Classical knots, quandles, categorical quandles, and invariants

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Joint work with:

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Plan

1. The fundamental quandle
2. Quandles and quandle cocycles
3. Strict 2-quandles
4. the fundamental 2 quandle
5. local arrow systems
6. relations to the fundamental group, commutator subgroup, etc.
Theorem
For any knot diagram $K$ and a coloring $\mathcal{C}$ of $K$ by a group $G$ that has a “suitable” normal subgroup $H$, there exists a strict 2-quandle $X$ with objects $O = G$, morphisms $M = H \rtimes G$, and a coloring $\tilde{\mathcal{C}}$ of a local arrow system of $K$ by $X$ that extends $\mathcal{C}$, in the sense that the restriction of $\tilde{\mathcal{C}}$ to $O = A$ coincides with $\mathcal{C}$. 
Main Results for today

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\( H = [\pi_1, \pi_1] \) — the commutator subgroup.
In this case we color the knot tautologically with the meridional generators, and color local arrows with specific elements in the commutator.
Definition

A quandle is a set $X$ that has a binary operation $\triangleleft$ such that

I. $\forall x \in X \ x \triangleleft x = x$. 

II. $\forall x, y \in X \ \exists ! z \in X$ such that $z \triangleleft x = y$. We write $z = y \triangleleft^{-1} x$. 

III. $\forall x, y, z \in X \ (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$. 
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Examples of Quandles

1. $G$ is a group, $a, b \in G$, 

$\quad \quad a \triangledown b = b^{-1}ab$: Conj.

2. $G$ is a group, $a, b \in G$, 

$\quad \quad a \triangledown b = b^{-1}ab$: Core.

3. $M$ is a $\mathbb{Z}[T, T^{-1}]$-module, 

$\quad \quad a \triangledown b = Ta + (1 - T)b$: LX-quandle.
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Quandle colorings and quandle cocycles

F(1,2) - F(1,3) + F(2,1) - F(2,4)

cocycle condition: \( F(a,b) + F(a*b,c) = F(a,b*c) + F(a*c, b*c) \)
Applications

- 2-cocycle invariants for classical knots detect knottedness (Eisermann);
- 3-cocycle invariants for classical knots can detect chirality (Fenn Rourke);
- 3-cocycle invariants for knotted surfaces can be used to give bounds on the number of triple points (Satoh);
- for example, the 2-twist-spun trefoil has at least 4 triple points when projected into 3-space (Satoh Shima);
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• hyperbolic volume can be interpreted as a quandle cocycle invariant (Inoue);
• Dijkgraaf-Witten invariants can be interpreted as a quandle cocycle invariants (Hatakanaka);
Category in $\mathcal{C}$
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Here $\mathcal{C}$ is either the category of groups or the category of quandles. If $A$ is an object in $\mathcal{C}$, then its underlying set is denoted by $|A|$. A category in $\mathcal{C}$ is constructed:

- $O, M \in \text{Obj } \mathcal{C}$
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- $O, M \in \text{Obj} \mathcal{C}$
- $s, t : M \to O$ and $i : O \to M$ are morphisms.
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- Note that
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  M \times_O M = \{(f_2, f_1) : s(f_2) = t(f_1)\} \in \text{Obj } \mathcal{C}.
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- $c : M \times_O M \to M$, 
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- $c : M \times_O M \to M$, composition denoted $c : (f_2, f_1) \mapsto f_2 \circ f_1$. 
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Category in $\mathcal{C}$ continued

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2. $s(f_2 \circ f_1) = s(f_1)$, $t(f_2 \circ f_1) = t(f_2)$;
3. $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$;
Category in $\mathcal{C}$ continued

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2. $s(f_2 \circ f_1) = s(f_1)$, $t(f_2 \circ f_1) = t(f_2)$;
3. $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$;
4. $(i(x) \circ f_1) = f_1$, and $(f_2 \circ i(x)) = f_2$
Suppose that $x, y \in O$. 

Category in $\mathcal{C}$ final
Suppose that $x, y \in O$. Then
\[ \text{hom}(x, y) = \{ f \in M : s(f) = x \& t(f) = y \}. \]
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Given a category \( \mathcal{C} \), a category in \( \mathcal{C} \):
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Given a category $\mathcal{C}$, a category in $\mathcal{C}$:
\[ (O, M, s, t, i, \circ)_\mathcal{C} \text{ or } (O, M, s, t, i, \circ) \]
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Given a category $\mathcal{C}$, a category in $\mathcal{C}$:

$(O, M, s, t, i, \circ)_{\mathcal{C}}$ or $(O, M, s, t, i, \circ)$
Let $\mathcal{C}_0$ and $\mathcal{C}_1$ denote categories in which the objects have underlying sets. Suppose that $F : \mathcal{C}_0 \to \mathcal{C}_1$ is a functor such that for any object $A$ in $\mathcal{C}_0$, the underlying set $|F(A)|$ is the induced image $F(|A|)$. 

Denote $F(X) = \hat{X}$. 

Lemma If $(O, M, s, t, i, \circ)$ is a category in $\mathcal{C}_0$, then $(\hat{O}, \hat{M}, \hat{s}, \hat{t}, \hat{i}, \hat{\circ})$ is a category in $\mathcal{C}_1$. 


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Lemma: If $(O,M,s,t,i,\circ)$ is a category in $\mathcal{C}_0$, then $(\hat{O},\hat{M},\hat{s},\hat{t},\hat{i},\hat{\circ})$ is a category in $\mathcal{C}_1$. 
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**Lemma**

If $(O, M, s, t, i, \circ)_{\mathcal{C}_0}$ is a category in $\mathcal{C}_0$, then $(\hat{O}, \hat{M}, \hat{s}, \hat{t}, \hat{i}, \hat{\circ})_{\mathcal{C}_1}$ is a category in $\mathcal{C}_1$. 
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Definition

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A crossed module is a quadruple $(G, H, \alpha, \tau)$, where $G$ and $H$ are groups, $\alpha : G \times H \to H$ defines an action of $G$ on $H$, $\tau : H \to G$ is a group homomorphism.
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Examples

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1. Let $H \triangleleft G$ denote a normal subgroup. $G$ acts on $H$ by conjugation; $\tau$ is the inclusion.

2. Let $H$ denote a group, and let $G = \text{Aut}(H)$; the map $\tau$ is the inclusion of $H$ in $G$ as inner automorphisms.
Crossed Modules = Cat in group

Given c.mod \((G, H, \alpha, \tau)\), def \((O, M, s, t, i, \circ)_G\)
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- target map \(t(h, g) = \tau(h)g\),
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Given c.mod $(G, H, \alpha, \tau)$, def $(O, M, s, t, i, \circ)_G$

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- the id. $i(g) = (1, g)$,
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- source: \(s(h, g) = g\),
- target map \(t(h, g) = \tau(h)g\),
- the id. \(i(g) = (1, g)\),
- comp. \(((h_2, \tau(g_1)h_1) \circ (h_1, g_1) = (h_2 h_1, g_1)\).
Apply the Conj functor

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\[(h_1, g_1) \triangleleft (h_2, g_2) = (\alpha(g_2^{-1}, h_2^{-1}h_1 \alpha(g_1, h_2)), g_2^{-1}g_1g_2)\]

\[= (\alpha(g_2^{-1}, h_2^{-1})\alpha(g_2^{-1}, h_1)\alpha(g_2^{-1}, \alpha(g_1, h_2)), g_2^{-1}g_1g_2)\]
Recap

- A (strict) category in the category of groups

The functor Conj: \text{groups} \rightarrow \text{quandles}.

A (strict) 2-quandle is a category in the category of quandles.

Under Conj a crossed module gives a strict 2-quandle.

There are other strict 2-quandles.
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- A (strict) category in the category of groups = crossed module.
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- A (strict) 2-quandle is a category in the category of quandles.
- Under $\text{Conj}$ a crossed module gives a strict 2-quandle.
- There are other strict 2-quandles.
Fundamental strict 2-quandle

Let $K$ denote an $n$-mfd.
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Let $K$ denote an $n$-mfd. embedded in an $(n + 2)$-mfd. $M$. 
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Let $K$ denote an $n$-mfd. embedded in an $(n + 2)$-mfd. $M$. We define a 2-quandle $\pi^{(2)}_Q(K) = \pi^{(2)}_Q(K, M)$:
Fundamental strict 2-quandle

Let $K$ denote an $n$-mfd. embedded in an $(n + 2)$-mfd. $M$. We define a 2-quandle
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- The quandle of objects: $O = \pi_Q(K)$,
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- The quandle of morphisms: $M$ hmtpy classes $a \cup \gamma \cup b$, 

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{a} \\
\end{array}
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- The quandle of objects: $O = \pi_Q(K)$,
- The quandle of morphisms: $M$ hmtpy classes $a \cup \gamma \cup b$, where $a, b$ are arcs “$\in \pi_Q(K)$.” $\gamma$ is an oriented arc between the feet of $a$ and $b$. 
Fundamental strict 2-quandle

Let $K$ denote an $n$-mfd. embedded in an $(n + 2)$-mfd. $M$. We define a 2-quandle $\pi_{Q}^{(2)}(K) = \pi_{Q}^{(2)}(K, M)$:

- The quandle of objects: $O = \pi_{Q}(K)$,
- The quandle of morphisms: $M$ hmtpy classes $a \cup \gamma \cup b$, where $a, b$ are arcs “$\in \pi_{Q}(K)$.” $\gamma$ is an oriented arc between the feet of $a$ and $b$. 
Fundamental strict 2-quandle

- $(b, \gamma, a) \in M$

- $s(b, \gamma, a) = a$

- $t(b, \gamma, a) = b$

- $i(a) = (a, c, a)$ where $c$ is the constant arc.

- Composition: $(a_2, \gamma_2, a_1) \circ (a_1, \gamma_1, a_0) = (a_2, \gamma_2 \circ \gamma_1, a_0)$
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Quandle operations

• On objects: same as $\pi_Q(K)$
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Local arrow systems

A primary local arrow of a crossing $\tau$ for a knot diagram $K$: 
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A primary local arrow of a crossing \( \tau \) for a knot diagram \( K \): A local arrow \( \rho_x \) of an (over-)arc \( x \) of \( K \) :
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A primary local arrow of a crossing $\tau$ for a knot diagram $K$: A local arrow $\rho_x$ of an (over-)arc $x$ of $K$: $x \gamma y$ $z \gamma x$ $\rho_x$ of $K$:
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A local arrow of a crossing $\tau$ is one of four arrows among the arrows $\gamma, \gamma_i, i = 1, 2, 3$. 
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$\gamma$, $\gamma_i$, $i = 1, 2, 3$. 
Local Arrow systems, cont.

A set $\mathcal{R} = \mathcal{R}_C \cup \mathcal{R}_A$ of local arrows over all crossings and over-arcs of $K$ (and inverse arrows) is called a local arrow system of $K$. 
Colorings of local arrows systems

Let $X = (O, M, s, t, \circ)$ denote a strict 2-quandle such that every morphism is invertible.
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Theorem

There exists a one-to-one correspondence between \( \text{Col}_R(K, X) \) and \( \text{Col}_R(K', X) \) for the diagrams before and after each Reidemeister move, where \( K \) and \( K' \) are diagrams before and after the move. In particular, the cardinality of \( \text{Col}_R(K, X) \) is a knot invariant.
To show

\[ \rho_x^{-1} \circ \gamma_3'^{-1} \circ \rho_z^{-1} \circ (\gamma_3 \triangleleft \rho_z) \circ (\rho_x \triangleleft \rho_z) \circ (\gamma_3' \triangleleft \rho_z)^{-1} \circ \rho_z \circ \gamma_3' = 1 \]

\[ x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_3 \triangleleft z)(x \triangleleft z)(\gamma_3' \triangleleft z)^{-1} z \gamma_3' = 1 \]

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\[ = x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_3 \triangleleft z)(x \triangleleft z)\gamma_1^{-1}(\gamma_2 \triangleleft z)^{-1} z \gamma_2 y \gamma_1 \]

\[ = x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_2 \triangleleft z)\gamma_1^{-1}(y \triangleleft z)^{-1} (\gamma_1 \triangleleft (y \triangleleft z))(\gamma_3 \triangleleft z)^{-1} z \gamma_2 y \gamma_1 \]

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\[ = x^{-1} \gamma_1^{-1} y^{-1} (\gamma_1' \triangleleft y)(x \triangleleft y)(\gamma_1 \triangleleft y)^{-1} y \gamma_1 \]

\[ = 1 \]
Theorem
For any link diagram $K$ and a coloring $\mathcal{C}$ of $K$ by a group $G$ that has a “suitable” normal subgroup $H$ there exists a strict 2-quandle $X$ with objects $O = G$, morphisms $M = H \rtimes G$, and a coloring $\tilde{\mathcal{C}}$ of a local arrow system of $K$ by $X$ that extends $\mathcal{C}$, in the sense that the restriction of $\tilde{\mathcal{C}}$ to $O = A$ coincides with $\mathcal{C}$. 
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Theorem
The Fox derivatives transform the fundamentally natural coloring to a coloring of the local arrow system by the strict Alexander 2-quandle $\mathcal{A}$. 
Let $G = \pi_1(K)$; let $H = [\pi_1, \pi_1] = \pi'$. Let $g_1$ and $g_1$ be the associated meridional generators, $h_1 = g_2g_1^{-1} \in \pi'$, and $h_3 = g_1^{-1}g_2$.
Conclusion

We can modify this coloring technique to get “vertex colors.” These are related to Alexander and Brigg’s definition of the Alexander polynomial. (Next slide)
Conclusion

We can modify this coloring technique to get “vertex colors.” These are related to Alexander and Brigg’s definition of the Alexander polynomial. (Next slide) Also, we have other new invariants of classical knots and of virtual knots that result.
\[-T + T^2 - T + (1-T) = 0\]

\[T^2 - 3T + 1 = 0\]
Thanks to the organizers and the audience!