Number Shapes

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Chapter 1

Introduction

Hello, boys and girls. My name is Professor Elvis P. Zap. That's not my real name, but I really am a math professor. Once I had a math professor who would say, "Zap!" every time he solved a problem. I thought that was really neat. Then when I became a math professor, I made up silly songs about college math classes. I sang one of these where I was teaching, and one of my students thought that was neat. So he wrote an article about what really happened to Elvis. The real Elvis was a singer who died a long time ago. My student said in his article that Elvis wanted to teach math, but he liked to sing still so now he sings about math. Well I am not the real Elvis either, but I figured only once in your life is someone, like, going to call you "Elvis," so you better run with it. I don't remember what the "P" stands for. I think it just sounds good. So I gave myself the pen name and stage name "Professor Elvis P. Zap." That way I can remember my math professor who I really liked, and my student who I also liked and he liked me too.

Some people choose pen-names and stage names that are different than their own. It is just something fun to do. You may have heard about a writer named Lewis Carrol. He wrote *Alice in Wonderland*. That is a really good book and it has a lot of math in it. Lewis Carrol is the pen name of Charles Dodgson who was a math professor in England. I guess I wanted to write this book with a pen-name because it is not the kind of math book I usually write.

So now I am a real math professor and I do real math. I also try to teach my own kids about math, and these are some of the things I have taught them. My kids are pretty good in math because they have a daddy who can teach them neat stuff. I think you probably have neat parents too. I mean, after all they bought you my book didn't they? Your parents can do a bunch of stuff that I can't do. Maybe they can do math, I don't really know. Anyway,I have shown the math in this book to my kids and they think it is pretty cool. Maybe they just tell me that because I am dad.

In this book I am going to show you some really neat patterns that you can make with numbers. Learning about these patterns will help you out as you get older and better in math. If you learn a lot about these number shapes you will be really good at arithmetic, and you will have learned some advanced math at a grade school level.

I like math because I like to see pretty patterns. A lot of the math that I do as a math professor involves really neat shapes and patterns that are really hard for people to see. My math friends also do really complicated stuff, and a lot of it involves complicated patterns. The patterns I will show you here are not so complicated. But you have to learn the easy stuff first.

A lot of what I show you here, I really want you to memorize. Memorizing is a good thing because it helps train your brain. A lot of things you already have memorized and your brain is smarter because of it. Here when you memorize number patterns you will learn relationships among different numbers, and these relationships are what our number system is made up as.

To help you memorize these things, I have left a bunch of math exercises at the end of the book. These you should do in order, and you should work them often. The ideas for the answers are already in the book. If you need to check you answers, you can ask a calculator. But first try to work them out with your brain.

Chapter 2

count-bys

2.1 count-by 1s

1										1
1	2									2
1	2	3								3
1	2	3	4							4
1	2	3	4	5						5
1	2	3	4	5	6					6
1	2	3	4	5	6	7				7
1	2	3	4	5	6	7	8			8
1	2	3	4	5	6	7	8	9		9
1	2	3	4	5	6	7	8	9	10	10

I don't really know you, and I don't know how much you know. So I am going to start-off pretty easy. I think you might know how to count-by 1s. I am pretty sure that your parents know how to count-by 1s, so they can help you with this part.

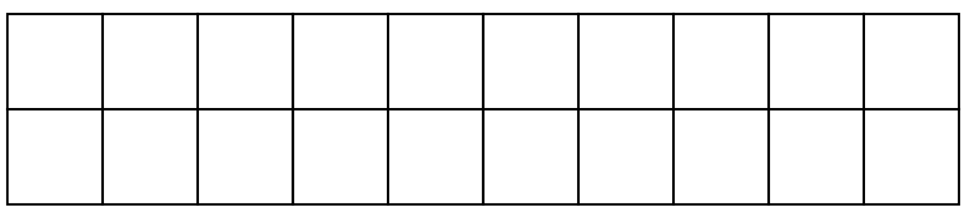
The number sequence

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

is what we get when we count-by 1s. In my pictures I always am counting little squares, but you can use count-by 1s to count anything: Legos, tomatoes, dolls, or paper airplanes. I like to count squares. They are easy for me to draw, and I can arrange them in shapes easily.

In my picture, you probably see that I made my squares into the shape of a stair-case. If we start with $1+2$ we get 3. So a staircase with 2 steps needs 3 squares. If we keep going, we get $1+2+3$ and when we count all the squares there, we get 6. In the same way $1+2+3+4$ is 10. If you have ever seen bowling pins, they are arranged like this. Do you wonder how many squares would be needed to make 100 steps?! The answer is 5050. You can find out just by counting each square, but there is an easier way. Later on, I'll show you the easier way.

2.2 count-by 2s



2 4 6 8 10 12 14 16 18 20

OK that last section was too easy wasn't it? Well except maybe for the stair-case question, but I'll show that trick soon. This section is not too bad either.

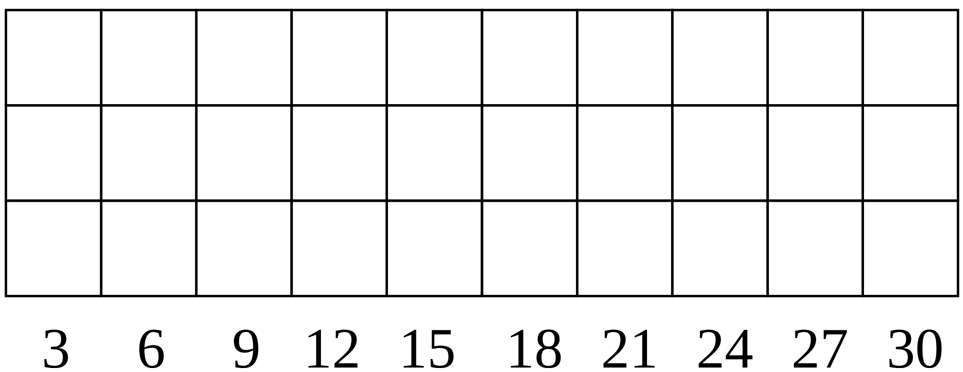
When you count-by 2 you say the sequence:

$$2, 4, 6, 8, 10, 12, 14, 16, 18, 20.$$

The numbers that you get to by counting by 2s are called the *even numbers*. If you have to count things that come in pairs it is easiest to count-by twos. Imagine Noah counting the animals as they get on the arc, "$2, 4, 6, \ldots$." People use a set of 3-dots to mean "and so on." That means that Noah would keep counting by 2s until ever pair of animals had gotten on the arc.

Some of you have parents who eat with chop-sticks. When you set the table with chop-sticks, you might count-by 2 to count the total number of sticks. Counting by 2s is not too hard.

2.3 count-by 3s



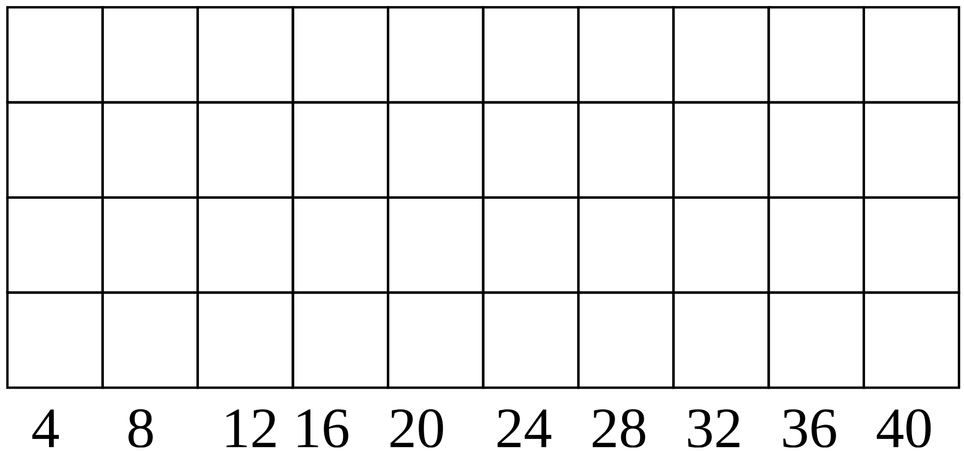
Most kids don't really learn to count-by 3s. But you are not most kids. You are a kid with a really neat book that is teaching you about number patterns. The count-by 3 sequence goes:

$$3, 6, 9, 12, 15, 18, 21, 24, 27, 30.$$

If you have to count things 3 at a time, that is how to do it. Maybe you eat with a fork and each fork has three tynes (the pokey things). Then if you want to know the number of tynes you can count-by 3s.

Now usually when people count-by 3s they forget where they are, so they might say something like $3, 6, 9, 12, 15, 18, 20, 22$. YIKES! Don't do that! When you count-by remember what you are counting by.

2.4 Count-by 4s



Counting by 4s is a little harder than counting by 2s but easier than counting by 3s. That is because you can count-by 2 and skip every other one. The sequence goes like this:

$$4, 8, 12, 16, 20, 24, 28, 32, 36, 40.$$

If you are counting the number of legs on the cows as you pass them by on the highway, you would count-by 4s. If you have a bunch of squares and you want to count the corners, you count-by 4s.

One reason that it is good to count-by 4s is that some times you want to know if 4 divides evenly into a big number. If you can count by 4s all the way to 100, then you can tell if 4 divides into the number evenly by checking if the last 2 digits can be gotten to by counting by 4s. Let me show you:

Count by 4s to 100.

$$4, 8, 12, 16, 20,$$

$$24, 28, 32, 36, 40,$$

$$44, 48, 52, 56, 60,$$

$$64, 68, 72, 76, 80,$$

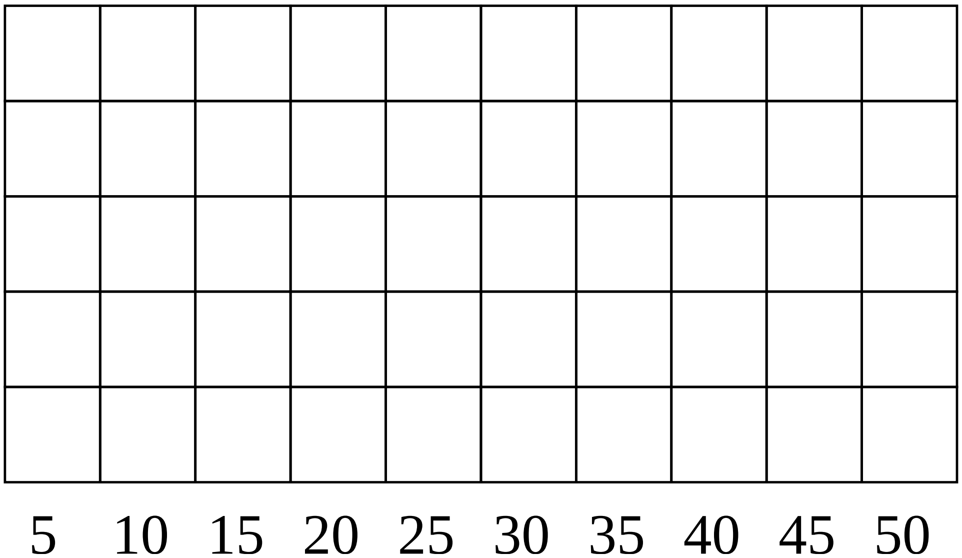
$$84, 88, 92, 96, 100.$$

Now you can see that 4 divides evenly into

$$472, 7872, 972$$

because the last two digits are 72 and you can get to 72 by counting by 4s.

2.5 count-by 5s



The next sequence is the easiest of them all. I bet even your older brothers and sisters know this one:

$$5, 10, 15, 20, 25, 30, 35, 40, 45, 50$$

The reason that count-by 5s is so easy is because the number names rhyme. I once wondered if there was a way of naming numbers so that each count-by sequence was a rhyming sequence. Well in a way they all are, but sometimes it is hard to hear the rhyming pattern because the sequence goes too long before the pattern repeats. We'll see that when we get to count-by 7s.

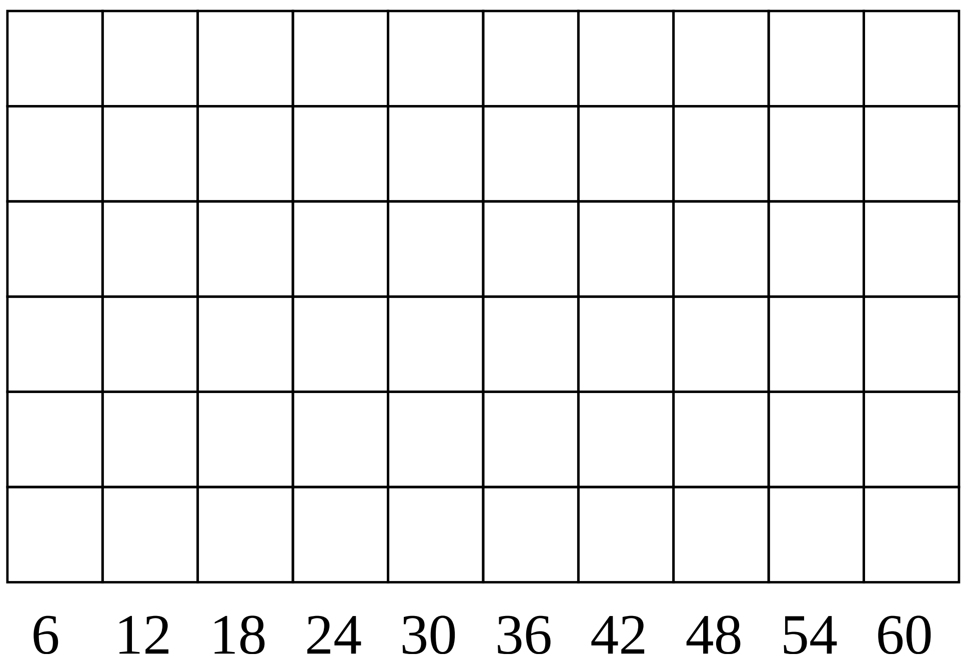
2.6 Review so far

The way I figure this, you probably read through the first sections and got to count-by 5s and stopped because things were pretty easy. These count-by sequences can be put into a little table. It looks like this:

1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50

As we go on we are going to learn a bunch of other count-by sequences. Some of these are not as easy as the ones you have learned so far. Be careful and remember the sequences as I have shown them to you.

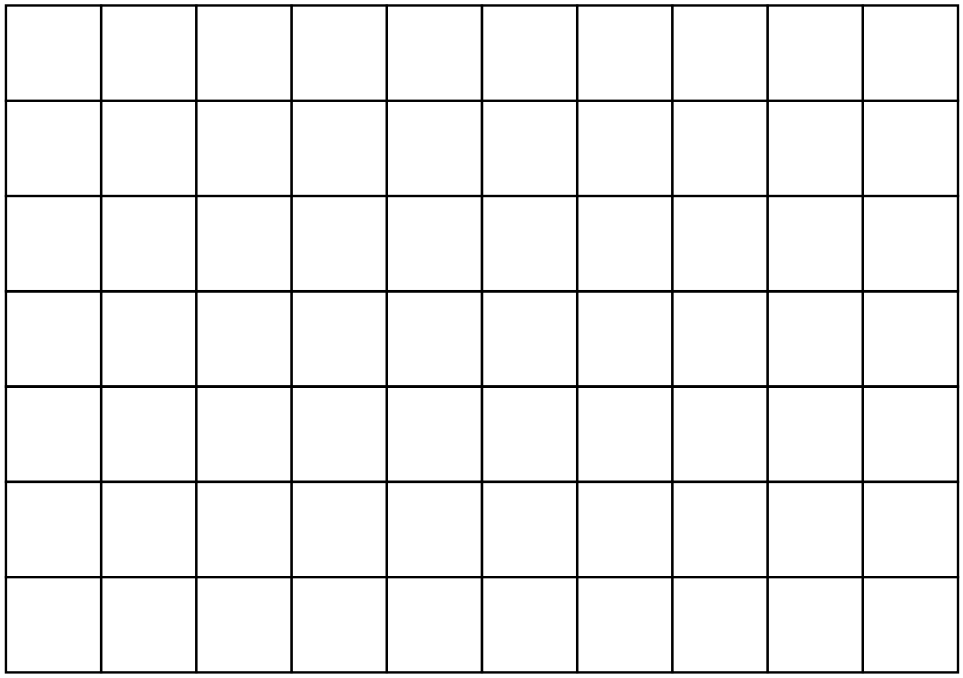
2.7 Higher count-bys



OK now we are going to count-by 6s:

$$6, 12, 18, 24, 30, 36, 42, 48, 54, 60.$$

You can count-by 3s and skip every other one. I think it is worth just learning the seqence. The ending numbers go $6, 2, 8, 4, 0, 6, 2, 4, 8, 0$. That is one of those rhymes I was telling you about. I am going to say more about that in a little while.



7 14 21 28 35 42 49 56 63 70

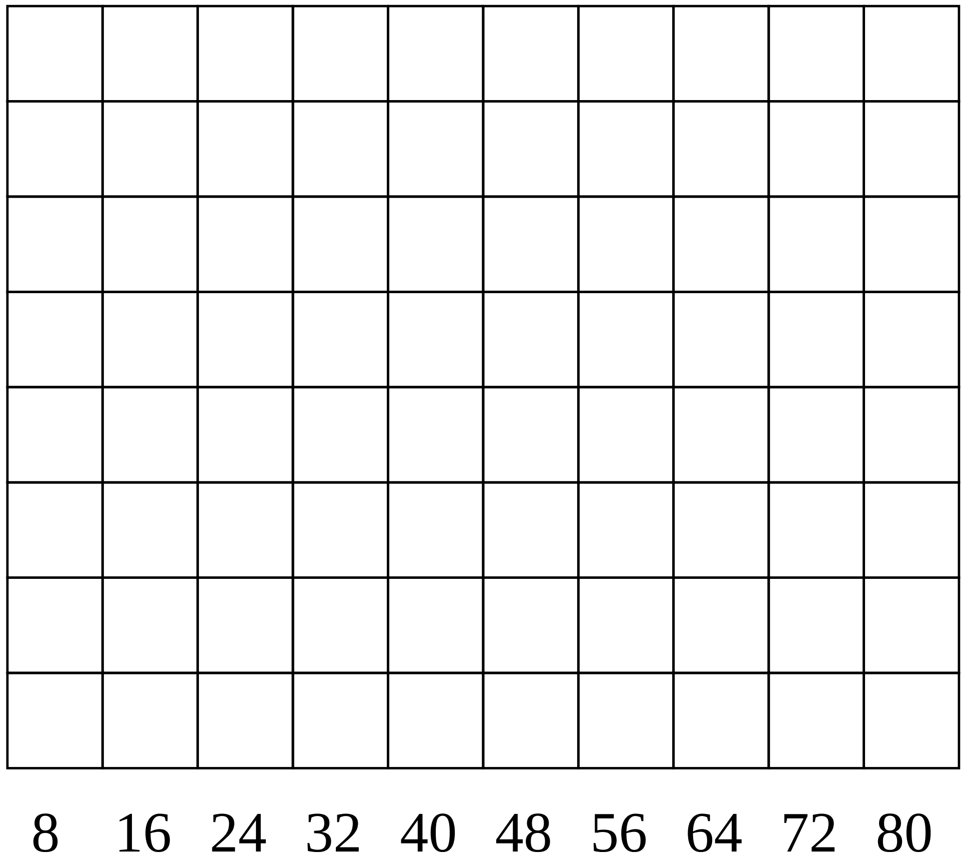
Counting by 7 is the trickiest one for me. The numbers at the end never repeat:

$$7, 14, 21, 28, 35, 42, 49, 56, 63, 70.$$

Let me show you our count-by table so far.

1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70

One thing you should see in this table is that the columns look sort of like the rows, but the columns aren't finished yet. The *columns* run up and down and the *rows* run left and right.

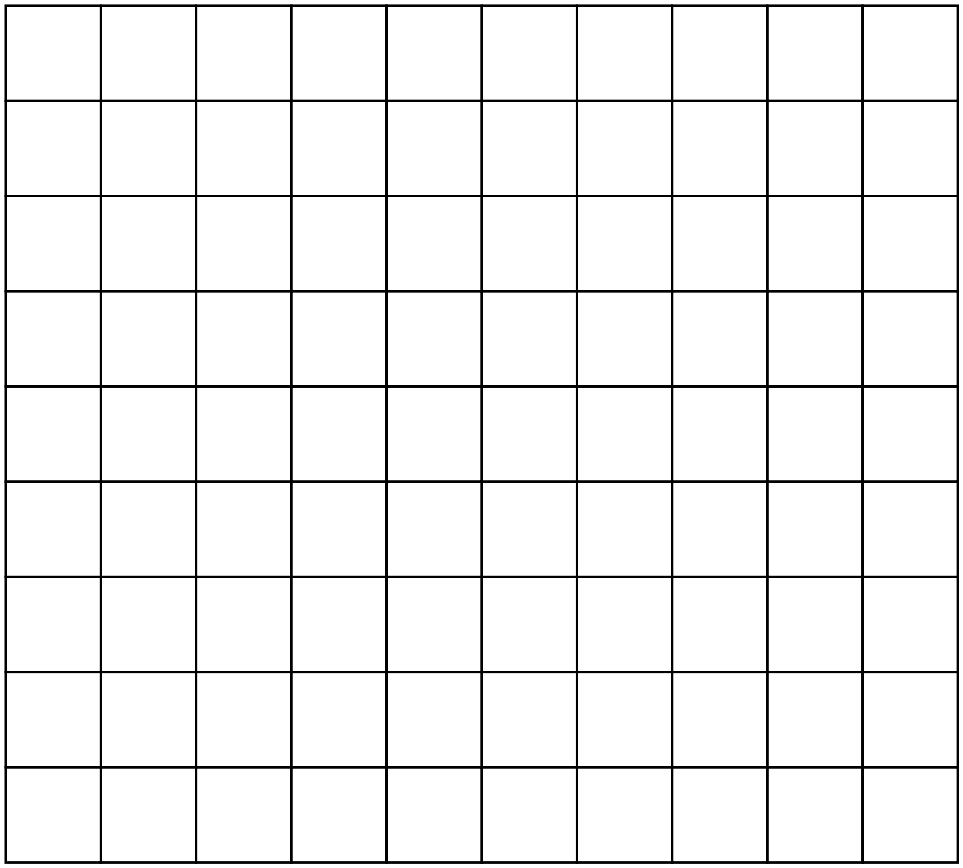


Let's go ahead and count-by 8s:

$$8, 16, 24, 32, 40, 48, 56, 64, 72, 80$$

Every number in the count-by 8s sequence is in one of our other sequences EXCEPT FOR 64, 72, and 80.

The number 64 is8 groups of 8 things, the number 72 is 8 groups of 9 things, and 80 is 8 groups of 10 things.



9 18 27 36 45 54 63 72 81 90

Now count-by 9s is the next easiest. That is because you just count-by 1s forwards and backwards at the same time

$$9, 18, 27, 36, 45, 54, 63, 72, 81, 90.$$

The "ones" place goes

$$9, 8, 7, 6, 5, 4, 3, 2, 1, 0$$

and the "ten's" place goes

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Let us look at the table with the count-by 10s all filled in.

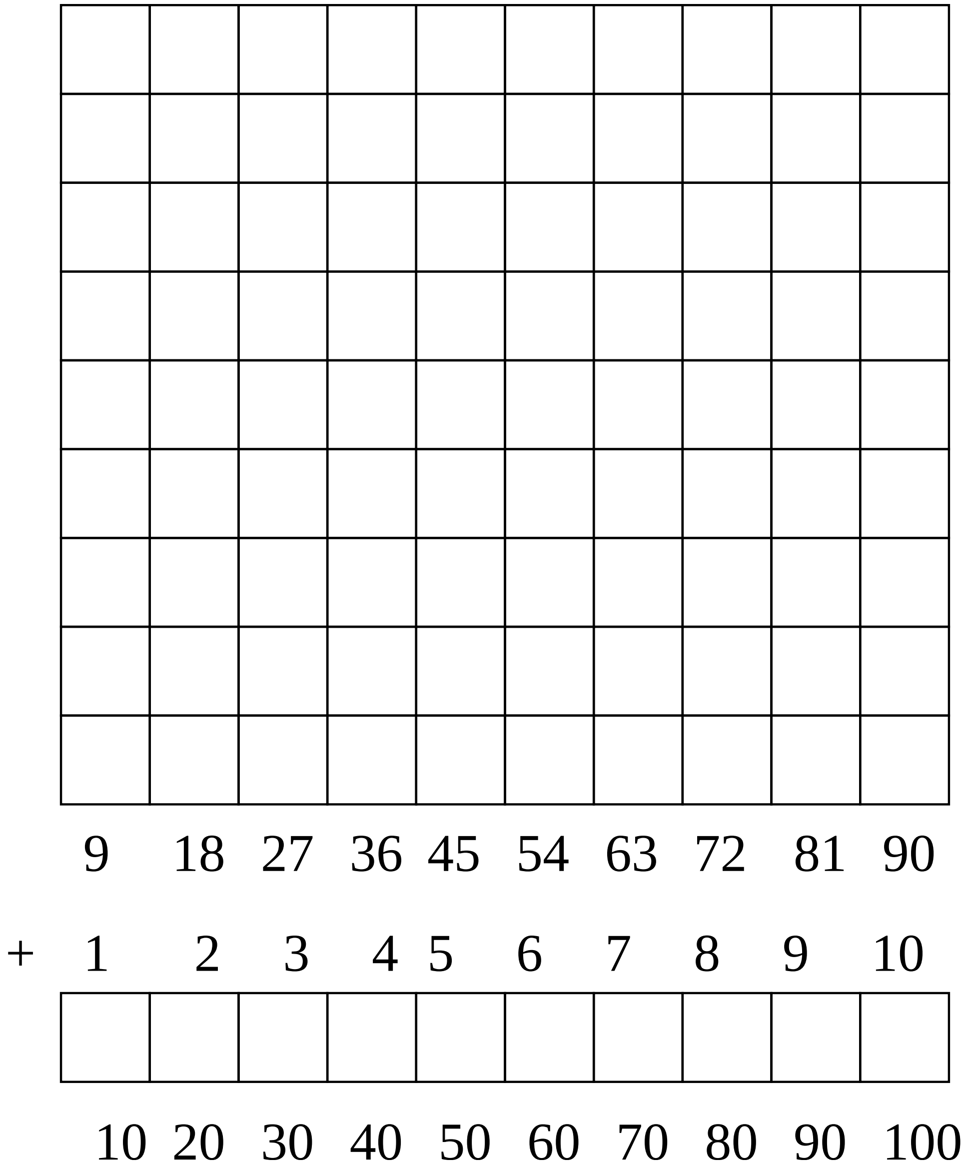
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

2.8 A few patterns

Numbers are filled with patterns, and lots of numbers can be put into nice shapes. The first thing I want to do in this section is to review the count-by 9s pattern and the count by 1s sequence.

1	1	2	3	4	5	6	7	8	9	10
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

When we add the count-by 1s to the count-by 9s we get the count by 10s. OK, it is not really hard to see why. We could have just put the blocks together.



2	2	4	6	8	10	12	14	16	18	20
8	8	16	24	32	40	48	56	64	72	80
10	10	20	30	40	50	60	70	80	90	100

But what if we did that with 2s and 8s. Same thing, right. OK but that means that count by 8s is like count by 2s backwards. Let's list the numbers backwards:

$$80, 72, 64, 56, 48, 40, 32, 24, 16, 8$$

The "ones" place is $0, 2, 4, 6, 8, \ldots$.

Now look at the 7s backwards:

$$70, 63, 56, 49, 42, 35, 28, 21, 14, 7, 0$$

Compare this to 3s forward

$$0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30.$$

The numbers in the 1 place are the same.

This works for 6s and 4s too. Try it!

When I was a kid people used to tell me I should learn my count-bys backwards and forwards. Now I know why. Because if I know my 3s backwards I can figure out my 7s forwards. A lot of times adults tell you to do things, and you don't really want to do them. When they are good things, like things about learning, then you should try. The adult may not tell you why he or she wants you to learn things, or they may say, "Because its good for you." Sometimes adults forget why you should learn things.

Because I am a math professor, I always am learning new things. Learning math things helps me learn about how people think about the world. Science people think a lot about how the world is put together, and they use math to explain these things. The reason that they use math is because it is pretty simple, and it is logical. People think that the world should be logical even when it is very complicated. So if you can think about math every day, you can learn more about things around you. You might even learn about how your brain works!

The last thing that I want to tell you in this section is what other people call count-bys. They call them your multiplication table. The table looks likes this:

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

It's just that same thing as our count-by table except that we called it times. If we want to know what 7 groups of 6 is, then we count by 6: $6, 12, 18, 24, 35, 42$ until we have 7 numbers in our list. To know this might be useful if there were 7 of us and each of us had 6 dollars in our pockets. Then we could know that we had 42 dollars among us.

The other thing that I want to tell you about count-bys is that usually we just memorize the table. But the easiest way of memorizing it is to keep counting-by. The reason is that if you do your count-bys you remember a lot of the table. As I go on, I am going to show more tricks. These help ups with our count-bys, but they also are really good when we multiply big numbers like 13 and 17.

Chapter 3

Square Numbers

The square numbers are really special. They are when you count by a number that many times. In other words,

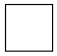
$$1 \times 1 = 1,$$

$$2 \times 2 = 4,$$

$$3 \times 3 = 9,$$

$$4 \times 4 = 16$$

When you have a big square, the number of little squares in it is a square number.



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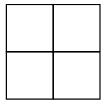

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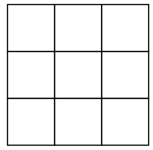

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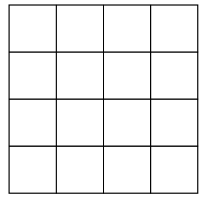

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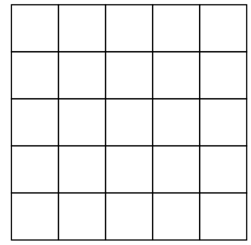

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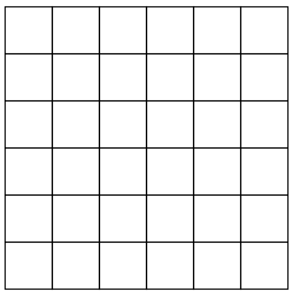

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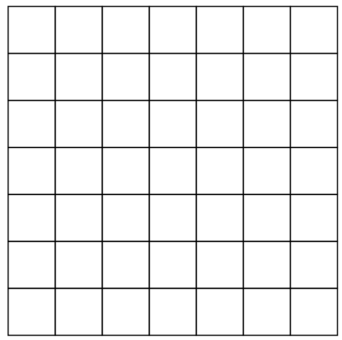

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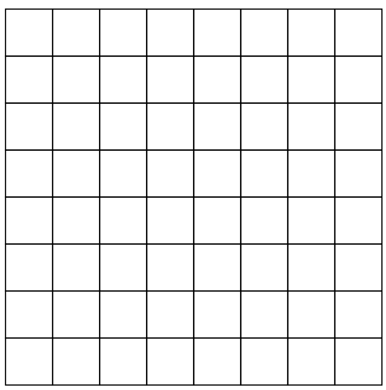

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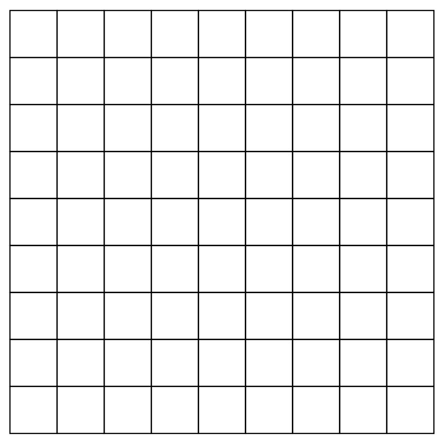

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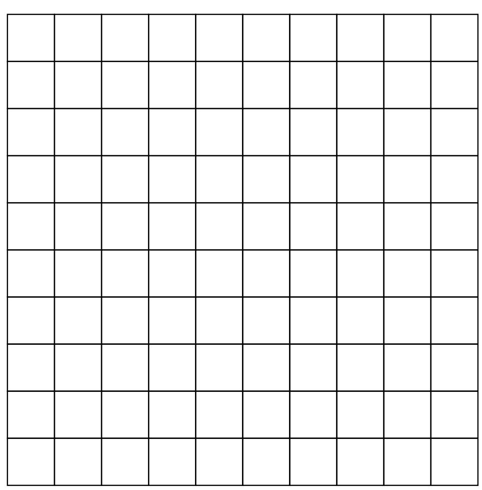


100

I like the list of square numbers. I like it so much I am going to tell you the first 25 square numbers.

$$1, 4, 9, 16, 25,$$

$$36, 49, 64, 81, 100,$$

$$121, 144, 169, 196, 225,$$

$$256, 289, 324, 361, 400,$$

$$441, 484, 529, 576, 625.$$

Just to help you out, this means that

$$13 \times 13 = 169.$$

There are some neat things here $14 \times 14 = 196$. It is just a coincidence that that has the digits switched from 13×13, but that coincidence makes it easy to remember. In the same way,

$$12 \times 12 = 144$$

and

$$21 \times 21 = 441.$$

If we wanted to we could count-by 21 all the way there:

$$21, 42, 63, 84, 105, 126, 147, 168, 189, 210,$$

$$231, 252, 273, 294, 315, 336, 357, 378, 399, 420, 441.$$

There might be an easier way of getting there, though.

Chapter 4

L-shaped numbers

Let me just start out this chapter telling you there is no such thing as an L-shaped number. It is just a word I made up for this lesson. The real word is *odd number*. But I want you to think about an odd number as begin L-shaped.

Every number is either even or odd. The even numbers are the numbers that we get when we count-by 2. The even numbers are

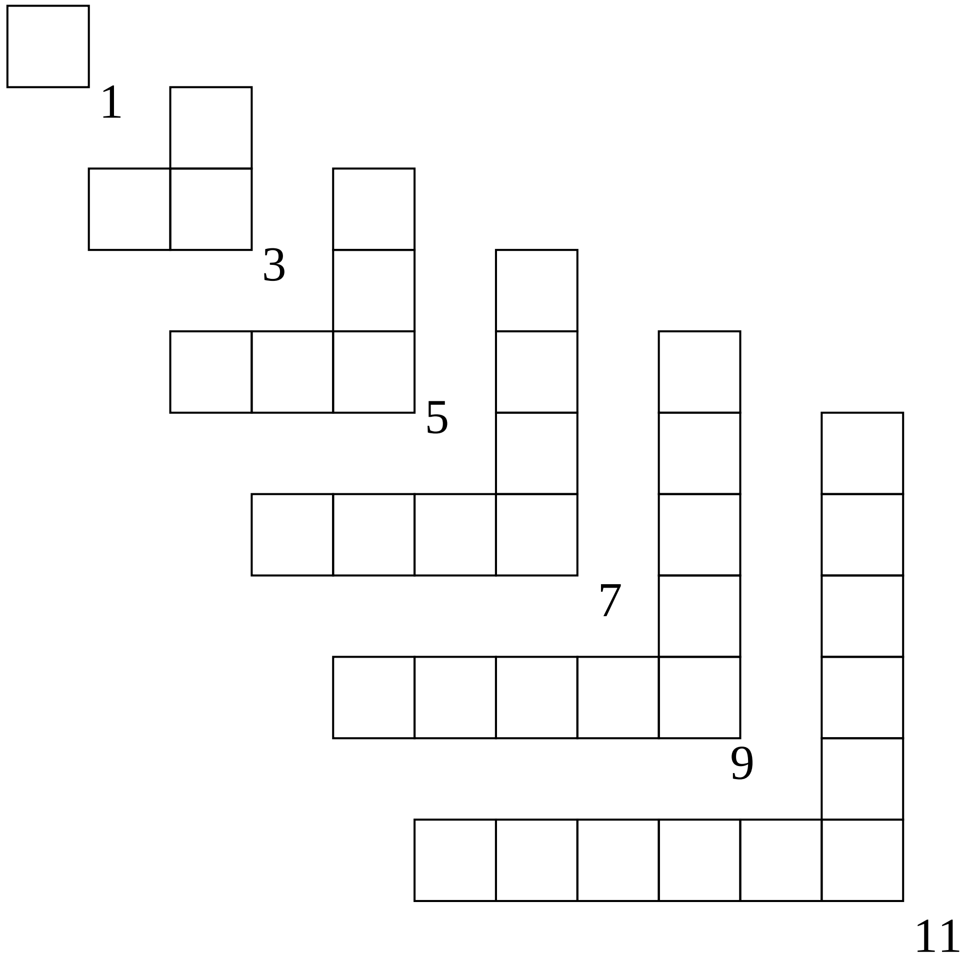
$$2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots \ .$$

Remember that the 3 dots means the sequence continues forever. The $4th$ dot is a period. It means the sentence ends there. The even numbers end in $2, 4, 6, 8,$ and 0. Everything else is odd. By the way, this means that 0 is even.

The odd numbers are

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \ldots \ .$$

They go on forever too. I want to arrange each odd number (except 1) in the shape of an L, like I do in this picture.



The L-shape has an odd number of squares on it. Every odd number is one more than an even number.

Now let us add up our odd numbers in order:

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

$$1 + 3 + 5 + 7 + 9 = 25$$

Hey, each time we add on another odd number we get a square number! That is because our odd numbers are all L-shaped. So we can put them back-to-back to make a square.

Let me show you an arithmetrick. An *arithmetrick* is a cool way of doing a really complicated computation. Suppose some mean teacher (like me) came up to you and said, “Young person, add up all the odd numbers from 1 to 57.”

Well, you could say, "I've read Professor Zap's book and I know it is a perfect square."

"But which perfect square," says the mean teacher.

Then you say, "Well $57 + 1 = 58$. And half of 58 is 29. So it must be 29×29."

"And exactly what is 29×29, young person?"

"Well, 30×30 is easy. That's 900. So the answer must be $900 - 59 = 841$." And you would be right!

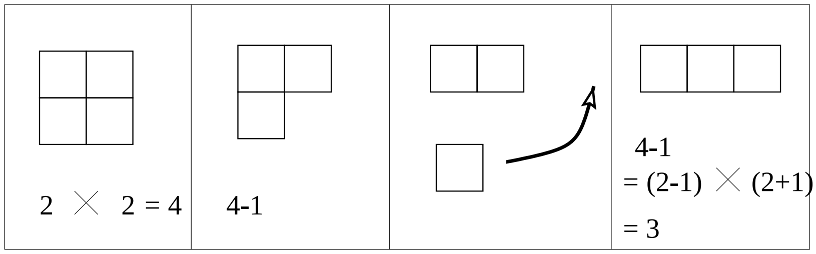
One of the things, you should know about people who write books is that they sometimes make the examples they work out easy because they already know the answer. Well in this case, I took a number that I thought would be really hard. I mean, I almost don't know anything about the number 57. So I started from 57 to write the problem.

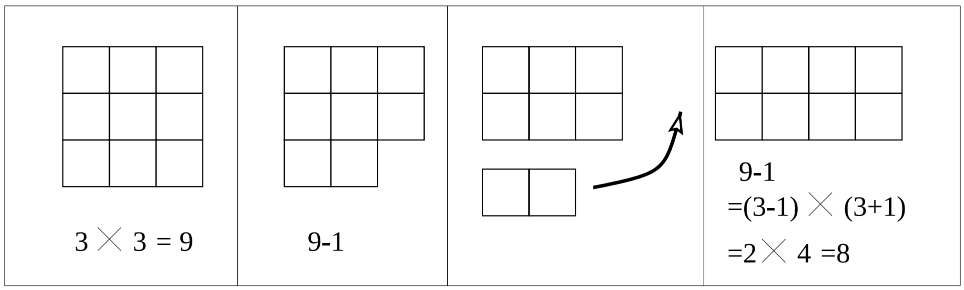
Chapter 5

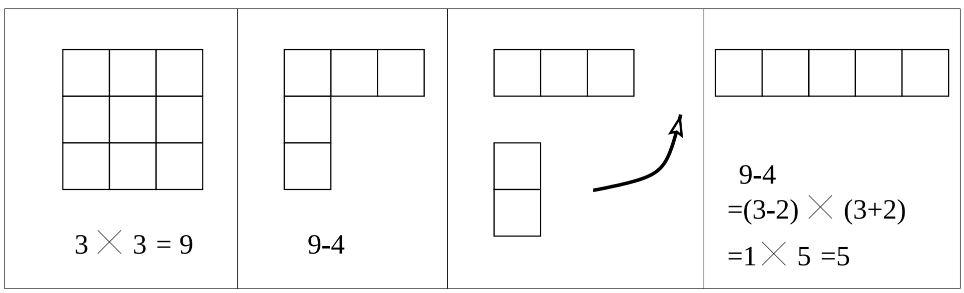
Rectangle Numbers

What happens when you take a square bite out of a square piece of cake? Let us try some experiments.

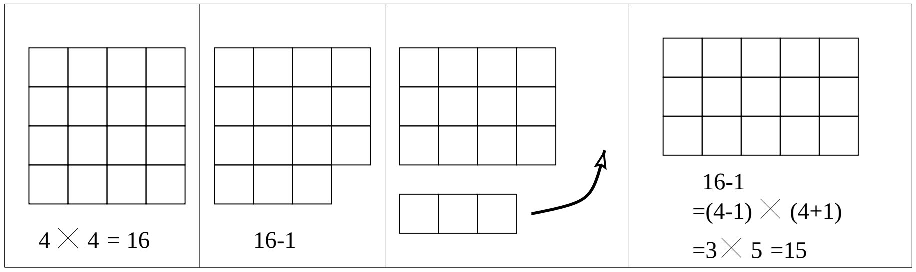
If our square cake has 4 pieces, and we take away 1 we have 3 left —an L-shaped number. If the cake has 9 pieces and we take away 1 we have 8 pieces left. The square number 9 is 3×3. The number 8 is 2×4. If we take the square 4 away from the square 9, then we get the L-shape 5. Let's keep going.

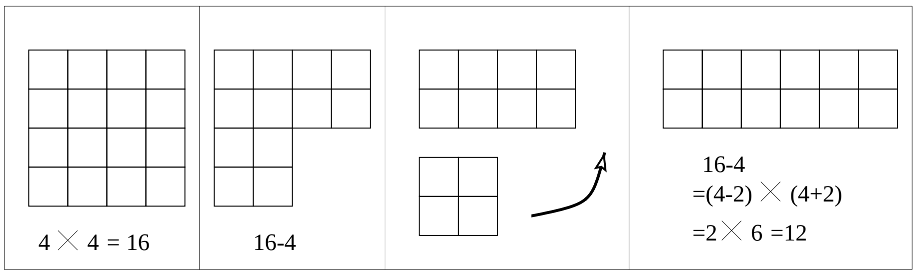


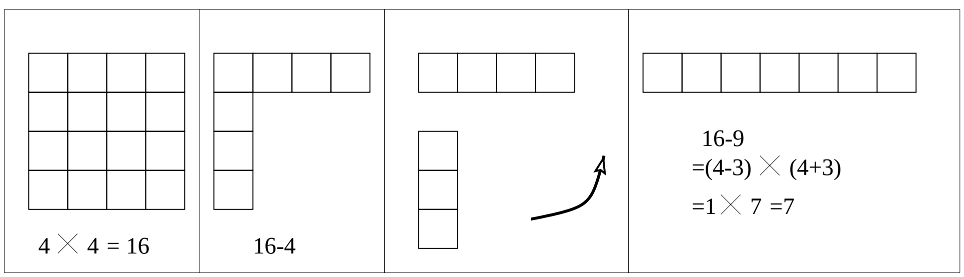




Take 1 away from 16. The number 16 is 4×4. And $16 - 1 = 15$. The left over cake is sized 3×5.







If we take 4 away from 16, then we get 12. Or

$$16 - 4 = 12.$$

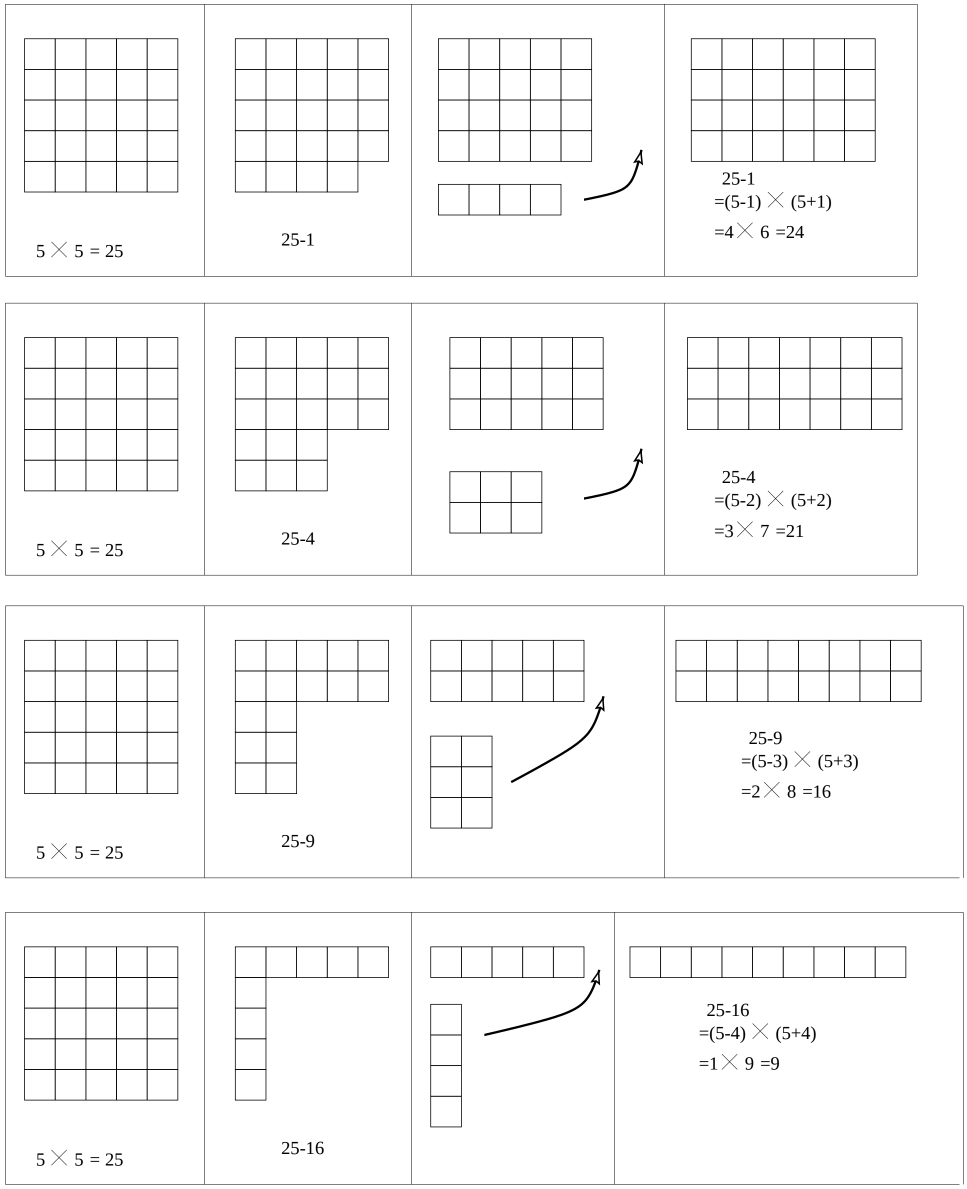
On one hand, $16 = 4 \times 4$, and $4 = 2 \times 2$.

$$4 \times 4 - 2 \times 2 = 2 \times 6 = (4 - 2) \times (4 + 2).$$

If we take 9 away from 16 we get 7. On the one hand, $16 = 4 \times 4$, and $9 = 3 \times 3$.

$$4 \times 4 - 3 \times 3 = 1 \times 7 = (4 - 3) \times (4 + 3).$$

Do you see the pattern in the numbers? What about in the pictures?



Let's go again. The difference in 25 and 1 is 24. The number 25 is 5×5; meanwhile $24 = 4 \times 6$. The difference between 25 and 9 is 16 which is $5 - 3 = 2$ times $5 + 3 = 8$.The pattern is that the difference between any two squares is a rectangle.

I'll say it again:

The difference between two square numbers is a rectangular number. The sides of the rectangle are the sum and difference between the two square roots.

You may have heard of square roots, and thought they are only for big kids. But you can tell me some square roots even now. You should know that the square root of 4 is 2 because $2 \times 2 = 4$. The square root of 144

is 12 because $12 \times 12 = 144$. In fact, if you have learned all your square numbers up to 625, then you know all those square roots.

Let me tabulate some facts to help you see that the difference between two squares is a rectangle.

Squares	Differences	Rectangles	Simplified
100	$100 - 1 = 99$	$= (10 - 1) \times (10 + 1)$	$9 \times 11 = 99$
–	$100 - 4 = 96$	$= (10 - 2) \times (10 + 2)$	$8 \times 12 = 96$
–	$100 - 9 = 91$	$= (10 - 3) \times (10 + 3)$	$7 \times 13 = 91$
–	$100 - 16 = 84$	$= (10 - 4) \times (10 + 4)$	$6 \times 14 = 84$
–	$100 - 25 = 75$	$= (10 - 5) \times (10 + 5)$	$5 \times 15 = 75$
–	$100 - 36 = 64$	$= (10 - 6) \times (10 + 6)$	$4 \times 16 = 64$
–	$100 - 49 = 51$	$= (10 - 7) \times (10 + 7)$	$3 \times 17 = 51$
–	$100 - 64 = 36$	$= (10 - 8) \times (10 + 8)$	$2 \times 18 = 36$
–	$100 - 81 = 19$	$= (10 - 9) \times (10 + 9)$	$1 \times 19 = 19$
81	$81 - 1 = 80$	$= (9 - 1) \times (9 + 1)$	$8 \times 10 = 80$
–	$81 - 4 = 77$	$= (9 - 2) \times (9 + 2)$	$7 \times 11 = 77$
–	$81 - 9 = 72$	$= (9 - 3) \times (9 + 3)$	$6 \times 12 = 72$
–	$81 - 16 = 65$	$= (9 - 4) \times (9 + 4)$	$5 \times 13 = 65$
–	$81 - 25 = 56$	$= (9 - 5) \times (9 + 5)$	$4 \times 14 = 56$
–	$81 - 36 = 45$	$= (9 - 6) \times (9 + 6)$	$3 \times 15 = 45$
–	$81 - 49 = 32$	$= (9 - 7) \times (9 + 7)$	$2 \times 16 = 32$
–	$81 - 64 = 17$	$= (9 - 8) \times (9 + 8)$	$1 \times 17 = 17$
64	$64 - 1 = 63$	$= (8 - 1) \times (8 + 1)$	$7 \times 9 = 63$
–	$64 - 4 = 60$	$= (8 - 2) \times (8 + 2)$	$6 \times 10 = 60$
–	$64 - 9 = 55$	$= (8 - 3) \times (8 + 3)$	$5 \times 11 = 55$
–	$64 - 16 = 48$	$= (8 - 4) \times (8 + 4)$	$4 \times 12 = 48$
–	$64 - 25 = 39$	$= (8 - 5) \times (8 + 5)$	$3 \times 13 = 39$
–	$64 - 36 = 28$	$= (8 - 6) \times (8 + 6)$	$2 \times 14 = 28$
–	$64 - 49 = 15$	$= (8 - 7) \times (8 + 7)$	$1 \times 15 = 15$
49	$49 - 1 = 48$	$= (7 - 1) \times (7 + 1)$	$6 \times 8 = 48$
–	$49 - 4 = 45$	$= (7 - 2) \times (7 + 2)$	$5 \times 9 = 45$
–	$49 - 9 = 40$	$= (7 - 3) \times (7 + 3)$	$4 \times 10 = 40$
–	$49 - 16 = 33$	$= (7 - 4) \times (7 + 4)$	$3 \times 11 = 33$
–	$49 - 25 = 24$	$= (7 - 5) \times (7 + 5)$	$2 \times 12 = 24$
–	$49 - 36 = 13$	$= (7 - 6) \times (7 + 6)$	$1 \times 13 = 13$
36	$36 - 1 = 35$	$= (6 - 1) \times (6 + 1)$	$5 \times 7 = 35$
–	$36 - 4 = 32$	$= (6 - 2) \times (6 + 2)$	$4 \times 8 = 32$
–	$36 - 9 = 27$	$= (6 - 3) \times (6 + 3)$	$3 \times 9 = 27$
–	$36 - 16 = 20$	$= (6 - 4) \times (6 + 4)$	$2 \times 10 = 20$
–	$36 - 25 = 11$	$= (6 - 5) \times (6 + 5)$	$1 \times 11 = 11$
25	$25 - 1 = 24$	$= (5 - 1) \times (5 + 1)$	$4 \times 6 = 24$
–	$25 - 4 = 21$	$= (5 - 2) \times (5 + 2)$	$3 \times 7 = 21$
–	$25 - 9 = 16$	$= (5 - 3) \times (5 + 3)$	$2 \times 8 = 16$
–	$25 - 16 = 9$	$= (5 - 4) \times (5 + 4)$	$1 \times 9 = 9$
16	$16 - 1 = 15$	$= (4 - 1) \times (4 + 1)$	$3 \times 5 = 15$
–	$16 - 4 = 12$	$= (4 - 2) \times (4 + 2)$	$2 \times 6 = 12$
–	$16 - 9 = 7$	$= (4 - 3) \times (4 + 3)$	1×7

Wow! That is a big table, and it contains a bunch of information. What is says is that when you subtract one square from another, you get a number that can be put in the form of a rectangle. The pictures show the same thing. So if we look along the diagonal of the table of count-bys we see the perfect squares. I put brackets [] around these. Then just above and below the diagonal are numbers that are one less than a perfect square. Numbers like 8,15, 35, 48, 63, 80, and 99. These numbers are rectangles where the length and width differ by 2. I put braces {} around them.

×	1	2	3	4	5	6	7	8	9	10
1	[1]	2	{3}	4	5	6	7	8	9	10
2	2	[4]	6	{8}	10	12	14	16	18	20
3	{3}	6	[9]	12	{15}	18	21	24	27	30
4	4	{8}	12	[16]	20	{24}	28	32	36	40
5	5	10	{15}	20	[25]	30	{35}	40	45	50
6	6	12	18	{24}	30	[36]	42	{48}	54	60
7	7	14	21	28	{35}	42	[49]	56	{63}	70
8	8	16	24	32	40	{48}	56	[64]	72	{80}
9	9	18	27	36	45	54	{63}	72	[81]	90
10	10	20	30	40	50	60	70	{80}	90	[100]

Numbers like 12 and 21 that are in the table along the lines formed by the braces are rectangle numbers like $16 - 4 = (4 - 2) \times (4 + 2) = 2 \times 6$ or $(25 - 4) = (5 - 2) \times (5 + 2) = 3 \times 7$.

Because I know that $24 \times 24 = 576$, I know that $23 \times 25 = 575$ and $22 \times 26 = 572$. This first fact is great to know if you happen to have 23 quarters, then you know you have 25 cents less than six dollar. I don't have an immediate use for the other fact, but it still is nice to know!

Now you can look along those lines and you can see differences of squares show up in the table. Some people are really good at computing times tables (count-by tables) by thinking about the difference between two squares.

Chapter 6

Triangle Numbers

OK, I have been writing a lot about count-bys, about the differences between squares, and adding together odd numbers. Now let's get back to the first picture. How could we add

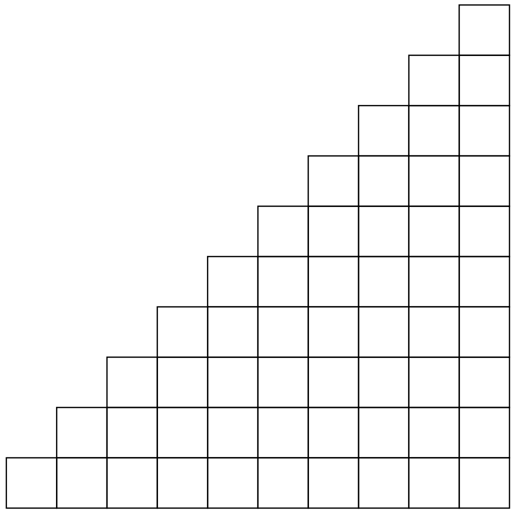
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10?$$

Well, the easy way is to write this forward and backward and then add things together:

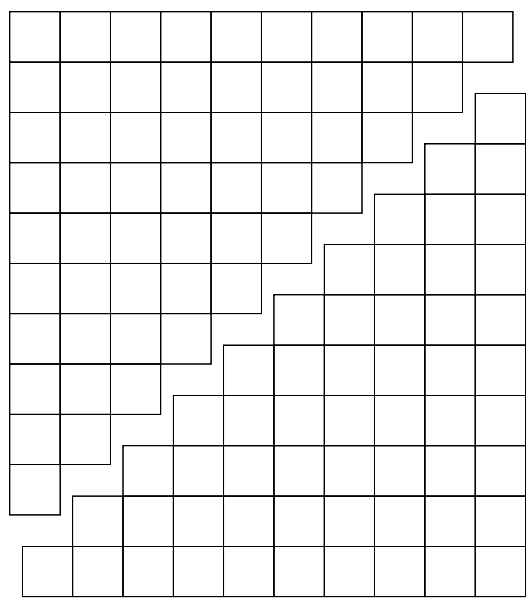
$$1\ +\ 2 +\ 3 +\ 4 +\ 5 +\ 6 +\ 7 +\ 8 +\ 9 + 10$$

$$10 +\ 9 +\ 8 +\ 7 +\ 6 +\ 5 +\ 4 +\ 3 +\ 2 +\ 1$$

$$11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11$$



1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10



1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10

+ 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1

=11+ 11 + 11+ 11 + 11+11 +11+11+11 +11=110

So,

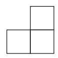
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 =55

But then we can count-by 11s and get $11, 22, 33, 44, 55, 66, 77, 88, 99, 110$.

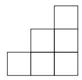
The answer is half of 110, and the number halfway in between is 55. The same trick works if we add all the numbers up to 100. But now we have to count by 101s. That is pretty tough. Halfway there is 5050.

The idea is that a triangle can be thought of as half of a rectangle, and rectangles can be gotten by count-bys.

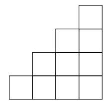
I have drawn some pictures of triangles, shown you how to put them together to get rectangles. If the triangle has its base of side length, 16, then the rectangle is of size 16×17. Since I know that $16 \times 16 = 256$ — it is a square number — I can add 16 to this to get 272. Then I take half of that to get 136. There are 136 squares in a triangular stair-case with base 16.



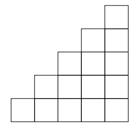
1 + 2



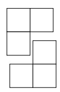
1 + 2 +3



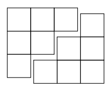
1 + 2 + 3 + 4



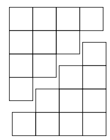
1 + 2 + 3 + 4 +5



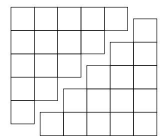
1 + 2
+2+1=6



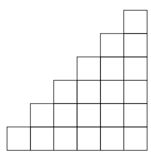
1 + 2 +3
+3+2+1=12



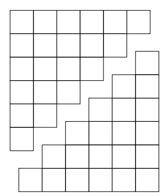
1 + 2 + 3 + 4
+4+3+2+1=20



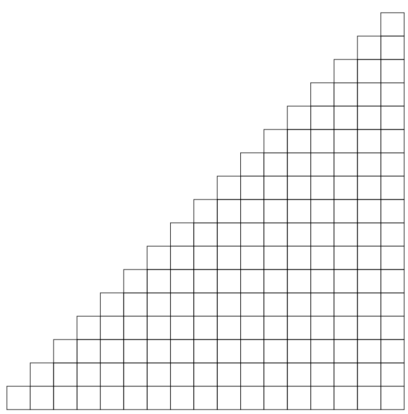
1 + 2 + 3 + 4 +5
+5+4+3+2+1=30



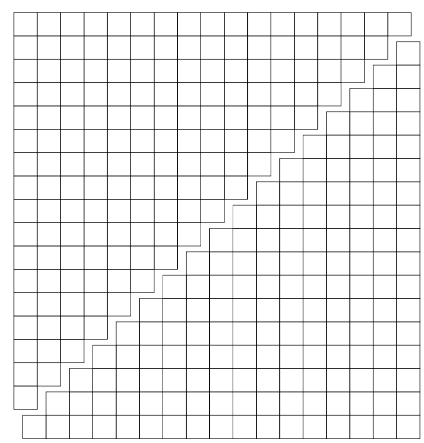
1 + 2 + 3 + 4 + 5 +6



1 + 2 + 3 + 4 + 5 +6
+6+5+4+3+2+1=42



1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 +13 + 14 + 15 + 16



1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 +13 + 14 + 15 + 16
+ 16 + 15 + 14 + 13 + 12 + 11+ 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1
=17+ 17 + 17 + 17 + 17 + 17 + 17 + 17 + 17 +17 + 17 + 17 + 17 + 17 + 17 +17 + 17

= 289+17 =306

Chapter 7

Exercises

Every morning, count-by 1s, 2s, 3s, $\dots$ 12s, up to 10×12.

After school, recite all of your perfect squares. On the first day, count

$$1, 4, 9, 16, 25.$$

On the second day count, $1, 4, 9, 16, 25, 36, 49, 64$. On the third day count by squares up to 144. On the fifth day count by squares up to 256.

In the shower or bath-tub, add $1 + 3$, $1 + 3 + 5$, $1 + 3 + 5 + 9$, as far as you can without pencil and paper.

After you finish your homework on a piece of paper, add $256 + 33$, $289 + 35$, $324 + 37$, $361 + 39$, $400 + 41$, $\dots$. Keep doing these until you are sure you know all the perfect squares up to 625.

Subtract $100 - 1, 100 - 4, 100 - 9, \dots$ .Then ask the questions what is 9×11, 8×12, 7×13. Keep doing this down to $4 - 1$. The next day start with $121 - 1$, $121 - 4$, and so forth. Do this all the way until you can figure out everything near 625.

If you can't subtract easily in your head, then think about the corresponding difference of squares. You can always work on paper. Learn these numbers before you use a caculator.

When you can do ALL of these exercises really quickly, then start thinking about perfect cubes: 1, 8, 27, 64, 125, 216. See if you can find more patterns! Good Luck!