A 4-dimensional proof of Heron’s Formula

J. Scott Carter
David Mullens

University of South Alabama

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Purpose

Give 4-d proof of Heron’s formula
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Give 4-d proof of Heron’s formula using Scissors Congruences.
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Thanks to organizers
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Thanks to my undergraduate advisor J. Scott Carter
Area of Polygons

$$A_{yt} = \frac{1}{2} pr$$
Area of Polygons

\[ A_{yt} = \frac{1}{2} pr, \quad A_p = pr \]
Area of Polygons

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Area of Polygons

\[ A_{yt} = \frac{1}{2}pr, \quad A_p = pr, \quad A_r = pr \]
Area of Polygons

\[ A_{yt} = \frac{1}{2}pr, \ A_p = pr, \ A_r = pr, \ A_{bt} = \frac{1}{2}pr \]
Heron’s Formula

\[ A_t = \sqrt{s(s - a)(s - b)(s - c)} \] where the semiperimeter, 
\[ s = \frac{(a + b + c)}{2} \]. We will keep \( a < b < c \).
Heron’s Formula

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Heron’s Formula can be written strictly in terms of its edges
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By squaring both sides and clearing the denominator, i.e.,
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Heron’s Formula can be written strictly in terms of its edges. By squaring both sides and clearing the denominator, i.e.,

\[ f(a, b, c) = 16A_t^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \]
Scissors Congruence

Polygons $A$ and $A'$ are *scissors congruent* if there exists polygon decomposition $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$ of $A$ and $A'$ respectively such that $P_i$ is congruent to $Q_j$. In short, two polygons are scissors congruent if one can be cut up and reassembled into the other. Let us denote scissors congruence by $A \sim_{sc} A'$. If we take the cross product of two polygons $A$ and $B$ in $\mathbb{R}^2$, then the resulting figure will exist in $\mathbb{R}^4$, i.e., $A \times B \in \mathbb{R}^4$ is a 4-d hypersolid.
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Higher Dimensional Polytopes
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\[ \square \times \square = \ \text{complex polytope} \]

\[ \triangle \times \triangle = \ \text{complex polytope} \]
Higher Dimensional Polytopes

\[
\begin{align*}
\text{square} \times \text{square} &= \text{polytope} \\
\text{triangle} \times \text{triangle} &= \text{polytope} \\
\text{square} \times \text{triangle} &= \text{polytope}
\end{align*}
\]
Scissors Congruence

Let $A, A', B, B'$ be polygons in $\mathbb{R}^2$, let $A \sim sc A'$, and let $B \sim sc B'$.
Scissors Congruence

Let $A, A', B,$ and $B'$ be polygons in $\mathbb{R}^2$, let $A \sim sc A'$, and let $B \sim sc B'$.

Then $\exists P_1, \ldots, P_n, Q_1, \ldots, Q_m$ such that,
Scissors Congruence

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$$A = P_1 \cup \cdots \cup P_n \text{ and } A' = \bigcup P_i$$
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$A = P_1 \cup \cdots \cup P_n$ and $A' = \bigcup P_i$

$B = Q_1 \cup \cdots \cup Q_m$ and $B' = \bigcup Q_j$
Scissors Congruence

Lemma 1: $A \times B \sim sc \ A' \times B'$
Scissors Congruence

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Lemma 2: The distributive law is a scissors congruence.
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Scissors Congruence

Distributive Law in $\mathbb{R}^2$

\[(a+b)c = ac + bc\]
Scissors Congruence

Distributive Law in $\mathbb{R}^2$, $\mathbb{R}^3$

$(a+b)c = ac + bc$

$(a+b)c^2 = ac^2 + bc^2$
Scissors Congruence

Distributive Law in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$

$$(a+b)c = ac + bc$$

$$(a+b)c^2 = ac^2 + bc^2$$

$$(a+b)c^3 = ac^3 + bc^3$$
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Lemma 1: $A \times B \sim sc A' \times B'$
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Since \( A = P_1 \cup \cdots \cup P_n \) and \( B = Q_1 \cup \cdots \cup Q_m \)
Scissors Congruence

**Lemma 1:** $A \times B \sim_{sc} A' \times B'$

Since $A = P_1 \cup \cdots \cup P_n$ and $B = Q_1 \cup \cdots \cup Q_m$ then $A \times B = (P_1 \cup \cdots \cup P_n) \times (Q_1 \cup \cdots \cup Q_m)$. 
Lemma 1: $A \times B \sim sc A' \times B'$

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We also have that $A' = \bigcup P_i$ and $B' = \bigcup Q_j$. 
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Hence, $\bigcup_{j=1}^{m} P_i \times Q_j = P_i \times B'$ and finally
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$\bigcup_{j=1}^{m} \bigcup_{i=1}^{n} P_i \times Q_j = A' \times B'$

Ergo, $A \times B \sim sc\ A' \times B'$. 
Scissors Congruence

Remember...
Scissors Congruence

Remember...

\[(q, r) \quad (0, 0) \quad (p, 0) \quad (p+q, r) \]
\[p - q \]
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Rectangle x Rectangle
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Triangle x Triangle
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Trapezoid x Triangle
Scissors Congruence

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Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Trapezoid x Trapezoid

Trapezoid x Trapezoid
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Parallelogram x Parallelogram
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

move tri x tri to ...
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

... here.

... here.
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Move trap x tri...
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

...here.

...here.
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

trap $x$ trap stays in place.
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

Move tri x trap ...
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.

...here.

...here.
Scissors Congruence

Example in $\mathbb{R}^4$ looks as follows.
Scissors Congruence

A Scissors Congruence proof of the Pythagorean Theorem looks as follows.

\[ a^2 + b^2 = c^2 \]
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Scissors Congruence

Why is this important?
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Because each time we see the sum of squares,

\[ z^2 = x^2 + y^2 \]
Scissors Congruence

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Because each time we see the sum of squares,

\[ z^2 = x^2 + y^2 \] and \[ v^2 = t^2 + u^2. \]
Scissors Congruence

Now it is important to note,
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\[ z^2 v^2 = (x^2 + y^2)(t^2 + u^2) \]
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\[ = x^2(t^2 + u^2) + y^2(t^2 + u^2) \]

is a scissor congruence by Lemma 2.
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\[ = x^2 t^2 + x^2 u^2 + y^2 t^2 + y^2 u^2 \]
Now it is important to note,

\[ z^2 u^2 = (x^2 + y^2) (t^2 + u^2) = x^2 (t^2 + u^2) + y^2 (t^2 + u^2) = x^2 t^2 + x^2 u^2 + y^2 t^2 + y^2 u^2 \]

is a scissor congruence by Lemma 2.
Scissors Congruence

Once again recall...

which tells us that we can write

\[ a^2 = (q^2 + r^2), \quad b^2 = ((p - q)^2 + r^2), \quad \text{and} \quad c^2 = p^2. \]
Back to Heron’s Formula

$$16A_t^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)$$
Back to Heron’s Formula

\[ 16A_t^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c) \]
\[ = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 \]
Back to Heron’s Formula

\[16A_t^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)\]
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Since \(a^2 = (q^2 + r^2)\), \(b^2 = ((p - q)^2 + r^2)\), and \(c^2 = p^2\),
Back to Heron’s Formula

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Since \( a^2 = (q^2 + r^2) \), \( b^2 = ((p - q)^2 + r^2) \), and \( c^2 = p^2 \), every piece of Heron’s Formula left over that contains an \( a \) and \( b \) can be thought of as
Back to Heron’s Formula

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Since \( a^2 = (q^2 + r^2) \), \( b^2 = ((p - q)^2 + r^2) \), and \( c^2 = p^2 \), every piece of Heron’s Formula left over that contains an \( a \) and \( b \) can be thought of as
Algebraic Proof

\[2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\]
Algebraic Proof

\[2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 = 2[(q^2 + r^2)((p - q)^2 + r^2)] + 2[((p - q)^2 + r^2)(p^2)]
+ 2[(q^2 + r^2)(p^2)]
- (q^2 + r^2)^2 - ((p - q)^2 + r^2)^2 - p^4\]
Algebraic Proof

\[ 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 \]
\[ = 2\left[(q^2 + r^2)((p - q)^2 + r^2)\right] + 2\left[((p - q)^2 + r^2)(p^2)\right] \]
\[ + 2\left[(q^2 + r^2)(p^2)\right] \]
\[ - (q^2 + r^2)^2 - ((p - q)^2 + r^2)^2 - p^4 \]
\[ = \left[2\left[(q^2 + r^2)((p - q)^2 + r^2)\right] + 2\left[((p - q)^2 + r^2)(p^2)\right] \right] \]
\[ - ((p - q)^2 + r^2)^2 \]
\[ + 2\left[(q^2 + r^2)(p^2)\right] - ((q^2 + r^2)^2 - p^4 \right] \]
Algebraic Proof

\[2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\]
\[= 2[(q^2 + r^2)((p - q)^2 + r^2)] + 2[((p - q)^2 + r^2)(p^2)] + 2[(q^2 + r^2)(p^2)] - (q^2 + r^2)^2 - ((p - q)^2 + r^2)^2 - p^4\]
\[= 2[(q^2 + r^2)((p - q)^2 + r^2)] + 2[((p - q)^2 + r^2)(p^2)] - ((p - q)^2 + r^2)^2\]
\[+ 2[(q^2 + r^2)(p^2)] - ((q^2 + r^2)^2 - p^4\]
\[= 4r^2p^2 + (p - q)^2\left[2(q^2 + r^2) + 2p^2 - (p - q)^2 - 2r^2\right] + 2(q^2 + r^2)r^2 - r^4 + 2q^2p^2 - (q^2 + r^2) - p^4\]
Algebraic Proof

\[ 4r^2p^2 + (p - q)^2\left[q^2 + 2pq + p^2\right] + 2q^2p^2 - q^4 - p^4 \]
Algebraic Proof

\[= 4r^2p^2 + (p - q)^2 \left[ q^2 + 2pq + p^2 \right] + 2q^2p^2 - q^4 - p^4 \]

\[= 4r^2p^2 + (p - q)^2 \left[ (q + p)^2 \right] - \left[ q^4 - 2q^2p^2 + p^4 \right] \]
Algebraic Proof

\[= 4r^2p^2 + (p - q)^2\left[q^2 + 2pq + p^2\right] + 2q^2p^2 - q^4 - p^4\]

\[= 4r^2p^2 + (p - q)^2\left[(q + p)^2\right] - \left[q^4 - 2q^2p^2 + p^4\right]\]

\[= 4r^2p^2 + \left[(p - q)^2(q + p)^2\right] - \left[(p - q)^2(q + p)^2\right]\]
Algebraic Proof

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\[= 4r^2p^2\]
Algebraic Proof

\[= 4r^2p^2 + (p - q)^2 \left[ q^2 + 2pq + p^2 \right] + 2q^2p^2 - q^4 - p^4\]

\[= 4r^2p^2 + (p - q)^2 \left[ (q + p)^2 \right] - \left[ q^4 - 2q^2p^2 + p^4 \right]\]

\[= 4r^2p^2 + \left[ (p - q)^2(q + p)^2 \right] - \left[ (p - q)^2(q + p)^2 \right]\]

\[= 4r^2p^2\]

So now we can see that \(16A_t^2 = 4r^2p^2 \Rightarrow A_t^2 = \frac{1}{4}r^2p^2\)
\[ A_t^2 = \frac{1}{4} r^2 p^2 \]

In fact Heron’s Formula tells us....
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