Chapter 1

Outline and Overview

1.1 Overview

This course is a self-contained survey of material that I would hope an educated technician can master. I am assuming a solid grasp of high school algebra and trigonometry that, for example, a professional teacher or practicing engineer might possess. Within the course we are going to "construct jet airplanes from bicycle parts." That is we are going to develop a host of examples of high-dimensional phenomena from a very elementary point of view. My intended audience is the experienced practitioner — the high school teacher who not only loves the craft of teaching, but also loves the subject of mathematics.

I mentioned my goal to a cynical young person the other day. This person thought that a high school teacher may not care because learning advanced mathematics does not necessarily directly affect how students are going to respond. I have an answer to this objection. Nearly every study on mathematics education shows that teachers who know more mathematics are more effective teachers. Knowing mathematics is not sufficient, but it is necessary. Students in this class may walk away from class confused, scratching their heads they will say, "I don't know WHAT he was talking about today." I think I know the mathematics that I am presenting, but I may not know how to present it.

I am asking the students in the class to be my co-conspirators. I want this material to be understandable to you, and to subsequent students. We will work on organization and exposition, as the course develops. I expect homework to be worked. And I expect questions and intellectual probing. Let us have fun with this material.

1.2 Outline

1. Number and Algebra
2. Lines and Circles
3. Triangles and Circles
4. Planes in Space
5. Coordinate space
6. Tetrahedra and Spheres
7. Structures in higher dimensions
8. Matrices and Operations
9. Quadratic Expressions
10. Further Explorations

Chapter 2

Number and Algebra

2.1 Counting

Counting is the most primitive mathematical function. It underlies our number sense, and accountability was probably the basis of written language. Literary traditions can easily be passed down by oral traditions. There is little harm in the embellishment that a story receives as it is handed from generation to generation. But an embellishment of the size of a plot of land, the number of sheep, or the size of a wedding dowry can cause great social unrest. Thus, accountability is a fundamental aspect of society.

The most primitive count is to count the elements of a specific set of objects: 1 widget, 2 widgets, 3 widgets, *etc.* Shortly into the process, the noun is dropped and the adjectives remain: 1, 2, 3, 4, 5. As teachers, we can get a little more mileage by reintroducing the nouns. Let us count by twelfths. In words, one twelfth, two twelfths, three twelfth, four twelfths, five twelfths, six twelfths, seven twelfths, eight twelfths, nine twelfths, ten twelfths, eleven twelfths, one. In symbols we have

$$1/12, 2/12, 3/12, 4/12, 5/12, 6/12, 7/12, 8/12, 9/12, 10/12, 11/12, 12/12$$

Or the reduced forms

$$1/12, 1/6, 1/4, 1/3, 5/12, 1/2, 7/12, 2/3, 3/4, 5/6, 11/12, 1.$$

When I exercise by swimming either 72 laps, or by walking for 1 hour (I try to keep my pace between 4 and 5 miles per hour), I am constantly recomputing how much of my exercise goal I have achieved, by dividing the number of laps or minutes completed by the total number. The 72 laps come from a 25 yard long pool. $72 \times 25 = 1,800$ yards. Since 1 mile = 5280 feet = 1760 yards, the 1,800 is a little over, but 72 is a highly divisible number. My ratchet wrench set would be more convenient *to me*, if it expressed all the diameters in thirty-seconds. Then I would not have to pause and contemplate if I needed a 5/32 inch wrench or a 7/32 inch wrench when a 1/4 inch wrench is too small.

There are other forms of counting. Most children learn to count to 10 by 1s and even to 20 by 2s. Seldom are they taught to count to 30 by 3s, 40 by 4s, 50 by 5s, 60 by 6s, 70 by 7s, eighty by 8s, or 90 by nines. Counting by is considered a bad habit when multiplication tables are to be memorized. And the memorization of mathematical facts can be a good thing. I know many interesting mathematical facts. Some I have memorized; some I derive. Some I can derive so fast, you can't tell that I did not recall it from memory. I am not sure what the harm is in counting by sevens to compute 7×6.

I have memorized all the squares from 1 to 25 — my high school geometry teacher taught me it was a good thing to do. It is not impressive to write them here. From these, I can tell you the squares from 26 to about 60 without much time delay. And from this set of 60 numbers I can tell you how to compute half of the products of pairs of numbers from 1 to 50. The algorithm is to compute products as differences of squares. When I am in my prime, you cannot tell that I did not memorize the result. My point here is that there should be a balance among enumeration, memorization, and derivation. All three are mathematical skills. All three are mathematical skills that can be translated directly into any aspect of human endeavor.

2.2 Arithmetic and Operation

Theorem 2.2.1 *Every rectangle can be expressed as a difference of squares.*

Before I go to the proof of the theorem, let me give some examples.

Example 2.2.2 A rectangle of size 7×9 has as its area 63 square units. The number 63 is 1 less than 64. That is $63 = 8 \times 8 - 1 \times 1$. More to the point $7 \times 9 = (8-1) \times (8+1) = 8^2 - 1^2$.

Example 2.2.3 Among the few perfect squares that I know $38^2 = 1,444$. Thus $37 \times 39 = 1,443$. The number 38 is a distance 12 from 50. The square of 12 is 144. The other number a distance 12 from 50 is 62. The square of 62 is $62^2 = 3844$. The number less than 100 that is a distance 12 from 100 is 88. Since 88 is divisible by 11 and the last two digits of the number are 44, the square of 88 must be 7744 which is indeed correct.

Suppose that $n \in \{1, 2, \ldots, 25\}$. Then $(50 \pm n)^2 = 50^2 \pm 100n + n^2 = 2500 \pm 100n + n^2$. Thus the squares of the numbers from 49 through 26 have their last two digits coinciding with the last two digits of the squares of the numbers from 1 through 24. The squares of the numbers from 51 through 74 have their last two digits coinciding with those of the numbers from 1 through 24. Similarly, $(100 - n)^2 = 10,000 - 200n + n^2$. So the last two digits of the numbers from 76 to 99 coincide with those of the numbers 24 through 1 in that order.

During the second class we will examine the squares of the numbers that are 17 away from 0, 50 and 100: 289, 1089, 4489, and 6889. Taken in smaller sets these squares will become more digestible.

Proof of Theorem. The illustration of this fact is given by drawing a rectangle that is length x on the horizontal side and length y on the vertical side. We may assume by rotating the figure if necessary that $x < y$. If $x = y$, then the rectangle is the difference between itself and the empty square. Since $x < y$, split the difference. Set $\beta = (y - x)/2$. Then $y = x + 2\beta$. Then $xy = (x + \beta)^2 - \beta^2$. Let us check this in a little more detail: $xy = x(x + 2\beta) = (x + \beta)^2 - \beta^2$. Geometrically, the horizontal rectangle of size $x \times \beta$ is cut from the top of the $x \times y$ rectangle and slide to the side of the larger slice of size $x \times (x + \beta)$. This almost makes a square of size $(x + \beta) \times (x + \beta)$. But there is a missing piece of size β^2.

The proof as it is being written is unsatisfactory to me. It is necessary to illustrate the idea with a plethora of examples, before going to the letters. I find the variety of proofs in Euclid book II, similarly dissatisfying. There the arguments are purely geometrical. So a good balance between geometry and algebra has to be maintained.

2.3 Quadratic Equations

Completing the square in general goes following the example:

$$y = 3x^2 + 5x - 17.$$

$$y = 3[\qquad\qquad\qquad] - 17.$$

$$y = 3[x^2 + 5/3x] - 17.$$

The rectangle that we want to express as a difference of squares is $x(x + 5/3)$.

$$y = 3[(\qquad\qquad)^2 - (\quad)^2] - 17.$$

$$y = 3[(x + 5/6)^2 - (5/6)^2] - 17.$$

$$y = 3(x + 5/6)^2 - 3(5^2)/6^2 - 17.$$

$$y = 3(x + 5/6)^2 - 5^2/(4 \cdot 3) - 17.$$

$$y = 3(x + 5/6)^2 + \left[\frac{-5^2 + 4 \cdot 3 \cdot (-17)}{4 \cdot 3}\right].$$

$$(y - \left[\frac{-5^2 + 4\cdot 3\cdot(-17)}{4\cdot 3}\right]) = 3(x+5/6)^2.$$

From this form we see the vertex of the parabola as $(-5/6, \left[\frac{-5^2+4\cdot3\cdot(-17)}{4\cdot3}\right])$, we can compute the x-intercepts, and we can mimic the computation to prove the quadratic formula. Observe, that the final form that I wrote was

$$(y-k) = A(x-h)^2.$$

So that the quadratic expression has been put into a "vertex/slope" form that is analogous to the point/slope form of the line. We will exploit similar analogies in the future.

The key to this calculation was the choice of the y-intercept as -17. That is, that value guaranteed x-intercepts even if these are irrational numbers. At this stage, it is good to approximate the roots using your table of squares that you have since memorized.

Now I mimic that calculation to determine the slope, vertex, and x-intercepts of the generic parabola:

$$y = Ax^2 + Bx + C.$$

$$y = A[\qquad\qquad] + C.$$

$$y = A[x^2 + \frac{B}{A}x] + C.$$

The rectangle that we want to express as a difference of squares is $x(x+B/A)$.

$$y = A[(\qquad\quad)^2 - (\quad)^2] + C.$$

$$y = A[(x+\frac{B}{2A})^2 - (\frac{B}{2A})^2] + C.$$

$$y = A(x+\frac{B}{2A})^2 - A\frac{B^2}{4A^2} + C.$$

$$y = A(x+\frac{B}{2A})^2 - \frac{B^2}{4A} + C.$$

$$y = A(x+\frac{B}{2A})^2 - \left[\frac{B^2-4AC}{4A}\right].$$

$$(y - (-\left[\frac{B^2-4AC}{4A}\right])) = A(x-(-\frac{B}{2A}))^2.$$

Observe, that the final form that I wrote was

$$(y-k) = A(x-h)^2.$$

From this form we see the vertex of the parabola as $(-\frac{B}{2A}, -\left[\frac{B^2-4AC}{4A}\right])$. The x-intercepts if they exist are obtained by solving for $y = 0$:

$$(\left[\frac{B^2 - 4AC}{4A}\right]) = A(x - (-\frac{B}{2A}))^2.$$

$$A(x - (-\frac{B}{2A}))^2 = \left[\frac{B^2 - 4AC}{4A}\right].$$

$$(x - (-\frac{B}{2A}))^2 = \left[\frac{B^2 - 4AC}{4A^2}\right].$$

$$(x - (-\frac{B}{2A})) = \pm\frac{\sqrt{B^2 - 4AC}}{2A}.$$

$$(x = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}.$$

Several issues occur when solving the quadratic formula. The first is dealing with radicals of non-perfect squares. The second is dealing with radicals of negative numbers. Both can be addressed by the formal introduction of roots of equations. I touch upon this in the in the next section.

2.3.1 The Basic Arithmetic Rules

The set of integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ is convenient to work with because it forms an *abelian group*: the sum of two integers is an integer, the addition is a commutative operation, it is associative, and every integer, a, has an additive inverse, $-a$. The additive inverse of an integer is diametrically opposite to the number on the number line. If a is to the left of 0, then $-a$ is to the right at the same distance. If a is to the right, $-a$ is to the left. The additive identity is 0.

(Some mathematicians are fussy about the terminology negative a or minus a. I am not. I think that numbers have many names. Most people like to call 4/32 one eighth. Sometimes the former is more useful. I parse the squares from 1 to 100 as so many hundreds. So for example $98^2 = 9604$ and I pronounce the latter ninety-six oh four precisely because that is how I computed it: four hundred less than 100 squared plus 4.)

There is also a multiplicative structure to the integers. Multiplication distributes over addition, and there is a multiplicative identity. Also there is cancelation: If $ab = ac$, then $b = c$. But the only non-zero integers with multiplicative inverses are ± 1. From the point of view of division, the integers are not the best place to work.

The rational numbers satisfy the *field axioms*:

1. (existence of operations) For every $a, b \in \mathbb{Q}$, the sum $a + b \in \mathbb{Q}$, and the product $ab \in \mathbb{Q}$.
2. (commutativity) For every $a, b \in \mathbb{Q}$, we have $a + b = b + a$ and $ab = ba$
3. (associativity) For every $a, b, c \in \mathbb{Q}$, we have $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
4. (identities) There are numbers 0 and 1 (with $0 \neq 1$), such that for every $a \in \mathbb{Q}$, the sum $a + 0 = a$, and $a1 = a$.
5. (inverses) For every $a \in \mathbb{Q}$, there is a rational number $-a$, such that $a + (-a) = 0$, and for every $a \neq 0$ there is a number a^{-1} such that $aa^{-1} = 1$.
6. Distributive laws: For every $a, b, c \in \mathbb{Q}$, we have $(a + b)c = ac + bc$, and $a(b + c) = ab + ac$.

There are a plethora of examples of fields available. These are sets that satisfy the list of axioms immediately above.

2.3.2 Modular arithmetic

Let me remind you how to do *modular arithmetic*: Over the integers $\mathbb{Z}$ consider an equivalence relation

$$a \equiv b \pmod{p}$$

if and only if the (positive) number p divides the difference $b - a$. Such an *equivalence relation* partitions the integers into equivalence classes. The usual representatives of these equivalence classes are $\{0, 1, \ldots, p-1\}$. To compute the equivalence class that a number n belongs to, divide n by p to obtain a quotient q and remainder r. That is $n = pq + r$ where $r \in \{0, 1, \ldots, p\}$. The remainder indicates the class in which n belongs. I think of the integers as forming an infinite piano, and that p is the number of notes in a scale. Two numbers are equivalent if they represent the same tone within the scale determined by p.

Equivalence classes can be added and multiplied by computing the sum or product of the representative, and then recomputing the remainder of the sum or product. The resulting set $\mathbb{Z}/(p\mathbb{Z}) = \mathbb{Z}_p = \mathbb{Z}/p$ is called the integers modulo p. This set is a field if and only if p is a prime number.

The notation $p\mathbb{Z}$ is meant to suggest all the multiples of the number p. Given two multiples, pa and pb, their sum is a multiple. And given any number n and any multiple pa of p, the product $n(pa)$ is also a multiple of p. A *non-empty set* that is closed under addition and subtraction and is "super-absorptive" with respect to multiplication is called an *ideal*. An ideal is called *prime* if it has no non-trivial sub-ideals. The trivial sub-ideals are the ideal itself and the set, $\{0\}$

that only has the signal element 0. The *prime numbers* correspond to the prime ideals.

My favorite *finite field* then is $\mathbb{Z}/(3\mathbb{Z})$ — the integers modulo 3. I am also fond of the integers mod 2. but I want to concentrate on $\mathbb{Z}_3$, here, now. We can, for example, consider polynomial equations over this field, and it does not take too long to discover that the quadratic equation $x^2 + 1 = 0$ has not solution over $\mathbb{Z}_3$. It doesn't take too long, because we only have to test, $0^2 + 1 \equiv 1$, $1^2 + 1 \equiv 2$, and $2^2 + 1 \equiv 2$. All the equivalences are modulo 3.

Exercise 2.3.1 *Consider the set* $\{a + bi : a, b \in \mathbb{Z}_3\}$ *where* $i^2 = 2$, *so* i *is an artificially adjoined square root of* -1. *Form an addition table and multiplication table for this set. Think of these* 9 *points as forming a lattice in the plane, and try to understand the multiplication geometrically.*

2.3.3 Field Extensions

A more general construction of field extensions goes as follows. It mimics the examples above. We begin with a field, for the time being you may assume that the field is the rational numbers $\mathbb{Q}$. The set of polynomials over the rational numbers is the set

$$\mathbb{Q}[x] = \{\sum_{j=0}^{n} a_j x^j : a_j \in \mathbb{Q}, \quad n \in \mathbb{N} \cup \{0\}\}$$

and x is an indeterminant. Ordinarily, we assume that $a_n \neq 0$, and then n is called the degree of the polynomial. Polynomials can be added, multiplied, there is a 0 polynomial, addition and multiplication are commutative and associative operations, and multiplication distributes over addition.

Moreover, there is a division algorithm for polynomials. The algorithm depends on the degree of the polynomial as a measure of size. In the polynomial ring $\mathbb{Q}[x]$, we can also form modular arithmetic. Here though, the interesting 'ideals' are the multiples of an irreducible polynomial. That is, we consider a polynomial such as $x^2 - 2$ that does not have a root in the rational numbers. Then we consider two polynomials to be equivalent if their difference is divisible by $x^2 - 2$. If we work through this carefully, we find that any polynomial is equivalent to one of the form $ax + b$ where a and b are rational numbers. So the set $\{a + bx : a, b \in \mathbb{Q}\}$ plays the role of the equivalence classes modulo p. Moreover, if we are given any polynomial of higher degree, we can compute its remainder mod $x^2 - 2$, by evaluating each x^2 factor to be 2. In class, we'll work a couple of examples to see why this works, but it is not very mysterious. Thus the set of remainders behaves exactly as if we had adjoined a $\sqrt{2}$ to the rational numbers.

The classical construction of this is to take the real numbers $\mathbb{R}$ and adjoin i where $i^2 = -1$. This construction gives the complex numbers. Observe that the geometric picture of the complex numbers is quite a bit different than that of the rational numbers adjoining a real square root. There are similarities that can be exploited, but ultimately, we understand real and complex numbers as having different structures.

Chapter 3

Lines and Circles

The material here may appear too basic. However, it is the material here that will be generalized and expanded upon. Please be patient with this exposition. As a college teacher, this is the material that I would hope that an incoming student would know intimately. Furthermore, I would hope that the students' technical skill here would be either without mistake, mistakes could be easily corrected, or that at most an occasional sign error might occur.

3.1 Pairs of Points

Exercise 3.1.1 *Choose between* 5 *and* 10 *pair of points* (x_1, y_1), (x_2, y_2) *and determine the following:*

1. Δx

2. Δy

3. $M = \frac{\Delta x}{\Delta y}$ *— the slope of the line determined by the points.*

4. *The equation of the line determined by these points in:*

   (a) *point-slope form*

   (b) *slope-intercept form*

   (c) *general form*

   (d) *intercept-intercept form*

   (e) *"$\Delta x \Delta y$" form.*

   (f) *parametric form from* (x_1, y_1) *to* (x_2, y_2) *at unit speed.*

5. *Determine a vector that is perpendicular to the line.*

6. *Determine a pair of unit vectors perpendicular to the line.* $(x_m, y_m) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ — *the midpoint between the points.*

7. $d = \sqrt{(\Delta x)^2 + \Delta y)^2}$ — *the distance between the points.*

8. *The equation of the circle that has these points along a diameter.*

9. *The intersection of this circle with each of the coordinate axes.*

I trust that an example will suffice to illustrate the ideas:

Example 3.1.2 For the pair of points $(x_1, y_1) = (-2, -4)$, $(x_2, y_2) = (5, 4)$ determine the following:

1. $\Delta x = (x_2 - x_1) = (5 - (-2)) = 7$

2. $\Delta y = (y_2, y_1) = (4 - (-4)) = 8$

3. $M = \frac{\Delta x}{\Delta y} = \frac{8}{7}$.

4. The equation of the line determined by these points in

   (a) point-slope form:
   $$(y - 4) = \frac{8}{7}(x - 5).$$

   (b) slope-intercept form:
   $$y = \frac{8}{7}(x - 5) + 4.$$
   $$y = \frac{8}{7}x + \frac{-5 \cdot 8}{7} + \frac{28}{7}$$
   $$y = \frac{8}{7}x + \frac{-12}{7}$$

   (c) general form
   $$12 = 8x - 7y$$

   (d) intercept-intercept form:
   $$\frac{x}{\frac{3}{2}} + \frac{y}{\frac{-12}{7}} = 1.$$

   Note $(3/2, 0)$, and $(0, -12/7)$ are the intercepts.

   (e) "$\Delta x \Delta y$" form.
   $$7(y - 4) = 8(x - 5)$$

(f) parametric form from (x_1, y_1) to (x_2, y_2) at unit speed.

$(5,4)t + (1-t)(-2,-4) = (5t - 2 + 2t, 4t - 4 + 4t) = (7t - 2, 8t - 4)$.

Note at $t = 0$, we get $(-2,-4)$ and at $t = 1$, we get $(5,4)$. In this form, it is worth solving for t at which the line intersects the x and y axes. For example, if $x = 0$, $t = 2/7$ and $y = 8 \cdot (2/7) - 4 = -12/7$; if $y = 0$, then $x = 3/2$.

5. Determine a vector that is perpendicular to the line. The natural vector to choose is $(8,-7)$.

6. Determine a pair of unit vectors perpendicular to the line.

$$\pm\frac{\sqrt{113}}{113}(8,-7)$$

$(x_m, y_m) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) = (3/2, 0)$

7. $d = \sqrt{(\Delta x)^2 + \Delta y)^2} = \sqrt{113} \approx 10.5$ — the distance between the points.

8. The equation of the circle that has these points along a diameter.

$$(x - \frac{3}{2})^2 + y^2 = \frac{113}{4}$$

9. The intersection of this circle with each of the coordinate axes. y-intercepts: $x = 0$

$$\frac{9}{4} + y^2 = \frac{113}{4}.$$
$$y^2 = \frac{104}{4}.$$
$$y = \pm\frac{\sqrt{104}}{2}.$$

x-intercepts: $y = 0$

$$(x - \frac{3}{2})^2 = \frac{113}{4}.$$
$$x = \frac{3}{2} \pm \frac{\sqrt{113}}{2}.$$

3.2 The unit circle

In general, the equation of a circle of radius r that is centered at the point (h,k) is given by the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

A circle is the set of points in the plane that is a fixed distance from a center point. This equation follows directly from the definition and from the distance formula in the plane.

We can view this equation as a translation and dilation of the unit circle $x^2 + y^2 = 1$. We translate x to the right by h. We translate y up by k. and we dilate both x and y by r.

In class, I will discuss trigonometric parameterizations of generic circles and ellipses.

Finally, I will discuss the stereographic parametrization of the unit circle, and show how to use this to construct Pythagorean triples.

3.2.1 Stereographic projection

3.2.2 Pythagorean Triples

3.2.3 Trigonometric Functions

3.3 Complex numbers revisited

3.3.1 The Exponential Function

3.3.2 The Problem of Inversion

Chapter 4

Triangles and Circles

4.1 Inscribed and Circumscribed

4.2 Points in Space

Chapter 5

Planes in Space

5.1 Triangles, circles, spheres

5.2 Intersections between planes

Chapter 6

Coordinate Space

6.1 Parametric Lines

6.2 parameterized Planes

6.3 More Intersections

6.4 Changing Points of View

Chapter 7

Tetrahedra and Spheres

7.1 Four points are given

7.2 Planes, circles, and intersections

Chapter 8

Structures in Higher Dimensions

8.1 the n-cube

8.2 the n-simplex

8.3 the n-sphere

8.4 Intersections

Chapter 9

Matrices and Operations

9.1 Low Dimensional Examples

9.2 Space and subspace

9.3 Circle Inversion revisited

9.4 Transformational Geometry

Chapter 10

Quadratic Expressions

10.1 Translations

10.2 Matrix Representations

10.3 Classification

10.4 Representation

Chapter 11

Further Exploration

11.1 Cubic forms

11.2 Abstract Tensor Formalisms