Classical knots, quandles, categorical quandles, and invariants

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Plan

1. The fundamental quandle
2. Quandles and quandle cocycles
3. Strict 2-quandles
4. the fundamental 2 quandle
5. local arrow systems
6. relations to the fundamental group, commutator subgroup, etc.
Main Results for today

Theorem
For any knot diagram $K$ and a coloring $\mathcal{C}$ of $K$ by a group $G$ that has a “suitable” normal subgroup $H$, there exists a strict 2-quandle $X$ with objects $O = G$, morphisms $M = H \rtimes G$, and a coloring $\tilde{\mathcal{C}}$ of a local arrow system of $K$ by $X$ that extends $\mathcal{C}$, in the sense that the restriction of $\tilde{\mathcal{C}}$ to $O = A$ coincides with $\mathcal{C}$. 
In the above “suitable” means that for any pair of elements \( g_1, g_2 \in G \) there exists an \( h_1 \in H \) such that \( h_1 g_1 = g_2 \), but we can get by with weaker conditions. The example that we have in mind is \( G = \pi_1(K) \) — the fundamental group. \( H = [\pi_1, \pi_1] \) — the commutator subgroup.

In this case we color the knot tautologically with the meridional generators, and color local arrows with specific elements in the commutator.
The Fundamental Quandle
Quandles

Definition
A quandle is a set $X$ that has a binary operation $\triangleleft$ such that
I. $\forall x \in X \quad x \triangleleft x = x$.
II. $\forall x, y \in X \quad \exists! z \in X$ such that $z \triangleleft x = y$. We write $z = y \triangleleft^{-1} x$.
III. $\forall x, y, z \in X \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
Examples of Quandles

1. $G$ is a group, $a, b \in G$, let $a \triangleleft b = b^{-1}ab$ : Conj.

2. $G$ is a group, $a, b \in G$, let $a \triangleleft b = ba^{-1}b$ : Core.

3. $M$ is a $\mathbb{Z}[T, T^{-1}]$-module, let $a \triangleleft b = Ta + (1 - T)b$ : LX-quandle.
Quandle colorings and quandle cocycles

F(1,2)-F(1,3)+F(2,1)-F(2,4)

cocycle condition: $F(a,b)+F(a*b,c)=F(a,b*c)+F(a*c, b*c)$
Applications

- 2-cocycle invariants for classical knots detect knottedness (Eisermann);
- 3-cocycle invariants for classical knots can detect chirality (Fenn Rourke);
- 3-cocycle invariants for knotted surfaces can be used to give bounds on the number of triple points (Satoh);
- for example, the 2-twist-spun trefoil has at least 4 triple points when projected into 3-space (Satoh Shima);
Applications continued

- 2-cocycles can be used to show that certain surfaces knots are not ribbon concordant (C. Saito);
- hyperbolic volume can be interpreted as a quandle cocycle invariant (Inoue);
- Dijkgraaf-Witten invariants can be interpreted as a quandle cocycle invariants (Hatakanaka);
Category in $\mathcal{C}$

Here $\mathcal{C}$ is either the category of groups or the category of quandles. If $A$ is an object in $\mathcal{C}$, then its underlying set is denoted by $|A|$. A category in $\mathcal{C}$ is constructed:

- $O, M \in \text{Obj } \mathcal{C}$
- $s, t : M \rightarrow O$ and $i : O \rightarrow M$ are morphisms.
- Note that $M \times_O M = \{(f_2, f_1) : s(f_2) = t(f_1)\} \in \text{Obj } \mathcal{C}$.
- $c : M \times_O M \rightarrow M$, composition denoted $c : (f_2, f_1) \mapsto f_2 \circ f_1$. 
1. $s(i(x)) = t(i(x)) = x$ for any $x \in O$;
2. $s(f_2 \circ f_1) = s(f_1)$, $t(f_2 \circ f_1) = t(f_2)$;
3. $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$;
4. $(i(x) \circ f_1) = f_1$, and $(f_2 \circ i(x)) = f_2$
Suppose that \( x, y \in O \). Then
\[
\text{hom}(x, y) = \{ f \in M : s(f) = x \text{ } \& \text{ } t(f) = y \}.
\]
Given a category \( \mathcal{C} \), a category in \( \mathcal{C} \):
\[(O, M, s, t, i, \circ)_{\mathcal{C}} \text{ or } (O, M, s, t, i, \circ)\]
Let $\mathcal{C}_0$ and $\mathcal{C}_1$ denote categories in which the objects have underlying sets. Suppose that $F : \mathcal{C}_0 \to \mathcal{C}_1$ is a functor such that for any object $A$ in $\mathcal{C}_0$, the underlying set $|F(A)|$ is the induced image $F(|A|)$. Denote $F(X) = \hat{X}$.

**Lemma**

If $(O, M, s, t, i, \circ)_{\mathcal{C}_0}$ is a category in $\mathcal{C}_0$, then $(\hat{O}, \hat{M}, \hat{s}, \hat{t}, \hat{i}, \hat{\circ})_{\mathcal{C}_1}$ is a category in $\mathcal{C}_1$. 
Crossed Modules

Definition

A crossed module is a quadruple \((G, H, \alpha, \tau)\), where \(G\) and \(H\) are groups, \(\alpha : G \times H \to H\) defines an action of \(G\) on \(H\), \(\tau : H \to G\) is a group homomorphism.

\[
\alpha(\tau(h), h') = hh'h^{-1},
\]

\[
\tau(\alpha(g, h)) = g\tau(h)g^{-1}.
\]
Examples

1. Let $H \triangleleft G$ denote a normal subgroup. $G$ acts on $H$ by conjugation; $\tau$ is the inclusion.
2. Let $H$ denote a group, and let $G = \text{Aut}(H)$; the map $\tau$ is the inclusion of $H$ in $G$ as inner automorphisms.
Crossed Modules = Cat in group

Given c.mod \((G, H, \alpha, \tau)\), def \((O, M, s, t, i, \circ)\)

- objects \(O = G\),
- morphisms \(M = H \rtimes G\) (with multiplication \((h_1, g_1) \cdot (h_2, g_2) = (h_1 \alpha(g_1, h_2), g_1g_2)\)),
- source: \(s(h, g) = g\),
- target map \(t(h, g) = \tau(h)g\),
- the id. \(i(g) = (1, g)\),
- comp. \(((h_2, \tau(g_1)h_1) \circ (h_1, g_1) = (h_2h_1, g_1)\).
Apply the Conj functor

For morphisms, under conj. we have

$$(h_1, g_1) \triangleleft (h_2, g_2) = (\alpha(g_2^{-1}, h_2^{-1} h_1 \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)$$

$$= (\alpha(g_2^{-1}, h_2^{-1}) \alpha(g_2^{-1}, h_1) \alpha(g_2^{-1}, \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)$$
Recap

- A (strict) category in the category of groups = crossed module.
- The functor Conj: groups → quandles.
- A (strict) 2-quandle is a category in the category of quandles.
- Under Conj a crossed module gives a strict 2-quandle.
- There are other strict 2-quandles.
Fundamental strict 2-quandle

Let $K$ denote an $n$-mfd. embedded in an $(n + 2)$-mfd. $M$. We define a 2-quandle $\pi_Q^{(2)}(K) = \pi_Q^{(2)}(K, M)$:

- The quandle of objects: $O = \pi_Q(K)$,
- The quandle of morphisms: $M$ hmtpy classes $a \cup \gamma \cup b$, where $a, b$ are arcs $\in \pi_Q(K)$. $\gamma$ is an oriented arc between the feet of $a$ and $b$. 
Fundamental strict 2-quandle

• \((b, \gamma, a) \in M \) — note: really an equiv. class.
• source: \(s(b, \gamma, a) = a\).
• target: \(t(b, \gamma, a) = b\).
• id: \(i(a) = (a, c, a)\) where \(c\) is the constant arc.
• Composition:
  \[(a_2, \gamma_2, a_1) \circ (a_1, \gamma_1, a_0) = (a_2, \gamma_2 \circ \gamma_1, a_0)\]
Fundamental strict 2-quandle

Quandle operations

- On objects: same as $\pi_Q(K)$
- On morphisms:
  \[(a_1, \gamma, a_0) \triangleleft (b_1, \delta, b_0) = (a_1 \triangleleft b_1, \gamma, a_0 \triangleleft b_0)\]
Local arrow systems

A primary local arrow of a crossing $\tau$ for a knot diagram $K$: A local arrow $\rho_x$ of an (over-)arc $x$ of $K$: A local arrow of a crossing $\tau$ is one of four arrows among the arrows $\gamma, \gamma_i, i = 1, 2, 3$. 
A set $\mathcal{R} = \mathcal{R}_C \cup \mathcal{R}_A$ of local arrows over all crossings and over-arcs of $K$ (and inverse arrows) is called a local arrow system of $K$. 
Let $X = (O, M, s, t, \circ)$ denote a strict 2-quandle such that every morphism is invertible. A distributive element $z \in M$ satisfies $(x \circ y) \circ z = (x \circ z) \circ (y \circ z) \ \forall x, y \in M$. 
Colorings of Local Arrows systems

We color a local arrow system by

\[ X = (O, M, s, t, \circ) \]

- Arcs are assigned objects
- Arrows are assigned morphisms
- Colors assigned to local arrows of arcs are assigned distributive elements
Theorem

There exists a one-to-one correspondence between $\text{Col}_R(K, X)$ and $\text{Col}_R(K', X)$ for the diagrams before and after each Reidemeister move, where $K$ and $K'$ are diagrams before and after the move. In particular, the cardinality of $\text{Col}_R(K, X)$ is a knot invariant.
To show

$$\rho_x^{-1} \circ \gamma_3^{-1} \circ \rho_z^{-1} \circ (\gamma_3 \triangleleft \rho_z) \circ (\rho_x \triangleleft \rho_z) \circ (\gamma_3' \triangleleft \rho_z)^{-1} \circ \rho_z \circ \gamma_3' = 1$$

$$x^{-1} \gamma_3^{-1} z^{-1} (\gamma_3 \triangleleft z)(x \triangleleft z)(\gamma_3' \triangleleft z)^{-1} z\gamma_3'$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_3 \triangleleft z)(x \triangleleft z)(\gamma_3' \triangleleft z)^{-1} z\gamma_2 \gamma_1$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_3 \triangleleft z)(x \triangleleft z)\gamma_1^{-1}(\gamma_2 \triangleleft z)^{-1} z\gamma_2 \gamma_1$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_3 \triangleleft z)(x \triangleleft z)\gamma_1^{-1}(y \triangleleft z)^{-1}(\gamma_2 \triangleleft z)^{-1} z\gamma_2 y \gamma_1$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_2' \triangleleft z)\gamma_1'(x \triangleleft z)\gamma_1^{-1}(y \triangleleft z)^{-1}(\gamma_1' \triangleleft (y \triangleleft z))(\gamma_3 \triangleleft z)^{-1} z\gamma_2 y \gamma_1$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_2 \triangleleft z)(y \triangleleft z)^{-1}(\gamma_1' \triangleleft (y \triangleleft z))((x \triangleleft y) \triangleleft z)(\gamma_3 \triangleleft z)^{-1} z\gamma_2 y \gamma_1$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_2' \triangleleft z)(y \triangleleft z)^{-1}(\gamma_1' \triangleleft (y \triangleleft z))((x \triangleleft y) \triangleleft z)(\gamma_3 \triangleleft z)^{-1} z\gamma_3(\gamma_1 \triangleleft y)^{-1} y \gamma_1$$

$$= x^{-1} \gamma_3'^{-1} z^{-1} (\gamma_2 \triangleleft z)(y \triangleleft z)^{-1}(\gamma_2' \triangleleft z)z\gamma_3(x \triangleleft y)(\gamma_1 \triangleleft y)^{-1} y \gamma_1$$

$$= x^{-1} \gamma_1^{-1} \gamma_2^{-1} z^{-1} (\gamma_2' \triangleleft z)(y \triangleleft z)^{-1}(\gamma_2' \triangleleft z)z\gamma_2'(\gamma_1' \triangleleft y)(x \triangleleft y)(\gamma_1 \triangleleft y)^{-1} y \gamma_1$$

$$= x^{-1} \gamma_1^{-1} y(\gamma_1' \triangleleft y)(x \triangleleft y)(\gamma_1 \triangleleft y)^{-1} y \gamma_1$$

$$= 1$$
Theorem

For any link diagram $K$ and a coloring $\mathcal{C}$ of $K$ by a group $G$ that has a “suitable” normal subgroup $H$ there exists a strict 2-quandle $X$ with objects $O = G$, morphisms $M = H \rtimes G$, and a coloring $\tilde{\mathcal{C}}$ of a local arrow system of $K$ by $X$ that extends $\mathcal{C}$, in the sense that the restriction of $\tilde{\mathcal{C}}$ to $O = A$ coincides with $\mathcal{C}$.

Theorem

The Fox derivatives transform the fundamentally natural coloring to a coloring of the local arrow system by the strict Alexander 2-quandle $A$. 
Let $G = \pi_1(K)$; let $H = [\pi_1, \pi_1] = \pi'_1$. Let $g_1$ and $g_1$ be the associated meridional generators, $h_1 = g_2 {g_1}^{-1} \in \pi'$, and $h_3 = {g_1}^{-1} g_2$. 

\[
g_1 \quad (h_1, g_1) \quad g_2 = h_1 g_1 \quad h_1 = g_2 {g_1}^{-1} \\
g_2 \quad (h_3, g_3) \quad g_3 = {g_2}^{-1} g_1 g_2 \quad h_3 = {g_1}^{-1} g_2
\]
Conclusion

We can modify this coloring technique to get “vertex colors.” These are related to Alexander and Brigg’s definition of the Alexander polynomial. (Next slide)
Also, we have other new invariants of classical knots and of virtual knots that result.
\[ -T + T^2 - T + (1-T) = 0 \]
\[ T^2 - 3T + 1 = 0 \]
Thanks to the organizers and the audience!