A non-involutory quandle that is connected and has a good involution

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Joint work with:

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Plan

1. Quandles
2. Involutory Quandles
3. Good Involution
4. Constructions of quandles
5. The dihedral quandle
6. Its extension
7. Conclusion

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III. $\forall x, y, z \in X \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

Example 1

Let $m = 2n + 1$. Consider the set of reflections in the dihedral group with $\triangleleft$ being conjugation.
We can let $R_m = \{1, 2, \ldots, m\}$ with
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$c = 2a - b$.
In other words, $b \triangleleft a = b \triangleleft^{-1} a$. Such a quandle is called *involutory*.

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2. $a \triangleleft \rho(b) = a \triangleleft^{-1} b$.

A quandle that possesses an involution that satisfies (1) and (2) is said to have a *good involution*.

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Cf. The fundamental quandle of a classical knot is isom. (as a quandle) to $(\pi_1(k), P, \mu_*)$ where P is the peripheral subgroup, and μ_* is the meridian at the base point which has been chosen to be a point on the tubular nbhd of k.

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$y \mapsto \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{bmatrix}$. So $\begin{array}{l} x \mapsto (e_1, e_m, \dots, e_2) \\ y \mapsto (e_m, e_1, \dots, e_{m-1}). \end{array}$

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2. $(G, H, a) = \tilde{R}_{2n+1}$ *is a connected non-involutory quandle with a good-involution.*
3. *There is a surjective quandle homom.* $F : \tilde{R}_{2n+1} \to R_{2n+1}$, *given by* $Ht \mapsto \langle 1, x \rangle |t|$.

Sketch of Proof

Let $\vec{\epsilon} = (\epsilon_1 e_1, \epsilon_2 e_2, \ldots, \epsilon_{2n+1} e_{2n+1})$. As long as $\prod_j \epsilon_j = 1$, the element $\vec{\epsilon} \in G_{2n+1}$.

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Proof sketch cont.

Let $\rho(Hu) =$
$H((-1)^n e_1, -e_2, \ldots - e_{n+1}, e_{n+2}, \ldots e_{2n+1})u.$

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Proof sketch cont.

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To show that $\tilde{R}_{2n+1}$ is not involutory, just check $(Ha \triangleleft Hb) \triangleleft Hb \neq Ha$.

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Let

$$\begin{aligned} I_- &= (-e_1, e_2, \ldots, e_{n+1}, -e_{n+2}, e_{n+3}, \ldots, e_{2n+1}) \\ &= a^2 b^{-n-1} a^2 b^{n+1} \end{aligned}$$

and

$$I_+ = (e_1, \ldots, e_n, -e_{n+1}, e_{n+2}, \ldots, e_{2n}, -e_{2n+1})$$

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and

$$ba = ab^{-1} I_+ \quad b^{-1} a = abI_-$$

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Easiest Example

The interesting thing about this example is that its symmetric quandle homology has a free summand.