these works is correct: it involves applications of topology, algebra, and analysis.

3.3 Taite and Thompson Revisted

The origins of the mathematical theory of knots includes hypotheses by Taite and Thompson (Lord Kelvin) that atoms are knotted vortices in the ether. There is no ether, and a table of knots does not resemble the periodic table of elements. However, the categorical interpretation of knots with $\cup$, $\cap$, X, $\overline{\mathsf{X}}$, and I as morphisms has an alternate physical interpretation. The morphism $\cup$ is the birth of a particle/anti-particle pair while $\cap$ corresponds to their mutual annihilation. The morphisms X and $\overline{\mathsf{X}}$ correspond to particles interchanging positions in a 1-dimensional space. Here there is a distinction X corresponds to "left moves in front of right" while $\overline{\mathsf{X}}$ corresponds to "right moves in front of left." In our 1-dimensional universe, this distinction is neither visible nor intuitive, but becomes part of a reasonable physical model.

Each process can be assigned an amplitude that depends on the "states" in which the particles find themselves. A given knot diagram, then is a vacuum-to-vacuum amplitude that is expressed as a state-sum. The Kauffman bracket model of the Jones polynomial is exactly of this form. This metaphorical situation would just be a bit of fancy but for two main points.

First, the Jones polynomial is constructed via projections of type-II_1 factors in von Neumann algebras which were constructed for the study of quantum mechanical systems. Second, in the quantum Hall effect particles are indeed vortices in a super-cooled fluid, and the dance of particles is induced via magnetic effects. In this case, the computation of the Jones polynomial is an aspect of quantum computing. So even though Taite and Thompson were wrong, they were sort of right.

Herein, I want to add to the original tangle category two additional 1-morphisms Y and $\curlywedge$ whose physical interpretation in our cartoon model physics correspond as follows. The 1-morphism Y corresponds to a particle that splits into two daughters, and the 1-morphism $\curlywedge$

corresponds to two particles colliding and emitting another particle. Next, I want to discuss how to create algebraic models from these simple scenarios.

4 Algebraification and Categorification

Before I begin this section, I need to talk a little about time's arrow. My conventions for composition of morphisms, so far, have followed the traditional function oriented convention in which the argument of the function is fed into it from the right. Diagrammatically, this means that the object of a categorical composition appears at the bottom of the diagram. The advantage in this convention is that when one reads from top to bottom recording $\cap$s, Xs, and so forth, the composition is prewritten by your recording thereof. In the Jones polynomial set-up, from the bracket model, the matrix composition is written left to right, and input vectors are column vectors on the right. At first, these conventions seem quite strange, and they are. In a braid scenario, marbles do not flow down the braid diagram in the direction of downward pointing arrows, but instead they percolate up the diagram.

In a little while, I am going switch conventions. The reason for the switch, I think, is that I am more interested in the region surrounding the knotting rather than the diagram.

Meanwhile, I will ignore $\cap$s and $\cup$s and concentrate efforts to understanding isotopy moves that involve X, Y, and $\curlywedge$. There are four such moves, and these are called the A-move (for associator), YI-move, IY-move, and III-move (or type-III move). In the categorical setting they are 2-morphisms of the form:

$$[(\mathsf{Y} \otimes \mathsf{I}) \circ (\mathsf{Y})] \overset{A}{\Longleftarrow} [(\mathsf{I} \otimes \mathsf{Y}) \circ (\mathsf{Y})],$$

$$[(\mathsf{I} \otimes \mathsf{X}) \circ (\mathsf{X} \otimes \mathsf{I}) \circ (\mathsf{Y})] \overset{YI}{\Longleftarrow} [(\mathsf{Y} \otimes \mathsf{I}) \circ (\mathsf{X})],$$

$$[(\mathsf{I} \otimes \mathsf{Y}) \circ (\mathsf{X})] \overset{IY}{\Longleftarrow} [(\mathsf{X} \otimes \mathsf{I}) \circ (\mathsf{I} \otimes \mathsf{X}) \circ (\mathsf{Y})],$$

and

$$[(\mathsf{I} \otimes \mathsf{X}) \circ (\mathsf{X} \otimes \mathsf{I}) \circ (\mathsf{I} \otimes \mathsf{X})] \overset{III}{\Longleftarrow} [(\mathsf{X} \otimes \mathsf{I}) \circ (\mathsf{I} \otimes \mathsf{X}) \circ (\mathsf{X} \otimes \mathsf{I})].$$

Here, I have suppressed the notation for the source and targets of the 1-morphisms involved.

I remark further that moves that involve $\overline{\mathsf{X}}$ can be obtained from those that involve X via judicious applications of the type-II 2-morphisms:

$$[\mathsf{I} \otimes \mathsf{I}] \overset{II\binom{+}{-}}{\Longleftarrow} [\mathsf{X} \circ \overline{\mathsf{X}}]$$

and

$$[\mathsf{I} \otimes \mathsf{I}] \overset{II\binom{-}{+}}{\Longleftarrow} [\overline{\mathsf{X}} \circ \mathsf{X}].$$

Each is an invertible 2-morphism and invertibility is expressed via type II bubble and saddle moves.

4.1 Fundamental Group

Now tangle diagrams that consist of compositions of $\cap$, $\cup$, X, $\overline{\mathsf{X}}$, Y, $\curlywedge$ and I are considered not only to be 1-morphisms in our category, but in addition they represent properly embedded trivalent graphs in $[0,1] \times [0,1] \times (-\epsilon, \epsilon)$ where both the vertical composition ($\circ$) and the horizontal composition ($\otimes$) are rescaled so that the entire diagram fits into a unit square. The endpoints of the graph are along the top and bottom faces, and the thickness $(-\epsilon, \epsilon)$ encapsulates the over/under crossing information such as when our dancers moved off the stationary line.

Such an embedding has a fundamental group — the fundamental group of the complement of the graph in the thickened window. This fundamental group remains unchanged when the 2-morphisms A, YI, IY, and III are applied. For the last three moves, the choice of diagram that represents the embedding is affected. In the case of A, the complement of a regular neighborhood of the graph is topologically equivalent.

Now let us assume that all of the legs of Y and X are oriented downwards. Small loops that encircle the legs of Y and X are considered to be based at the eye of the reader. In this way, the generators of the fundamental group appear as superscripts and subscripts: ${}^{a}\mathsf{Y}^{b}_{ab}$ and ${}^{a}_{b}\mathsf{X}^{b}_{a*b}$ where $a * b = b^{-1}ab$. That is, the loops at the top legs of the vertex of Y merge to be the product. Similarly, at X the northwest to southeast arc crosses in front of the remaining arc, so the bottom right leg is the conjugation of the top left arc, labeled a, by the over-arc labeled b, namely $b^{-1}ab$. These relations can be visualized as null homotopies whose principal pieces are the triangle dual to the Y and the square dual to the X. Arcs encircling the legs of the Y and the X appear as small arrows below the legs, and the triangle or square tracks the important part of the homotopies.

The A, YI, IY, and III relations hold at the level of the fundamental group because the label on the right-most leg of these diagrams is

$$\begin{aligned}
A: &\quad (ab)c &=& \ a(bc), \\
YI: &\quad (ab) * c &=& \ (a * c)(b * c), \\
IY: &\quad (a * b) * c &=& \ a * (bc), \\
III: &\quad (a * b) * c &=& \ (a * c) * (b * c).
\end{aligned}$$

The first relation is, of course, associativity for group multiplication. And the last relation is called *self-distributivity.* Furthermore, by looking at the triangles dual to the Ys in the figures and squares dual to the Xs in the figures, we see that the relations that are satisfied are dual respectively to a tetrahedron, a prism, a prism, and a cube. These four figures are the different ways of decomposing a 3-dimensional ball as the product of simplices. into

4.2 Group and Quandle

Given a group G its conjugation operation defines a structure called a *quandle* which satisfies three axioms that correspond to the Reidemeister moves:

$$\begin{array}{lrrcl} I: & (\forall a): & a * a & = & a \\ II: & (\forall a, b)(\exists c): & c * b & = & a \\ III: & (\forall a, b, c): & (a * b) * c & = & (a * c) * (b * c). \end{array}$$

Now both groups and quandles have homology theories that are defined in terms of their intrinsic multiplications. Roughly speaking, the group homology is a theory based upon simplices while the quandle homology is based upon cubes.

A result of Joyce and Matveev is that the fundamental quandle of a knot complement is a complete invariant (up to orientation reversing homeomorphism) of the complement. The quandle encodes both the fundamental group and a peripheral subgroup. The four axioms above (A, YI, IY, III) indicate some of how group and quandle interact. The equalities in the case of knotted objects in 3-space are isomorphisms when considered as surfaces (or foams) in 4-space. Here we want to measure them.

4.3 Measuring Isomorphisms — Homology

We start from a slicing into k pieces of the interval of consecutive integers that starts at 1:

$$\langle 1, 2, \ldots, \ell_1 \rangle \langle \ell_1 + 1, \ldots, \ell_1 + \ell_2 \rangle \cdots$$
$$\left\langle \sum_{i=1}^{j-1} \ell_i + 1, \ldots, \sum_{i=1}^{j} \ell_i \right\rangle \cdots \left\langle \sum_{i=1}^{k} \ell_i + 1, \ldots, n \right\rangle.$$

Such a slice corresponds to a decomposition of the n-ball into a product of simplices. There are 2^{n-1} ways to make such slices.

The integers here are subscripts that indicate group elements. We define a boundary operator on such slices as follows. First, use the Leibniz rule:

$$\partial(PQ) = (\partial P)Q + (-1)^{|P|} P(\partial Q)$$

when $P = \langle s+1, \ldots, s+k \rangle$ and $Q = \langle s+k+1, \ldots, s+k+j \rangle$. Then, in each homogeneous factor apply the group homology boundary operator:

$$\begin{aligned} \partial\left(\langle s+1, s+2, \ldots, s+r \rangle\right) &= *(s+1)\langle s+2, \ldots, s+r \rangle \\ &\quad + \sum_{j=1}^{r-1} (-1)^j \langle s+1, \ldots, (s+j) \cdot (s+j+1), \ldots, s+r \rangle \\ &\quad + (-1)^r \langle s+1, \ldots, s+r-1 \rangle. \end{aligned}$$

The prefix $*(s+1)$ in the first term indicates that this element operates, via the $*$ product on all of the factors that are to the left. In case, $r=1$, the formula reads

$$\partial\langle s+1\rangle = *(s+1)\langle\ \rangle - \langle\ \rangle$$

since the element $(s+1)$ is thrown out of the front of the expression enclosed within $\langle\rangle$ and also thrown out of the back of the expression. When it leaves the front, it pays a "bus fare:" $*(s+1)$. If it is thrown out of the back, then there is no fare.

Now this boundary operator satisfies $\partial \circ \partial = 0$ precisely when the multiplications $\cdot$ and $*$ satisfy the relations A, YI, IY, and III. That is an observation that is essentially due to Przytycki. Furthermore, we have:

- The complex agrees with the group homology chain complex at the level in which there is no slicing.

- The complex agrees with the quandle homology complex when there is maximal slicing.

- The boundaries of the chains $\langle 1,2,3\rangle$, $\langle 1,2\rangle\langle 3\rangle$, $\langle 1\rangle\langle 2,3\rangle$, and $\langle 1\rangle\langle 2\rangle\langle 3\rangle$ correspond exactly to the "source" colored chains represented by the Xs and Ys in the moves that correspond to the A, YI, IY, and III axioms.

- These also correspond to the (signed) boundaries of the corresponding products of simplices.

- The same phenomena occur in higher dimensions.

4.4 Higher Order Relations

There are eight slices of the interval of length 4. They are $\langle 1,2,3,4\rangle$, $\langle 1,2,3\rangle\langle 4\rangle$, $\langle 1,2\rangle\langle 3,4\rangle$, $\langle 1,2\rangle\langle 3\rangle\langle 4\rangle$, $\langle 1\rangle\langle 2,3,4\rangle$, $\langle 1\rangle\langle 2,3\rangle\langle 4\rangle$, $\langle 1\rangle\langle 2\rangle\langle 3,4\rangle$, and $\langle 1\rangle\langle 2\rangle\langle 3\rangle\langle 4\rangle$. These correspond to eight possible moves to 2-foams. The move that corresponds to $\langle 1,2,3,4\rangle$, changes the topological type of the foam, corresponds to a 4-simplex (with boundary the $(3,2)$ move), and corresponds to a group 3-cocycle condition. The remaining moves maintain the topological type of the foams, and each corresponds to a product of simplices.

The moves are depicted within the illustrations. More can be deciphered here. Namely, as the foams are projected, they too behave as models for more sophisticated particle interactions. Meanwhile, the world lines of the seams and double curves allow for more complicated Feymann-type diagrams of particles. Seams can represent one type of particle, and double lines can represent another.

5 Foams and Further Studies

Here, let me itemize some prospects.

- I have outlined a homology theory here, but we have not been able to compute any interesting cycles except in the most simple cases.
- The theory of 2-foams is more delicate than that of 2-knots. Twist spinning, in particular, is more interesting.
- In a certain sense, construction of foams and recognizing non-trivial cycles are the same problem. Except that non-trivial cycles will likely be manifested as analogues of virtual knots.
- As the singular sets of higher dimensional knots and foams are projected, one imagines that the world lines will appear as particle-like interactions. This is perhaps the most speculative aspect. We know knot theory is a toy model of physics, but maybe the higher dimensional examples are better toys. Think of the difference between Pong and Super-Mario Bros.
- The moves to foams are informed by both Stasheff polytopes and permutahedra. There appear to be some interesting combinatorial structures that inter-related commutativity and associativity. They too deserve study.

Thank you for your attention.