Coloring Knot Diagrams, Knotted Surfaces, and Quandles

J. Scott Carter

Masahico Saito

1. Recent Collaborators

2. Fox Colorings

3. Braid Example

4. Burau Matrices and Alexander Polynomials

5. Quandles

6. Key Examples

7. Counting Colorings

8. More General Functions

9. Knotted Surfaces

Recent other Collaborators

Mohammed Elhamdadi

Matias Graña

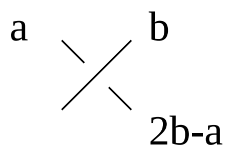
Dan Jelsovsky

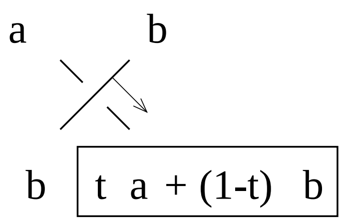
Seiichi Kamada

Laurel Langford

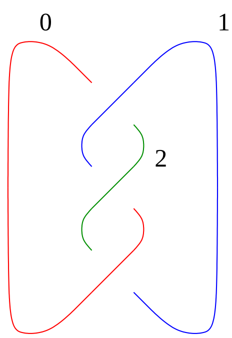
Shin Satoh

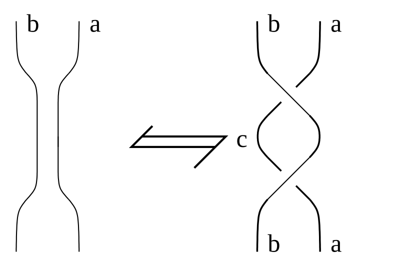
Fox Colorings

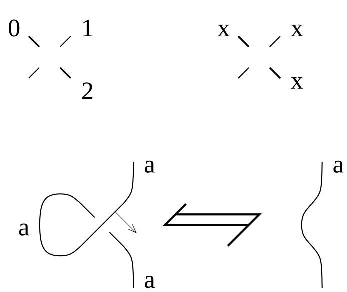


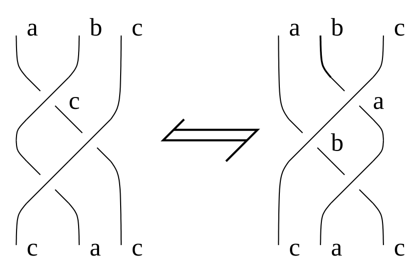


$$(a,b)\begin{bmatrix}0 & t\\ 1 & 1-t\end{bmatrix}$$

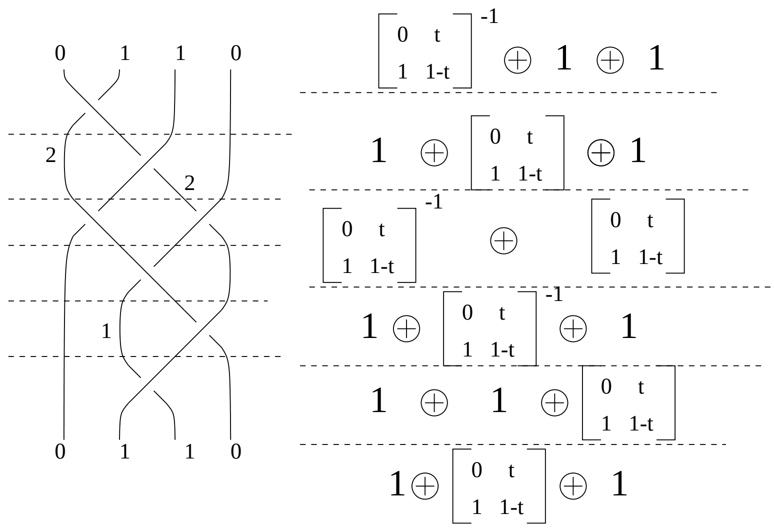








Braid Example



Burau Matrices and Alex. Polyn.

$$\begin{bmatrix} \frac{(1-t)^2}{t^2} & t & 1+\frac{(1-t)^2}{t^3}-t & -\left(\frac{1-t}{t^3}\right)+\frac{1-t}{t} \\ -\left(\frac{1-t}{t}\right) & 0 & -\left(\frac{1-t}{t^2}\right) & t^{-2} \\ t & (1-t)\,t^2 & (1-t)^2\,t & (1-t)^2 \\ 0 & (1-t)\,t^2 & t+(1-t)^2\,t & (1-t)^2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -2 & 0 \\ 2 & 0 & -2 & 1 \\ -1 & 2 & -4 & 4 \\ 0 & 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} b \\ a \\ a \\ b \end{bmatrix} = \begin{bmatrix} -3a+4b \\ -2a+3b \\ -2a+3b \\ -3a+4b \end{bmatrix}$$

- σ — an n-string braid.

- $B(\sigma)$ — the $n \times n$ matrix exemplified above.

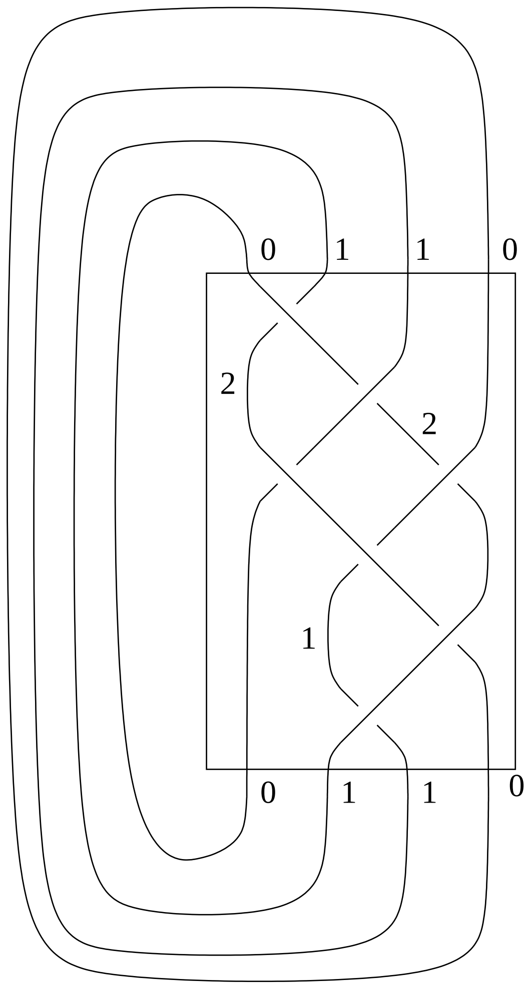
- Consider the matrix of $(n-1) \times (n-1)$ minors of $B(\sigma) - I$

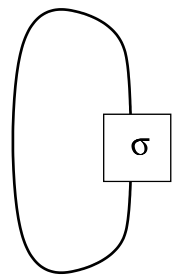
- $\Delta(\sigma)$ any one entry of this matrix.

- For Example, $\Delta(t) = 2 - 3\,t + 3\,t^2 - 3\,t^3 + 2\,t^4$

- NB: Δ is well def'd. up to $\pm t^{\pm 1}$.

- $\Delta(t)$ is called the Alexander Polynomial of the knot $\widehat{\sigma}$.





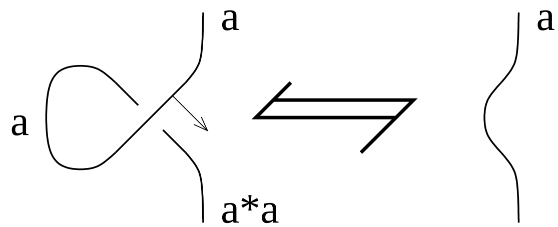
$\hat{\sigma}$

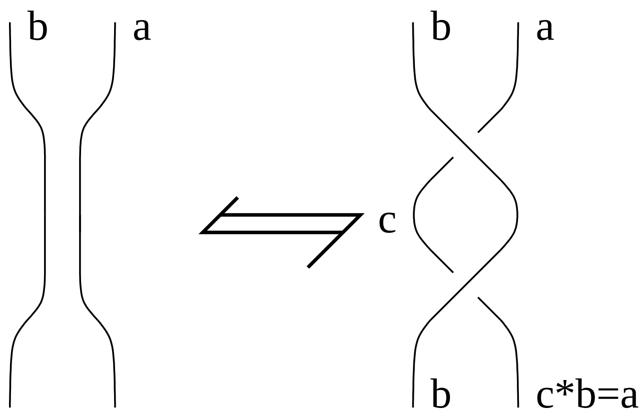
- $|\Delta(-1)|$ called the determinant of the knot, related to possible colorings.

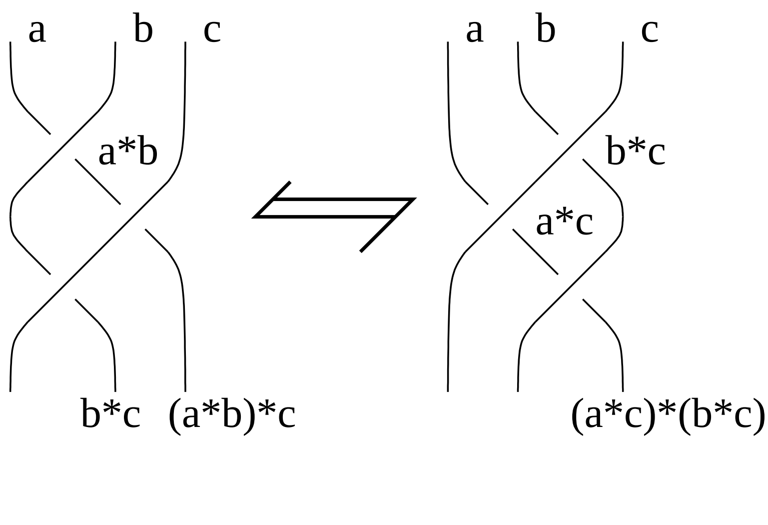
Quandles

A QUANDLE is a set, Q, with a binary operation $*$ defined such that

- $a * a = a$
- $\forall a, b \in Q \ \exists ! c \in Q$ s.t. $a = c * b$
- $(a * b) * c = (a * c) * (b * c)$





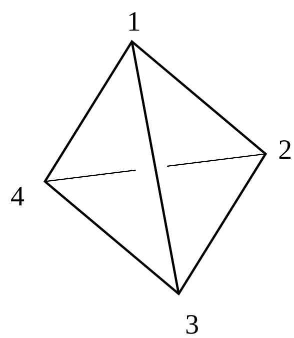


Examples:

- $a * b = ta + (1-t)b$

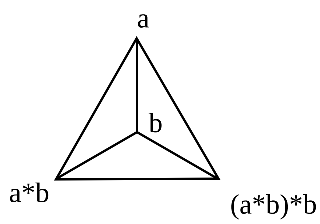
- at $t = -1$, $a * b = 2b - a$ — read modulo n

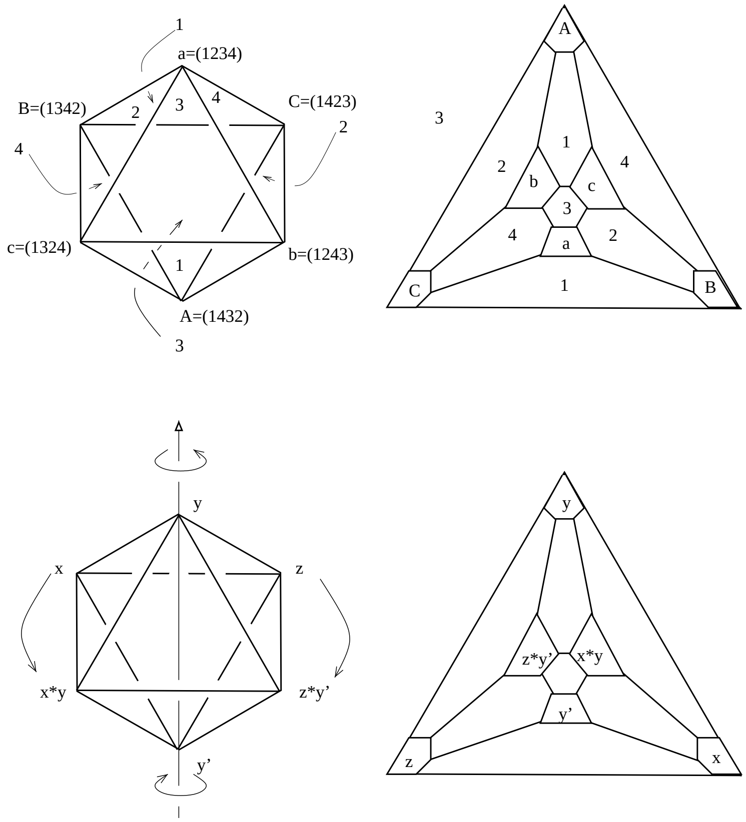
- More generally, $S \subset G$— a group. $bab^{-1} \in S$ if $a, b \in S$. On S define $a * b = bab^{-1}$.



a*b= rotate a counter clockwise
about the vertex b

Row*Col	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4





Row*Col	a	b	c	A	B	C
a	a	c	B	a	C	b
b	C	b	a	c	b	A
c	b	A	c	B	a	c
A	A	C	b	A	c	B
B	c	B	A	C	B	a
C	B	a	C	b	A	C

Row*Col	α	β	γ	α	β	γ
α	α	γ	β	α	γ	β
β	γ	β	α	γ	β	α
γ	β	α	γ	β	α	γ
α	α	γ	β	α	γ	β
β	γ	β	α	γ	β	α
γ	β	α	γ	β	α	γ

The example points to the possibility of *Quandle Extensions*.

- S is a set; X is a quandle

- $\alpha : X \times X \to S^{S \times S}$

- Define a product $*$ on $S \times X$ via

$$(s,x)*(t,y) = (\alpha_{(x,y)}(s,t), x*y).$$

- Direct calculation shows

$$((s,x)*(t,y))*(u,z))$$

$$= ((s,x)*(u,z))*((t,y)*(u,z))$$

$$\Leftrightarrow$$

$$\alpha_{(x*y,z)}(\alpha_{(x,y)}(s,t),u)$$

$$= \alpha_{(x*z),(y*z)}(\alpha_{(x,z)}(s,u), \alpha_{(y,z)}(t,u))$$

- This can be verified via Reid. moves.

- $\Longrightarrow$ there is a cohomology theory of quandles analogous to group cohomology.

Counting Colors

- Important knot invariant: the number of colorings by a fixed quandle.

- ¿Why important? Graña and Pregel, Lopes have used colorings and generalizations from cohom. thy to distinguish many classical knots and knotted surfaces.

- One way to count

  — V a vector space with basis X, a quandle.

  — $R^{a,b}_{c,d}$ a map $V^{\otimes 2} \rightarrow V^{\otimes 2}$

  — $R^{a,b}_{c,d} = \begin{cases} 1 & \text{if } b = c \ \& \ d = a * b \\ 0 & \text{else} \end{cases}$

Then quandle rule III $\Rightarrow$ R satisfies the YBE:

$$R^{a,b}_{d,e} R^{e,c}_{g,h} R^{d,g}_{i,j} = R^{b,c}_{d,e} R^{a,d}_{i,g} R^{g,e}_{j,h}$$

Then the number of colorings is evaluated as a state-sum invariant using this R matrix.

$R^{a,b}_{c,d}$ can be written as $\phi(a,b)$. Then

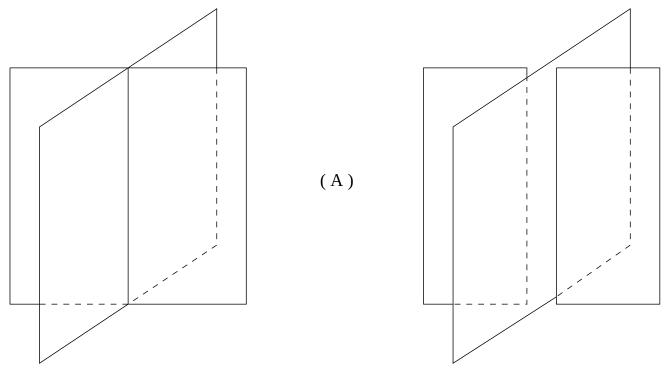
$$\phi(a,b)\phi(a*b,c)\phi(b,c) = \phi(b,c)\phi(a,c)\phi(a*c,b*c)$$

Observations

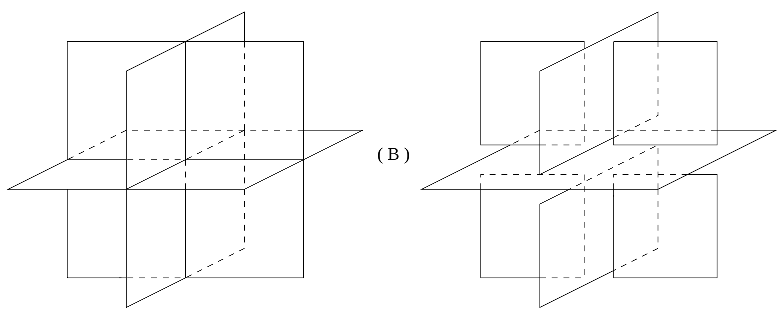
— This is a cocycle condition similar to that for α def'd above.

— There is a cohom. thy [FRS] for which knot invariants can be constructed.

— An appropriate version def's inv. for knotted surfaces.



(A)



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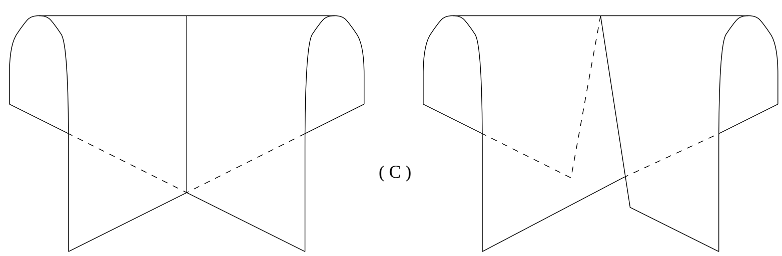

(C)