Local pictures for knotted foams and G-families of quandles

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Intelligence of Low Dimensional Topology
joint work with Atsushi Ishii and Masahico Saito
preliminaries

1. Thanks to the organizers
2. Brain-pool Trust
3. joint with Atsushi Ishii and Masahico Saito
4. Example knotted graph
5. Example of a knotted foam
6. Main result
7. Introducing knotted foams
8. $G$-families of quandles
9. boundaries of chains
10. The model space $Y^n$
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This talk was supported by the Ministry of Education Science and Technology (MEST) and the Korean Federation of Science and Technology Societies (KOFST).
• The above movie was the first twist of a 2-twist-spin of the knotted trivalent graph $5_2$.
• This is an example of a knotted foam.
• We have computed that the corresponding knotted foam has a non-trivial cocycle invariant by using a variation upon Mochizuki’s 3-cocycle.
The previous slide contained the main result.

**Theorem**

There is a non-trivial cocycle invariant of knotted foams.
The principle features of trivalent graphs are crossings and vertices. These are depicted here.

\[
\begin{align*}
\begin{array}{ll}
1 & 2 \\
\includegraphics[width=0.25\textwidth]{crossing_1.png} & \includegraphics[width=0.25\textwidth]{crossing_2.png}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ll}
<1,2> & <1><2> \\
\includegraphics[width=0.25\textwidth]{crossing_3.png} & \includegraphics[width=0.25\textwidth]{crossing_4.png}
\end{array}
\end{align*}
\]

is shorthand for...

is shorthand for...
Each move to a trivalent graph can be interpreted as an “atomic piece” of a surface in 4-dimensional space.
\langle 1,2 \rangle \langle 3 \rangle
G-family of quandles

after Ishii-Iwakiri-Jang-Oshiro. $G$ is a group. $Q$ is a set. $\forall g \in G \exists$ binary operation $\lhd_g$ on $Q$ s.t.

- $a \lhd_g a = a$
- $(a \lhd_g b) \lhd_h b = a \lhd_{gh} b,$
- $\lhd_g^{-1} = (\lhd_g)^{-1}$, $a \lhd_1 b = a,$
- $(a \lhd_g b) \lhd_h c = (a \lhd_h c) \lhd_{h^{-1}gh} (b \lhd_h c).$

($\forall a, b, c \in Q$, and $\forall g, h \in G$).
Examples

1. Let $H$ be a group. $G = \text{Aut}(H)$. Define $a \triangleleft_s b = s(ab^{-1})b$.

2. Specifically, $Q = (\mathbb{Z}/(p))^n$ — row vectors $G = \text{SL}(n, \mathbb{Z}/(p))$, and $a \triangleleft_M b = aM + b - bM$. 
Lemma

Let \((Q, G)\) denote a \(G\) family of quandles on \(Q\). Then \(G \times Q\) is a quandle under the binary operation \((g, a) \triangle (h, b) = (h^{-1}gh, a \triangle_h b)\).

\(G \times Q\) is a dynamical extension of \(\text{Conj}(G)\) with the dyn. cocycle \(\alpha : G \times G \rightarrow Q^{Q \times Q}\) given by \(\alpha_{g,h}(x, y) = a \triangle_h b\). In particular,

\[
\alpha_{g\triangle h, k}(\alpha_{g,h}(a, b), c) = \alpha_{g\triangle k, h\triangle k}(\alpha_{g,k}(a, c), \alpha_{h,k}(b, c)).
\]
(a, b, c ∈ Q, and g, h, k ∈ G).

⟨1, 2, 3⟩ ↔ ((g, a), (h, a), (k, a))

⟨1, 2⟩⟨3⟩ ↔ ((g, a), (h, a); (k, b))

⟨1⟩⟨2, 3⟩ ↔ ((g, a); (h, b), (k, b))

⟨1⟩⟨2⟩⟨3⟩ ↔ ((g, a); (h, b); (k, c))
Some obvious relations
Some obvious relations
Some obvious relations
Some obvious relations
Some obvious relations
Some obvious relations
Boundaries

\[ \partial \langle j + 1, j + 2, \ldots, j + k \rangle \]
\[ = \langle j + 1 \rangle \langle j + 2, \ldots, j + k \rangle \]
\[ + \sum_{\ell=1}^{k-1} (-1)^\ell \langle j + 1, \ldots, (j + \ell) \rangle \langle j + \ell + 1, \ldots, j + k \rangle \]
\[ + (-1)^k \langle j + 1, \ldots, j + k - 1 \rangle. \]

\[ \partial (PQ) = (\partial P)Q + (-1)^{\text{dim}P} P(\partial Q). \]

In part,

\[ \partial \langle j + 1 \rangle = \langle j + 1 \rangle_\rhd - _. \]
Boundaries

\[ \partial <1,2> = <2>-<1 \cdot 2> + <1> \]

\[ \partial <1><2>= <2>-<2> - <1 \triangle 2> + <1> \]
\[ \delta \langle 1, 2, 3 \rangle = \langle 2, 3 \rangle - \langle 1 \cdot 2, 3 \rangle + \langle 1, 2 \cdot 3 \rangle - \langle 1, 2 \rangle. \]
\[ \partial <1,2><3> = \]
\[ (\partial <1,2>) <3> + <1,2> \partial <3> \]
\[ = <2><3> - <1 \cdot 2><3> + <1><3> + <1><3> - <1,2> + <1<3, 2<3> \]

**Boundaries**
\[ \partial \langle 1 \rangle \langle 2,3 \rangle = \langle 2,3 \rangle - \langle 2,3 \rangle - \langle 1 \rangle \langle 2 \rangle - \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle + \langle 1 \rangle \langle 2 \cdot 3 \rangle \]
\[ \partial (\langle 1\rangle\langle 2\rangle\langle 3\rangle) = \]
\[ \langle 2\rangle\langle 3\rangle - \langle 2\rangle\langle 3\rangle - \langle 1\rangle\langle 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle\langle 2\rangle\langle 3\rangle - \langle 1\rangle\langle 2\rangle \]
Boundaries, Commentary

In general, we can take boundaries of lin. combos. of exps. of the form:

\[ \langle 1, \ldots, j_1 \rangle \langle j_1 + 1, \ldots, j_1 + j_2 \rangle \]

\[ \cdots \langle \sum_{\ell=1}^{k-1} j_\ell + 1, \ldots, \sum_{\ell=1}^{k} j_\ell \rangle. \]

These expressions correspond to products of simplices. For example, $Y$ corresponds to a triangle, $X$ corresponds to a square, and the moves to trivalent graphs correspond to a tetrahedron, prism, prism, and cube.
The boundaries of the expressions involving 4 variables correspond to the fundamental moves to foams. In the next few slides, I will express these in movie move terms.
The Space $Y^n$

Let $\Delta^{n+1} = \{ \vec{x} \in \mathbb{R}^{n+2} : \sum x_i = 1 \& 0 \leq x_i \}$ denote the standard simplex. The space $Y^n \subset \Delta^{n+1}$ is defined as follows: $Y^0 = (\frac{1}{2}, \frac{1}{2})$. Take $\Delta^n_j = \{ \vec{x} \in \Delta^{n+1} : x_j = 0 \}$. Embed a copy, $Y^{n-1}_j \subset \Delta^n_j$. Cone $\bigcup_{j=1}^{n+2} Y^{n-1}_j$ to the barycenter $b = \frac{1}{n+2}(1, 1, \ldots, 1)$ of $\Delta^{n+1}$.

$$Y^n = C \left( \bigcup_{j=1}^{n+2} Y^{n-1}_j \right).$$
$Y^0$, $Y^1$, and $Y^2$
Definition of $n$-foams

Every point $y \in Y^n$ has a nbhd. that is homeom. to $Y^{n-k} \times D^k$. The union of these points is called the $k$-stratum — the union of these is a set of $\binom{n+2}{k}$ disks of dimension $k$, for $k = 1, \ldots, n$.

An $n$-foam is a top. sp. $X$ for which each pt. $x \in X$ has a nbhd homeom. to a nbhd. of a point in $Y^n$. 
Local pictures of knottings of an $n$-foam

Let $(j_1, j_2, \ldots, j_k)$ denote an ordered partition of $n + 1$.
For example, when $n + 1 = 3$, the partitions are (3), (2, 1), (1, 2), (1, 1, 1). When $n = 4$, the partitions are (4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), and (1, 1, 1, 1).
For each such partition, we construct a local picture of a crossing as follows:
Let \((j_1, j_2, \ldots, j_k)\) denote an ordered partition of \(n + 1\). We write

\[\langle 1, \ldots, j_1 \rangle \langle j_1 + 1, \ldots, j_1 + j_2 \rangle \]

\[
\cdots \langle \sum_{\ell=1}^{k-1} j_\ell + 1, \ldots, \sum_{\ell=1}^k j_\ell \rangle,
\]

and consider \(\prod_{\ell=1}^k \Delta j_\ell\) as a fixed embedding in \(\mathbb{R}^{n+1}\). Embed

\[Y_{j_\ell-1} \subset \Delta j_\ell.\]
Local pictures of knottings

Now take

$$\bigcup_{\ell=1}^{k} (\Delta^{j_1} \times \cdots \times Y^{j_{\ell-1}} \times \cdots \times \Delta^{j_k}) \subset \mathbb{R}^{n+1} \times \{\ell\}$$,

and project this into $\mathbb{R}^{n+1}$. The factor $\ell$ is in the $(n + 2)$nd coordinate and represents the relative height of each $Y$. 
\[
\left[(Y^{j_1 - 1} \times \Delta^{j_2} \times \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}) \subset \mathbb{R}^{n+1} \times \{1\}\right], \\
\left[(\Delta^{j_1} \times Y^{j_2 - 1} \times \cdots \times \Delta^{j_\ell} \times \cdots \times \Delta^{j_k}) \subset \mathbb{R}^{n+1} \times \{2\}\right], \\
\cdots \\
\left[(\Delta^{j_1} \times \Delta^{j_2} \times \cdots \times Y^{j_\ell - 1} \times \cdots \times \Delta^{j_k}) \subset \mathbb{R}^{n+1} \times \{\ell\}\right], \\
\cdots \\
\left[(\Delta^{j_1} \times \Delta^{j_2} \times \cdots \times \Delta^{j_\ell} \times \cdots \times Y^{j_k - 1}) \subset \mathbb{R}^{n+1} \times \{k\}\right].
\]

These project to a 0-dimensional multiple point in \(\mathbb{R}^{n+1}\).
Summary Statements

- For each ordered partition \((j_1, \ldots, j_k)\) of \((n + 1)\), there is a 0-dimensional multiple point of an \(n\)-foam.
- The multiple point (when colored) represents a chain in the homology of a \(G\)-family of quandles.
- The sum of these chains over all multiple points is a cycle.
- The boundary of such a chain is a cycle in one dimension lower.
Summary Statements

- The 0-dimensional multiple point represents an essential isotopy move for the lower dimensional foam.
- Quandle cocycle invariants can be defined for knotted $n$-foams.
- More work is needed in constructing interesting knotted $n$-foams.
- Invariants of foams simultaneously generalize Dijkgraaf-Witten invariants and quandle cocycle invariants.
Thank you very much

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ありがとうございます

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