What is Mathematics?

J. Scott Carter*
University of South Alabama
Mobile, Alabama

January 15, 2007

Mathematics is the manipulation of abstract symbols according to specific rules. As such, mathematics is a language, but it differs from other languages in its universal nature and its applicability to human endeavors. The symbols that are manipulated usually represent quantities or geometric objects. The quantities can be as specific as numbers, or they can be abstract and variable. The quantities manipulated often are the rules of manipulation themselves. A mathematician is a linguist who constructs symbols and words from their conceptual roots. These symbols represent an ever increasingly intricate web of thoughts woven from the fibers of primary ideas.

In logic, axioms are chosen, consequences derived from them, and even though those axioms might initially appear reasonable, unreasonable consequences sometimes result, whence axioms are re-examined. Those axioms that result in unusual consequences delight mathematicians, who are attuned to the paradoxes inherent in the system.

A paradox of the twentieth century is Godel's theorem which states that any finite axiomatic system that includes the axioms of standard arithmetic is either inconsistent or incomplete. In the latter case, there are statements that are true but cannot be proven within the system. Mathematicians, other than some logicians, tend to smile at Godel's paradox, but continue their own endeavors towards that objective truth that mathematics affords them. Pure reason allows the questioning and manipulation of the postulates to reflect the truth.

Mathematics is the objective science of pure reason. Some might say that this ability to reason mathematically is a characteristic that humans have that is not readily apparent in other animals. Mathematics will be the first language of communication between us and other sentient beings when such communications occur.

Art, music, and language also involve the manipulation of abstract symbols according to specific rules, but each has slightly differing purposes. Artists manipulate the icons of the predominant cultures. Even though art involves aesthetic objectives, it is neither right nor wrong. It can please us, it can disgust us, it can provoke us, it can cause us to think, but it

*Supported in part by NSF Grants DMS #0301095, #0603926

is morally neutral. Music too is morally neutral, and both art and music are usually created to evoke emotional response. Mathematics always evokes an intellectual response. Its moral stance is that of truth: If one accept the axioms, then one must accept the conclusions. If the conclusions are repugnant or paradoxical, then the axioms must be re-examined, or the nature of the paradox must be distinguished by the careful introduction of symbols.

Here are some apparent mathematical paradoxes. "Every irreducible module is completely reducible." This statement is not only true, but obviously so, once the meanings of irreducible module and completely reducible module are understood. A typical object in mathematics is "a finite dimensional semi-simple complex Lie algebra." Another is a "simplicial complex." The juxtaposition of the words "simple" and "complex" invites a nonsensical interpretation. More disturbingly, the word "complex" in these two contexts represent disparate ideas. Yet, within the context of mathematical discourse, we do not fret over those apparent contradictions. These are familiar objects.

The mathematician's vocabulary is somewhat limited. During a seminar, one mathematician discussed closed curves on a closed surface and whether or not the image of non-closed curves could be compact. The speaker went on to show that the non-closed curves formed a compact subset of the closed surface, and therefore were closed and bounded. The discussion was centered around the dynamics of continuous surface automorphisms; a continuous map being one in which the inverse image of closed sets is closed. The audience protested. The problem is that the standard terminologies all use the same word to indicate different (but related) concepts.

These terminological paradoxes are due to the fact that the language has not had time to evolve. They are not foundational. They amount to a word choice that was made to expedite the conceptual framework. In time, different words will be chosen. The two thousand year old Euclidean geometry allows for the distinctions among rays, lines, segments, and so forth. That language has evolved.

Language is developed to be representational. Syntax and grammar are culturally determined as are a set of reasonable axioms, but it is possible to communicate without careful syntax. Communication within a language transcends dialects, pronunciations, and word placement. Contrastingly, Mathematical communication is dependent upon the proper and precise placement of symbols, and even though there are different dialects in the mathematical sciences, there are translations between dialects.

A mathematical rule such as, "the limit of a product is the product of the limits," points to a fundamental linguistic property. Mathematics allows the commutation of one set of symbols past another. Even an associative rule $(ab)c = a(bc)$ involves commuting the parentheses and the letters in the expression. Mathematical elegance, such as the differential form expression for Stokes's theorem

$$\int_{\partial M} \omega = \int_M d\omega,$$

is achieved by the careful choice of symbols (with an appropriate level of ambiguity) so that the theorem or axiom can be stated as a commutation of symbols. Here the symbol ∂

suggests an analogy between the boundary of the manifold and the partial derivative that is a constituent of the differential $d\omega$. Many interesting aspects of modern mathematics have been discovered by considering these commutations of symbols as processes, subject to their own set of rules. The is the *functorial point of view*.

Elementary school students, who have learned a suitable set of arithmetical facts, learn that these facts transcend specific cases. An observation such as $8 \times 10 = 9 \times 9 - 1 \times 1$ is a consequence of the more general: $(x-y)(x+y) = x^2 - y^2$. The algebraic fact, leads to quadratic surfaces: $z = x^2 - y^2$. And their study leads to a discussion of quadratic forms,

$$(x_1, x_2, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

or the study of rational points on elliptic curves. Such forms are related to geometries, and one can directly compare geometries by examining how they can be deformed into each other. The functorial point of view is to study the abstractions, the relations between objects, or the relationships among the relations.

Here is a further example. From the counting numbers $1, 2, 3, \ldots$ one abstracts the negative numbers, the rational numbers, and the real numbers. Each is understood as a set of equivalence classes of its predecessor. To understand the real numbers, one begins to study the family of functions from the reals to the reals. To understand the connectivities, one studies continuous functions. To understand analytic properties, one studies integrable functions. Algebraic properties are studied by means of homomorphisms, and homomorphisms involve the commutation of one symbol (the function) past another (the binary operation). In all cases, the structural aspects are studied by examining functions that preserve the structure. Next the families of functions are studied in and of themselves. And sets and functions between sets are studied via functors between categories. Functors are studied by natural transformations, and so forth.

Higher degrees of abstraction result in further analogies. These analogies can relate two different mathematical objects, they can relate to the external world, and analogies are formed between analogies. Analogies, then, become the objects of study.

The purposes of these mathematical endeavors are manifold. The primary purpose is to solve real world problems. The secondary purpose is to construct a mathematical world whose internal consistency is so devastatingly beautiful that its intrinsically simple internal structure will shed light on the inner complexities of the external world. On the other hand, the mathematical world is the world of pure thought.

Therefore, mathematical studies are studies of self-realization. Mathematical truths transcend the thinker. A mathematical conversation will lead all parties to the same conclusion. The mathematical thoughts that I think will be thoughts with which you will agree, and vice versa. The truths come from introspection, but they remain objective. And this paradox is the greatest and most splendid of them all: You and I can come to the same objective truth by thinking independently and subjectively. Once a truth is established and articulated within the language of mathematical symbolism, it is universally true — or true within the axiomatic system upon which we agree. In this way, subjective reality matches objective reality.

Mathematics is the language of abstraction. Metaphors are constructed between analogies. Paradoxes are dissected and resolved by the careful application of language and by making distinctions among the analogies. It is engendered from the real world of the mind, from the world of human observation and introspection. It is the application of a logically consistent set of rules and grammatical constructions to create more intricate rules and constructions. These constructions are used to model the world of external observation, and they do so with remarkable precision. The language of reason solves problems and puzzles, accounts for differences, creates puzzles, and analyzes the questions that arise from introspection.