Categorical Quandles and Knots

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Knots in Washington XXX
Joint work with:

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Plan

1. State Main Results
2. Motivation
3. Categories in Groups and Quandles
4. The Functors Conj and Core
5. Review $SU(2)$
6. Example computations $TSU(2)$
7. Non-group examples
8. The Fundamental 2-quandle.

Main Results

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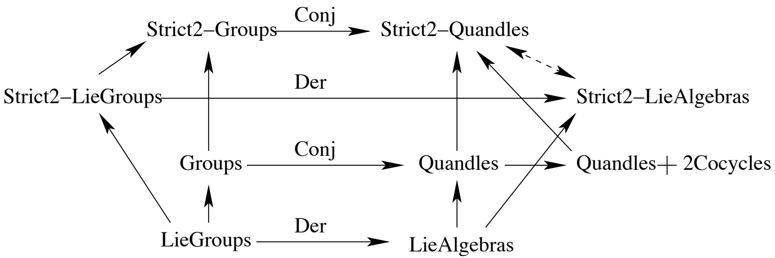
& some more stuff.

Motivations



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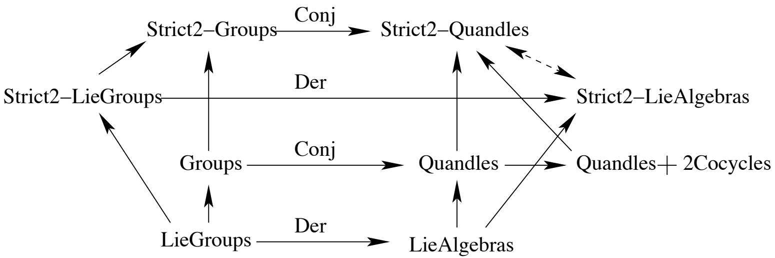

Relations via 3-cocycles ?!?!?!

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III. $\forall x, y, z \in X \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

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Category in $\mathcal{C}$

Here $\mathcal{C}$ is either the catgory of groups or the category of quandles. If A is an object in $\mathcal{C}$, then its underlying set is denoted by $|A|$. A *category in* $\mathcal{C}$ is constructed:

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3. $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$;
4. $(i(x) \circ f_1) = f_1$, and $(f_2 \circ i(x)) = f_2$

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Lemma

If $(O, M, s, t, i, \circ)_{\mathcal{C}_0}$ is a category in $\mathcal{C}_0$, then $(\hat{O}, \hat{M}, \hat{s}, \hat{t}, \hat{i}, \hat{\circ})_{\mathcal{C}_1}$ is a category in $\mathcal{C}_1$.

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3. Another example below.

Crossed Modules = Cat in group

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- comp. $((h_2, \tau(g_1)h_1) \circ (h_1, g_1) = (h_2h_1, g_1)$.

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$$= (\alpha(g_2^{-1}, h_2^{-1}) \alpha(g_2^{-1}, h_1) \alpha(g_2^{-1}, \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)$$

Stuff everyone should know about $SU(2)$

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then $g = x + y\hat{\imath} + z\hat{\jmath} + w\hat{k}$.

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Brackets: $\mathbf{i} \times \mathbf{j} = [\mathbf{i}, \mathbf{j}] = \mathbf{k}$, and $\mathbf{i} \times \mathbf{i} = \mathbf{0}$.

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$$\begin{pmatrix} x^2 + y^2 - z^2 - w^2 & 2(yz - xy) & 2(wy + xz) \\ 2(yz + xw) & x^2 + z^2 - y^2 - w^2 & 2(wz - xy) \\ 2(yw - xz) & 2(xy + zw) & x^2 + w^2 - y^2 - z^2 \end{pmatrix}.$$

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The axis is $y\hat{\imath} + z\hat{\jmath} + w\hat{k}$. Let $x = \cos(\theta)$, then the angle of rotation about this axis is $\pm 2\theta$.

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So in the corresponding category, $M = TSU(2)$.

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Consequently, we need to know . . .

$$(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}, \cos(s)\hat{\jmath} + \sin(s)\hat{k})$$

$$\triangleleft(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}, \cos(t)\hat{\jmath} + \sin(t)\hat{k})$$

$$= ((2a_2 - a_1)\,\hat{\imath}$$

$$+ [b_1\cos(2t)$$

$$+2\sin(s-2t)(c_2\cos(s) - b_2\sin(s)) + c_1\sin(2t)]\,\hat{\jmath}$$

$$+ [\sin(2t)(b_1 + b_2\cos(2s) - b_2 + c_2\sin(2s))$$

$$+ \cos(2t)(-b_2\sin(2s) - c_1 + c_2\cos(2s) + c_2)]\,\hat{k},$$

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- center arc next page

$$\left(a_2\hat{\imath} + \left(\frac{-b_2+\sqrt{3}c_2}{4} + \frac{\sqrt{3}}{2}(2c_2-c_1)\right)\hat{\jmath}\right.$$

$$\left. - \left(\frac{1}{2}(2c_2-c_1)+\frac{\sqrt{3}}{4}(-b_2+\sqrt{3}c_2)\right)\hat{k},\right.$$

$$\left.\frac{-1}{2}\hat{\jmath} - \frac{\sqrt{3}}{2}\hat{k}\right).$$

So the space of quandle representations for the trefoil into $TSU(2)$ has 4 free parameters.

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This will be called an *Alexander 2-quandle*.

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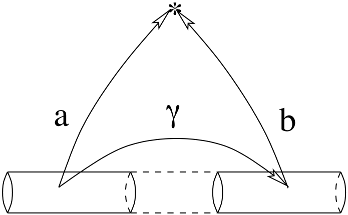

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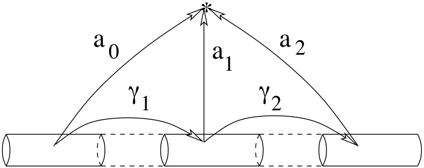
- Composition:
  $(a_2, \gamma_2, a_1) \circ (a_1, \gamma_1, a_0) = (a_2, \gamma_2 \circ \gamma_1, a_0)$


*
a 0
a 1
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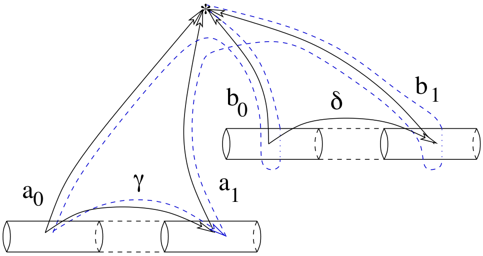

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- Masahico will discuss the virtual case as well as presentations for the fund. 2-quandle.