Categorical Quandles and Knots

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Knots in Washington XXX
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Plan

1. State Main Results
2. Motivation
3. Categories in Groups and Quandles
4. The Functors Conj and Core
5. Review $SU(2)$
6. Example computations $TSU(2)$
7. Non-group examples
8. The Fundamental 2-quandle.

Main Results

Theorem

The functors Conj and Core, when applied to a category object in the category of groups, give strict 2-quandles.

Theorem

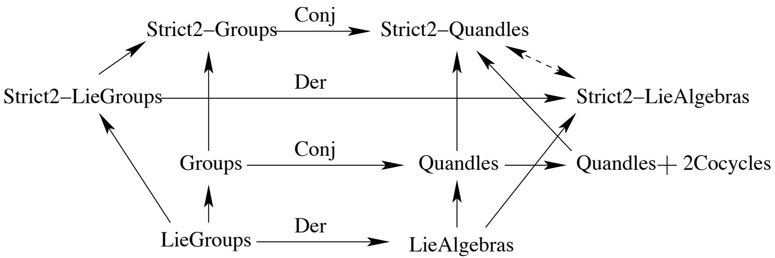
There are strict 2-quandles that don't come directly from groups.

Theorem

There is a strict 2-quandle that comes from the crossed module structure on $TSU(2)$. $\mapsto$ knot theory invariant.

& some more stuff.

Motivations



Relations via 3-cocycles ?!?!?!

Quandles

Definition

A *quandle* is a set X that has a binary operation $\triangleleft$ such that

I. $\forall x \in X \quad x \triangleleft x = x$.

II. $\forall x, y \in X \ \ \exists! z \in X$ such that $z \triangleleft x = y$. We write $z = y \triangleleft^{-1} x$.

III. $\forall x, y, z \in X \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

Examples of Quandles

1. G is a group, $a, b \in G$, let $a \triangleleft b = b^{-1}ab$: Conj.
2. G is a group, $a, b \in G$, let $a \triangleleft b = ba^{-1}b$: Core.
3. M is a $\mathbb{Z}[T, T^{-1}]$-module, let $a \triangleleft b = Ta + (1 - T)b$: LX-quandle.

Category in $\mathcal{C}$

Here $\mathcal{C}$ is either the catgory of groups or the category of quandles. If A is an object in $\mathcal{C}$, then its underlying set is denoted by $|A|$. A *category in* $\mathcal{C}$ is constructed:

- $O, M \in \text{Obj } \mathcal{C}$
- $s, t : M \to O$ and $i : O \to M$ are morphisms.
- Note that $M \times_O M = \{(f_2, f_1) : s(f_2) = t(f_1)\} \in \text{Obj } \mathcal{C}$.
- $c : M \times_O M \to M$, *composition* denoted $c : (f_2, f_1) \mapsto f_2 \circ f_1$.

Category in $\mathcal{C}$ continued

1. $s(i(x)) = t(i(x)) = x$ for any $x \in O$;
2. $s(f_2 \circ f_1) = s(f_1)$, $t(f_2 \circ f_1) = t(f_2)$;
3. $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$;
4. $(i(x) \circ f_1) = f_1$, and $(f_2 \circ i(x)) = f_2$

Category in $\mathcal{C}$ final

Suppose that $x, y \in O$. Then
$\hom(x, y) = \{f \in M : s(f) = x \;\&\; t(f) = y\}$.
Given a category $\mathcal{C}$, a category in $\mathcal{C}$:
$(O, M, s, t, i, \circ)_{\mathcal{C}}$ or $(O, M, s, t, i, \circ)$

Let $\mathcal{C}_0$ and $\mathcal{C}_1$ denote categories in which the objects have underlying sets. Suppose that $F : \mathcal{C}_0 \to \mathcal{C}_1$ is a functor such that for any object A in $\mathcal{C}_0$, the underlying set $|F(A)|$ is the induced image $F(|A|)$. Denote $F(X) = \hat{X}$.

Lemma

If $(O, M, s, t, i, \circ)_{\mathcal{C}_0}$ is a category in $\mathcal{C}_0$, then $(\hat{O}, \hat{M}, \hat{s}, \hat{t}, \hat{i}, \hat{\circ})_{\mathcal{C}_1}$ is a category in $\mathcal{C}_1$.

Crossed Modules

Definition

A *crossed module* is a quadruple (G, H, α, τ), where G and H are groups, $\alpha : G \times H \to H$ defines an action of G on H, $\tau : H \to G$ is a group homomorphism.

$$\alpha(\tau(h), h') = hh'h^{-1},$$

$$\tau(\alpha(g, h)) = g\tau(h)g^{-1}.$$

Examples

1. Let $H \triangleleft G$ denote a normal subgroup. G acts on H by conjugation; τ is the inclusion.
2. Let H denote a group, and let $G = \mathrm{Aut}(H)$; the map τ is the inclusion of H in G as inner automorphisms.
3. Another example below.

Crossed Modules = Cat in group

Given c.mod (G, H, α, τ), def $(O, M, s, t, i, \circ)_{\mathcal{G}}$

- objects $O = G$,
- morphisms $M = H \rtimes G$ (with multiplication $(h_1, g_1) \cdot (h_2, g_2) = (h_1 \alpha(g_1, h_2), g_1 g_2)$),
- source: $s(h, g) = g$,
- target map $t(h, g) = \tau(h)g$,
- the id. $i(g) = (1, g)$,
- comp. $((h_2, \tau(g_1)h_1) \circ (h_1, g_1) = (h_2 h_1, g_1)$.

Apply the Core functor

For morphisms, under conj. we have

$$(h_1, g_1) \triangleleft (h_2, g_2) = (\alpha(g_2^{-1}, h_2^{-1} h_1 \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)$$

$$= (\alpha(g_2^{-1}, h_2^{-1}) \alpha(g_2^{-1}, h_1) \alpha(g_2^{-1}, \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)$$

Stuff everyone should know about $SU(2)$

Consider $g = \begin{pmatrix} x+iy & z+iw \\ -z+iw & x-iy \end{pmatrix}$ where $x^2+y^2+z^2+w^2=1$. Let

$$\hat{\imath} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \hat{\jmath} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \hat{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};$$

then $g = x + y\hat{\imath} + z\hat{\jmath} + w\hat{k}$.

More stuff: $su(2)$

Any vector $v \in su(2)$ may be written as
$v = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. Observe $\hat{\imath} \cdot \hat{\jmath} = \hat{k}$, *etc.*
In particular, $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = -1$.
Standard basis: $\mathbf{i} = \hat{\imath}/2$, $\mathbf{j} = \hat{\jmath}/2$, and $\mathbf{k} = \hat{k}/2$
Brackets: $\mathbf{i} \times \mathbf{j} = [\mathbf{i}, \mathbf{j}] = \mathbf{k}$, and $\mathbf{i} \times \mathbf{i} = \mathbf{0}$.

More stuff: $su(2)$

Conjugate an algebra element by $g = x + y\hat{\imath} + z\hat{\jmath} + w\hat{k} \in SU(2)$. The matrix of this rotation (in the ordered basis $(\hat{\imath}, \hat{\jmath}, \hat{k})$) is

$$\begin{pmatrix} x^2 + y^2 - z^2 - w^2 & 2(yz - xy) & 2(wy + xz) \\ 2(yz + xw) & x^2 + z^2 - y^2 - w^2 & 2(wz - xy) \\ 2(yw - xz) & 2(xy + zw) & x^2 + w^2 - y^2 - z^2 \end{pmatrix}.$$

The axis is $y\hat{\imath} + z\hat{\jmath} + w\hat{k}$. Let $x = \cos(\theta)$, then the angle of rotation about this axis is $\pm 2\theta$.

The tangent bundle of $SU(2)$

In general, a Lie group and its Lie alg. define a crossed module. Here $G = SU(2)$, acts upon the abelian group $H = su(2)$ (vector space) via conj. $\alpha(g,h) = ghg^{-1}$. And $\tau : H \to G$ is (guesses?) the trivial map.
So in the corresponding category, $M = TSU(2)$.

The binary dihedral group

Consider the Hopf link $\in SU(2)$.

$$S^1_{1,i} = \{e^{i\theta}\} \quad \cup \quad S^1_{j,k} = \{e^{i\phi}\hat{\jmath}\}.$$

This link is a subgroup of $SU(2)$. A representation of the fundamental group of a 2-bridge knot into $SU(2)$ is conjugate to one that lands upon $S^1_{j,k}$. We may pick one generator to land at $\hat{\jmath}$.

Consequently, we need to know . . .

$$(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}, \cos(s)\hat{\jmath} + \sin(s)\hat{k})$$

$$\triangleleft(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}, \cos(t)\hat{\jmath} + \sin(t)\hat{k})$$

$$= ((2a_2 - a_1)\,\hat{\imath}$$

$$+ [b_1 \cos(2t)$$

$$+2\sin(s - 2t)(c_2\cos(s) - b_2\sin(s)) + c_1\sin(2t)]\,\hat{\jmath}$$

$$+ [\sin(2t)(b_1 + b_2\cos(2s) - b_2 + c_2\sin(2s))$$

$$+ \cos(2t)(-b_2\sin(2s) - c_1 + c_2\cos(2s) + c_2)]\,\hat{k},$$

$$\cos(2t - s)\hat{\jmath} + \sin(2t - s)\hat{k})$$

Example, the right-handed trefoil

Up to conjugation there is one representation into $SU(2)$ in which one arc is colored $\hat{\jmath}$. In contrast, when coloring by $su(2) \rtimes SU(2)$,

- Left top arc $(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}, \hat{\jmath})$,
- Right top arc $(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}, \cos(t)\hat{\jmath} + \sin(t)\hat{k})$.
- $t = \pm 2\pi/3$
- $a_1 = a_2$,
- $b_1 = \frac{1}{2}\left(-b_2 + \sqrt{3}c_2\right)$.
- center arc next page

$$\left(a_2\hat{\imath} + \left(\frac{-b_2 + \sqrt{3}c_2}{4} + \frac{\sqrt{3}}{2}(2c_2 - c_1) \right) \hat{\jmath} \right.$$

$$- \left(\frac{1}{2}(2c_2 - c_1) + \frac{\sqrt{3}}{4}(-b_2 + \sqrt{3}c_2) \right) \hat{k},$$

$$\left. \frac{-1}{2}\hat{\jmath} - \frac{\sqrt{3}}{2}\hat{k} \right) .$$

So the space of quandle representations for the trefoil into $TSU(2)$ has 4 free parameters. We have a similar computation for the figure eight knot.

Alexander strict 2-quandle

Let A denote a $\mathbb{Z}[T, T^{-1}]$-module. Let $O = A$; let $M = A \times A$.

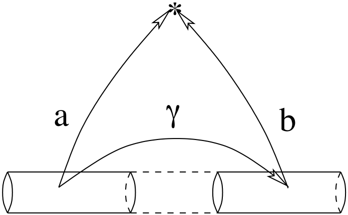
- On O we have $a \triangleleft b = Ta + (1 - T)b$;
- On M, we have $(a_1, a_2) \triangleleft (b_1, b_2) = (Ta_1 + (1 - T)b_1, Ta_2 + (1 - T)b_2)$.
- $i(a) = (0, a)$;
- $s(a, b) = b$;
- $t(a, b) = a + b$.
- $(a, c + d) \circ (c, d) = (a + c, d)$.

This will be called an *Alexander 2-quandle*.

Fundamental strict 2-quandle

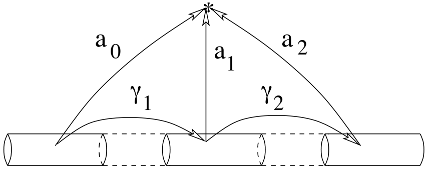
Let K denote an n-mfd. embedded in an $(n+2)$-mfd. M. We define a 2-quandle $\pi_Q^{(2)}(K) = \pi_Q^{(2)}(K, M)$:

- The quandle of objects: $O = \pi_Q(K)$, (describe)
- The quandle of morphisms: M hmtpy classes $a \cup \gamma \cup b$, where a, b are arcs "$\in \pi_Q(K)$." γ is an oriented arc between the feet of a and b.



Fundamental strict 2-quandle

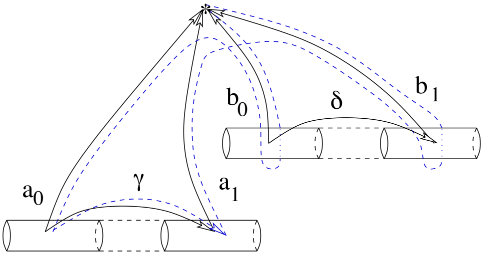
- $(b, \gamma, a) \in M$ — note: really an equiv. class.
- source: $s(b, \gamma, a) = a$.
- target: $t(b, \gamma, a) = b$.
- id: $i(a) = (a, c, a)$ where c is the constant arc.
- Composition: $(a_2, \gamma_2, a_1) \circ (a_1, \gamma_1, a_0) = (a_2, \gamma_2 \circ \gamma_1, a_0)$


*
a 0
a1
a 2
γ1
γ2

Fundamental strict 2-quandle

Quandle operations

- On objects: same as $\pi_Q(K)$
- On morphisms:
  $(a_1, \gamma, a_0) \triangleleft (b_1, \delta, b_0) = (a_1 \triangleleft b_1, \gamma, a_0 \triangleleft b_0)$



Closing remarks

- We have a working def of a strict 2-quandle
- We don't have the non-strict case, but we hope this is coordinated by quandle 3-cocycles.
- We have some interesting parameters when we extend representations to $TSU(2)$.
- The fund. 2-quandle does not appear to give more info. in the classical case.
- Masahico will discuss the virtual case as well as presentations for the fund. 2-quandle.