Quandles and Groups

This example is due to Joyce [**?**] and Matveev[**?**].

Let G be a group, H a subgroup, $s : G \to G$ an automorphism such that for each $h \in H$ $s(h) = h$. Define a binary operation $*_s = *$ on G by, $a * b = s(ab^{-1})b$. Then $*$ defines a quandle stucture on G. Axioms I and II are easily verified. For Axiom III, we have

$$(a * b) * c = s(s(ab^{-1})bc^{-1})c = s^2(a)s^2(b^{-1})s(b)s(c)^{-1})c$$

while

$$(a*c)*(b*c) = s(s(ac^{-1})c(s(bc^{-1})c)^{-1})s(bc^{-1})c = s^2(a)s^2(c^{-1})s(c)[s^2(b)s^2(c^{-1})s(c)]^{-1}]s(b)s(c^{-1})c$$

$$= s^2(a)s^2(c^{-1})s(c)s(c)^{-1}[s^2(c^{-1})^{-1}]s^2(b)^{-1}s(b)s(c^{-1})c = s^2(a)s^2(b^{-1})s(b)s(c)^{-1}c.$$

This passes to a well-defined quandle structure on G/H that is given by $Ha*Hb = Hs(ab^{-1})b$. In particular, if $z \in Z(H) \cap H$ where $Z(H) = \{z \in G : zh = hz \text{ for all } h \in H\}$, then $Ha * Hb = Hab^{-1}zb$ defines a quandle structure. Let us denote the resulting quandle by (G, H, z).

If X is a quandle, then $\mathrm{Aut}(X)$ is the group of (right) automorphisms of X. That is, $\phi \in \mathrm{Aut}(X)$ means that ϕ is bijective and $(x * y)\phi = (x)\phi * (y)\phi$. Suppose that X is *homogeneous* in the sense that $\mathrm{Aut}(X)$ acts transitively: For every $x, y \in X$ there is a $\phi \in \mathrm{Aut}(X)$ such that $y = (x)\phi$. Suppose that z is a fixed element of X and consider $H \subset \mathrm{Aut}(X)$ to be the stabilizer of z. That is $H = \{h \in \mathrm{Aut}(X) : (z)h = z\}$. Then there is a quandle isomorphism $X \cong (\mathrm{Aut}(X), H, z)$.

Proof. Let $x \in X$, and chose ϕ_x so that $x = (z)\phi_x$. Given ϕ_x and ϕ'_x that map z to x, we see that $(z)\phi'\phi^{-1} = z$. So $\phi'\phi^{-1} \in H$. On the other hand, if $h \in H$, then $x = (z)\phi_x = zh\phi_x$. Therefore, any element of $H\phi_x$ maps z to x. We have a function, $E : \mathrm{Aut}(X) \to X$, $E(\phi) = (z)\phi$. To see that E is a quandle homorphism, compute:

$$E(\phi * \psi) = (z)((\phi\psi^1) * z)\psi = (z)(\phi\psi^{-1})\psi * (z)\psi = E(\phi) * E(\psi).$$

The quandle map E passes to a bijection form $(\mathrm{Aut}(X), H, *z)$ to X. This completes the proof.

The fundamental quandle

Suppose that $k : M^n \to \mathbb{R}^{n+2}$ is either smooth or PL locally flat. Let $N(k)$ denote an open tubular neighborhood of $k(M)$. The *fundamental quandle* $\pi_Q(k)$, of a codimension 2 embedding; to be the set of homotopy classes of maps $\alpha : ([0,1], \{0,1\}, 1) \to (\mathbb{R}^{n+2} \setminus N(k), \partial N(k) \cup \{y_0\}, \{y_0\})$ where y_0 is a fixed base point pretty close to the boundary of the tubular neighborhood. The homotopies between such maps are required to have their bottom boundaries on $\partial N(k)$ and their top boundaries fixed at the base point ($H(t, 0) \in \partial N(k)$ while

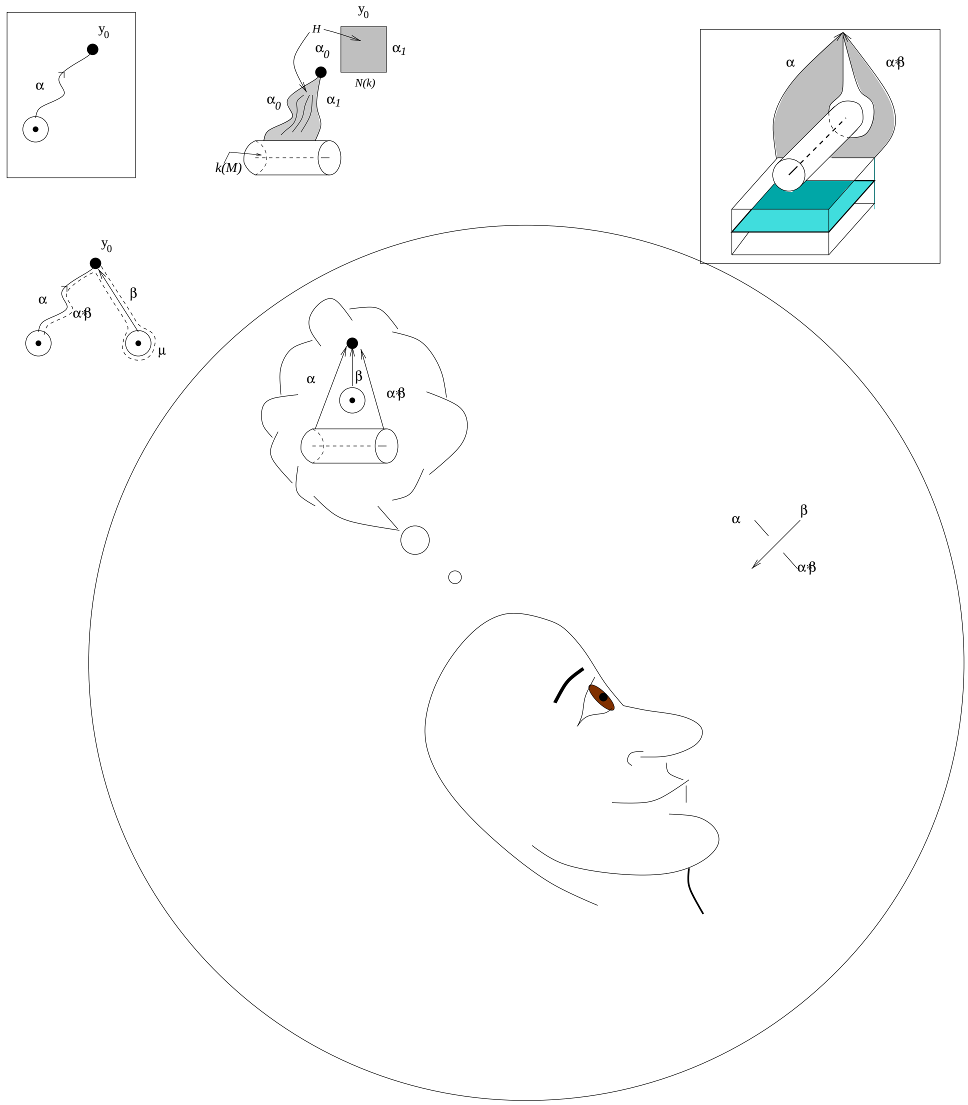

Figure 1: Thoughts about the fundamental quandle

$H(s, 1) = y_0$ for all $s \in [0, 1]$, and $H(t, i) = \alpha_i(t)$ for $i = 0, 1$). If α and β are such paths, then there is a unique oriented meridian μ_β that passes through the initial point of β. The quandle product is defined to be the path composition $\alpha * \beta = \alpha\beta^{-1}\mu_\beta\beta$. Here μ_β is the meridian that intersects the path β at β_0

Take for example, the case when $G = \pi_1(S^3 \setminus k(S^1))$. Then $G \subset \text{Aut}(\pi_Q(k))$. The action of $\alpha \in \pi_1$ on a path z from the tubular neighborhood to the base point is given by path multiplication: $z \cdot \alpha = z\alpha$. Let $z \in \pi_Q$ denote a fixed path, and let H denote the choice of peripheral subgroup in which longitudinal classes and meridional classes are determined by following the arc defining z to the base point as in Fig. **??**. Then the elements of H fix z since the endpoint of z can follow as it likes around a path on the tubular neighborhood that defines an element of H. See Fig. 3. Moreover these are precisely the elements that

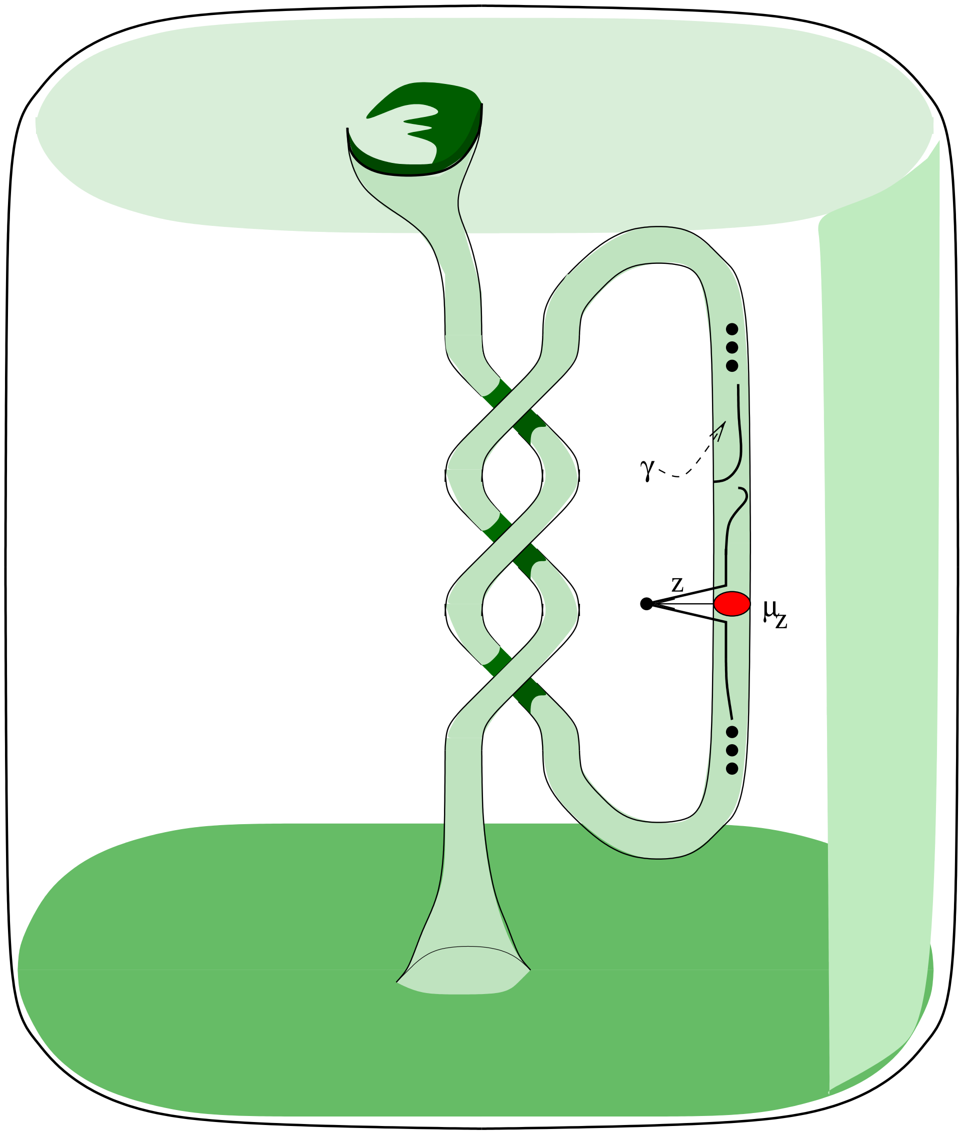

Figure 2: The peripheral determined by z

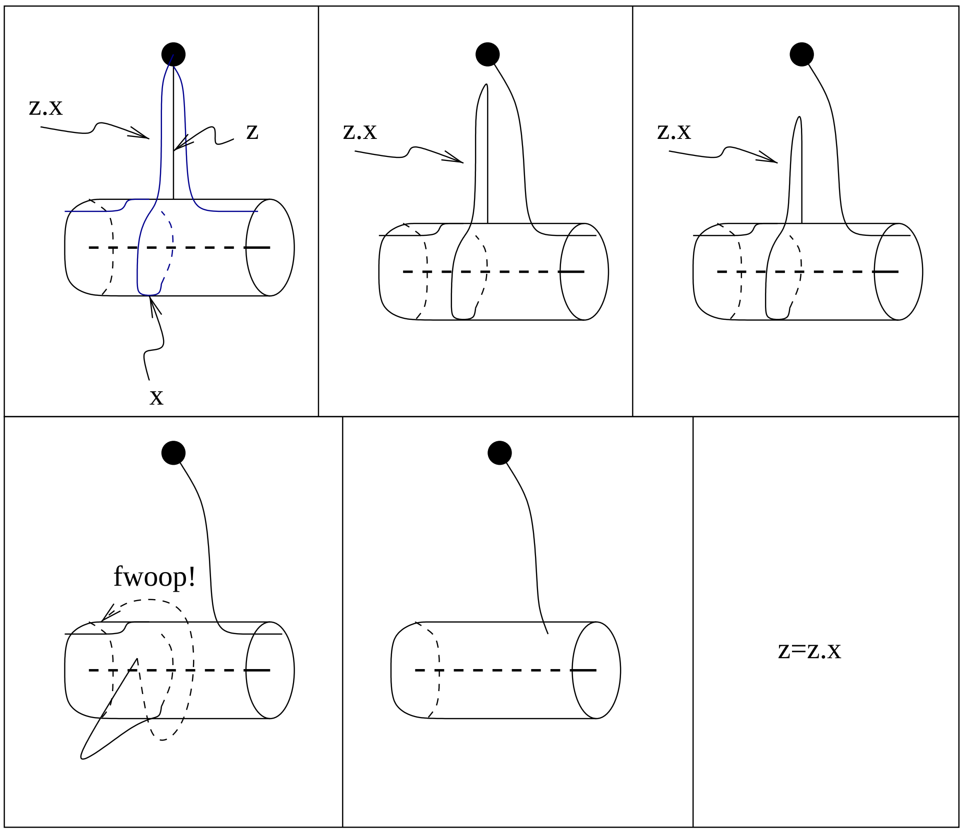

Figure 3: The action of the peripheral subgroup

fix z because any element that fixes z would result in the initial point of z moving around the tubular neighborhood. So a loop that fixes z is homotopic to a loop on the tubular neighborhood. Using the above quandle isomorphism, we have

Theorem: $\pi_Q(k) = (\pi_1(S^3 \setminus k(S^1)), P, \mu_z)$ where μ_z is the meridonal element in the fundamental group whose path to the base point follows along x.

Groups that are associated to a quandle

Let X be any quandle. Let $F(X)$ denote the free group generated by X. Then $F(X)$ acts on X as follows. First, we introduce the $\bar{*}$-operation. For any $x, y \in X$ there is a unique element $z \in X$ such that $z * x = y$. Let this element be denoted by $z = x\bar{*}y$. Then $(x\bar{*}y) * x = y$. If $x \in X$ also denotes the generator in $F(X)$ and $\cdot$ denotes the action of $F(X)$ on X, then $y \cdot x = y * x$, furthermore $y \cdot x^{-1} = y\bar{*}x$, and in general if $w = w_1 w_2 \in F(X)$ acts as $y \cdot (w_1 w_2) = (y \cdot w_1) \cdot w_2$ and this in turn can be written in terms of $\bar{*}$, $*$ and the generators in X. Thus there is a map from $F(X)$ to a subgroup of the group of symmetries of X. Let K denote the kernel of this map. Then $F(X)/K$ is isomorphic to a subgroup of the automorphism group of the quandle. In fact since the action of $F(X)$ is induced by the quandle product, $F(X)/K$ is isomophic to a subgroup of the inner automorphism group. The *inner automorphism group* $\mathrm{Inn}(X)$ consist of the set of automorphisms induced by $\phi_y : x \mapsto x * y$ for each $y \in X$.

Define the *enveloping group* of a quandle to be the group, $G(X)$ that is generated by the elements of X subject to the set of relations $a * b = bab^{-1}$ for all $a, b \in X$. The action of $F(X)$ also passes to an action of $G(X)$ with kernel K', and so we have in general $G(X)/K'$ is isomorphic to a subgroup of $\mathrm{Inn}(X)$.

Note that for a quandle X we have have a map $\partial : X \to \mathrm{Aut}(X)$ given by $y \mapsto \phi_y$ where, as above, $(x)\phi_y = x * y$. Note also, that $\partial((x)\phi) = \phi^{-1}\partial(x)\phi$. Thus a quandle is also a crossed G-set where $G = \mathrm{Aut}(X)$, but it has additional structure.

Example

Let QS_4 denote the quandle, $\mathbb{Z}_2[t, t^{-1}]/(t^2+t+1)$. Observe that this example is the 4-element field, but its quandle structure is given by $a * b = ta + (1 + t)b$. Letting $[0] = 0$, $[1] = 1$, $[2] = t$, and $[3] = t + 1$, we have the following quandle table:

$R * C$	[0]	[1]	[2]	[3]
[0]	[0]	[3]	[1]	[2]
[1]	[2]	[1]	[3]	[0]
[2]	[3]	[0]	[2]	[1]
[3]	[1]	[2]	[0]	[3]

Now $\mathrm{Aut}(X) \subset \Sigma_4 = \Sigma_4(0,1,2,3)$, is generated by (123), (032), (013), and (021). So $\mathrm{Aut}(X)$ is the alternating group, $A_4 = A_4(0,1,2,3)$. Let $H = \mathrm{Stab}(\{0\}) = \{1, (132), (123)\}$. The action of [0] as an automorphism, is $\partial[0] = (123)$. The stablizer of 0 in A_4 is $H = \{1, (132), (123)\}$. And the coset space is $\{H, H(031), H(02)(13), H(013)\}$.

Review of Group Cohomology

Consider the short exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{q} \mathbb{Z}_p \to 0$$

where q is the quotient map. Consider the set-theoretic section $s[j] = j$ where we take the usual representative $j \in \{0, \ldots, p-1\}$ for the coset. Since $qs[j] = [j]$, the difference $(s[x+y] - (s[x] + s[y]))$ is divisible by p. Define a function

$$f(x,y) = (s[x+y] - (s[x] + s[y]))/p.$$

Then $f(x,y)$ satisfies a cocycle condition:

$$f(x,y) + f(x+y,z) = f(x,y+z) + f(y,z)$$

since either side simplifies to $\frac{1}{p}(f(x+y+z) - f(x) - f(y) - f(z))$.

The group cocycle condition can be understood in terms of the diagrammatic representation of associativity, as indicated in Fig. 4.

Quandle Cohomology

Example

For QS_4, there is a function,

$$\phi(x,y) = \begin{cases} 0 & \text{if } \ x = [3], y = [3], \ \ \text{or} x = y \\ 1 & \text{otherwise} \end{cases}$$

that takes values in $\mathbb{Z}_2$ and that satisfies the following condition:

$$\phi(a,b) + \phi(a*b,c) = \phi(a,c) + \phi(a*b, b*c).$$

Such a function that also satisfies $\phi(x,x) = 0$ for every $x \in X$ is called a quandle 2-cocycle.

- Quandle 2-cocycles are to the Reidemeister type III move as group 2 cocycles are to the associativity condition.
- Quandle 2-cocycles are related to extensions of quandles by groups.

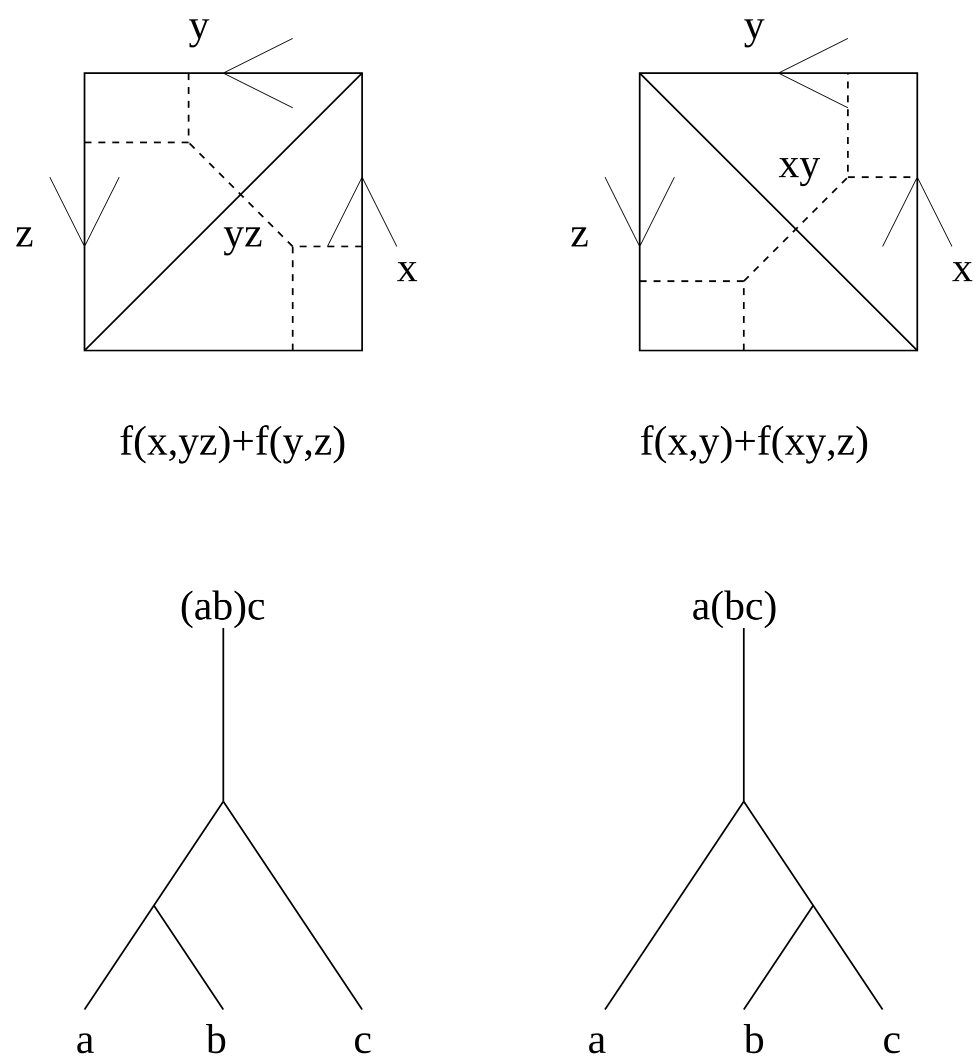

Figure 4: The cocycle condition and associativity

- Quandle 2-cocycles give invariants of knots that are expressed in state-sum form.

Let X denote a quandle, and let $C_n(X)$ denote the free Abelian group generated by n-tuples of elements $(x_1, x_2, \ldots, x_n)$ for which $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, n-1$. Define a boundary operator $\partial : C_n(X) \to C_{n-1}(X)$ on basis elements by

$$\partial_{n-1}(x_1, x_2, \ldots, x_n) \sum_{i=1}^{n} (-1)^i \left[(x_1 * x_i, \ldots x_{i-1} * x_i, x_{i+1}, \ldots, x_n) - (x_1, \ldots, \hat{x}_i, \ldots, x_n) \right].$$

It is easy to show that $\partial_{n-1} \circ \partial_n = 0$. Therefore we define, $H_n^{\mathrm{Q}}(X) = \mathrm{Ker}(\partial_{n-1})/\mathrm{Im}(\partial_n)$.

We represent generating chains by colored normally oriented crossings: the under arc away from which the normal to the over arc points receives the color, x_1, the over arc receives color x_2 and the other under arc receives color $x_1 * x_2$. In this way, colored knot diagrams can be used to represent 2-cycles. However, not every 2-cycle needs to have a planar representation.