Algebraic Deformations and Solutions to the YBE

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Outline

1. Overview

2. Yang-Baxter Solutions
   - Lie Algebra
   - Hopf Algebra
   - Frobenius Algebra
   - Duality pairing algebra

3. cocycles

4. Conclusion
Summary of results

Theorem

There is a cohomology theory in which both quandle cocycles and Lie algebra cocycles give non-trivial cocycles.
Theorem

*There is a cohomology theory in which both quandle cocycles and Lie algebra cocycles give non-trivial cocycles.*

Theorem

*In analogous theories for Hopf Algebras, Frobenius Algebras, and duality pairing algebras deformation cocycles can be used to construct YBE solutions.*
Anticipated applications

1. Connections between quantum invariants and fundamental group.
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2. Diagrammatically defined cocycles related to aspects of physical categorifications.
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Coalgebra

\[ Y = Y \quad \text{Co-assoc.} \]

\[ Y = Y \quad \text{Co-unit} \]
Let $\mathfrak{g}$ denote a Lie Alg. over a field $\mathbb{F}$. 
Lie Algebras

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Lie Algebras

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Lie Algebras

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Let $\mathfrak{g}$ denote a Lie Alg. over a field $\mathbb{F}$. Consider $N = \mathbb{F} \oplus \mathfrak{g}$. Then $N$ is a cocommutative coalgebra with counit $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$, $\Delta(1) = 1 \otimes 1$. The function $q : N \otimes N \rightarrow N$ defined by $q((a, x) \otimes (b, y)) = (ab, bx + [x, y])$ can be used to define a solution to the Yang-Baxter relation. [Woronowicz] Studied by Crans in her dissertation.
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$\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$

$\Delta(1) = 1 \otimes 1$

$\epsilon(1) = 1, \epsilon(x) = 0$ for $x \in \mathfrak{g}$. 

The function $q: N \otimes N \rightarrow N$ defined by

$q((a, x) \otimes (b, y)) = (ab, bx + [x, y])$

satisfies a coalgebra self-distributive relation AND can be used to define a solution to the Yang-Baxter relation. [Woronowicz] Studied by Crans in her dissertation.
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Lie Algebras

Let \( g \) denote a Lie Alg. over a field \( F \). Consider \( N = F \oplus g \). Then \( N \) is a cocommutative coalgebra with counit

\[
\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{for} \quad x \in g
\]

\[
\Delta(1) = 1 \otimes 1
\]

\[
\epsilon(1) = 1, \quad \epsilon(x) = 0 \quad \text{for} \quad x \in g.
\]

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\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
Lie alg, self distrib, and YBE

\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = \]
Lie alg, self distrib, and YBE

\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = \]
\[ q((ab + bx + [x, y]) \otimes (c, z)) = abc + bcx + c[x, y] + b[x, z] + [[x, y], z] \]
Lie alg, self distrib, and YBE

\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = \]
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\[ q(q \otimes q)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta)((a + x) \otimes (b + y) \otimes (c + z)) \]
$$q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) =$$

$$q((ab + bx + [x, y]) \otimes (c, z)) = abc + bcx + c[x, y] + b[x, z] + [[x, y], z]$$

$$q(q \otimes q)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta)((a + x) \otimes (b + y) \otimes (c + z)) =$$

$$q(q \otimes q)(1 \otimes \tau \otimes 1)((a + x) \otimes (b + y) \otimes \{(c + z) \otimes 1 + 1 \otimes z\})$$
$q(q \otimes 1)(a, x) \otimes (b, y) \otimes (c, z) = q((ab + bx + [x, y]) \otimes (c, z)) = abc + bcx + c[x, y] + b[x, z] + [x, y, z]$

$q(q \otimes q)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta)((a + x) \otimes (b + y) \otimes (c + z))$

$= q(q \otimes q)(1 \otimes \tau \otimes 1)((a + x) \otimes (b + y) \otimes \{(c + z) \otimes 1 + 1 \otimes z\})$

$= q(q \otimes q)((a + x) \otimes (c + z) \otimes (b + y) \otimes 1 + (a + x) \otimes 1 \otimes (b + y) \otimes z$
$$q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) =$$

$$q((ab + bx + [x, y]) \otimes (c, z)) = abc + bcx + c[x, y] + b[x, z] + [x, y, z]$$

$$q(q \otimes q)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta)((a + x) \otimes (b + y) \otimes (c + z))$$

$$= q(q \otimes q)(1 \otimes \tau \otimes 1)((a + x) \otimes (b + y) \otimes \{(c + z) \otimes 1 + 1 \otimes z\})$$

$$= q(q \otimes q)((a + x) \otimes (c + z) \otimes (b + y) \otimes 1 + (a + x) \otimes 1 \otimes (b + y) \otimes z$$

$$= q((ac + cx + [x, z]) \otimes (b + y)) + q((a + x) \otimes [y, z])$$
\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = \]
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\[ q(q \otimes q)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta)((a + x) \otimes (b + y) \otimes (c + z)) \]
\[ = q(q \otimes q)(1 \otimes \tau \otimes 1)((a + x) \otimes (b + y) \otimes \{(c + z) \otimes 1 + 1 \otimes z\}) \]
\[ = q(q \otimes q)((a + x) \otimes (c + z) \otimes (b + y) \otimes 1 + (a + x) \otimes 1 \otimes (b + y) \otimes z \]
\[ = q((ac + cx + [x, z]) \otimes (b + y)) + q((a + x) \otimes [y, z]) = \]
\[ (abc + bcx + c[x, y] + b[x, z] + [[x, z], y]) + [x, [y, z]] \]
\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = \\
q((ab + bx + [x, y]) \otimes (c, z)) = abc + bcx + c[x, y] + b[x, z] + [[x, y], z] \\
q(q \otimes q)(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \Delta)((a + x) \otimes (b + y) \otimes (c + z)) \\
= q(q \otimes q)(1 \otimes \tau \otimes 1)((a + x) \otimes (b + y) \otimes \{(c + z) \otimes 1 + 1 \otimes z\}) \\
= q(q \otimes q)((a + x) \otimes (c + z) \otimes (b + y) \otimes 1 + (a + x) \otimes 1 \otimes (b + y) \otimes z \\
= q((ac + cx + [x, z]) \otimes (b + y)) + q((a + x) \otimes [y, z]) = \\
(abc + bcx + c[x, y] + b[x, z] + [[x, z], y]) + [x, [y, z]] \\
R_q((a, x) \otimes (b, y)) = (b, y) \otimes (a, x) + (1, 0) \otimes (0, [x, y]) \]
Remarks

Crans’s dissertation discusses this in relation to her construction of so-called Lie 2-algebras. In particular, the Lie cocycle, $\langle a, [b, c] \rangle$ is used to construct a solution to the Zamolodchikov equation.
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Remarks

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Crans’s dissertation discusses this in relation to her construction of so-called Lie 2-algebras. In particular, the Lie cocycle, \( \langle a, [b, c] \rangle \) is used to construct a solution to the Zamolodchikov equation. In a weird geometric sense (or *a posteriori*) this solution makes sense. The 3-cocycle condition corresponds to the boundary of a 4-cube, and the ZE is to the 4-cube as the YBE is to the 3-cube. The \( R \)-matrix is diagrammatically related to the YBE solution as indicated.
Hopf Algebra

- Associativity
- Coassociativity
- Compatibility
- Unit
- Counit
- Unit is a coalgebra hom
- Counit is an algebra hom
- Antipode condition

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Algebraic Deformations and Solutions to the YBE
For example the adjoint map in a Hopf algebra is given by:
Example: group algebra

- $G$ is a group
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
- $\mu(a, b) = ab$
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
- $\mu(a, b) = ab$
- $\Delta(a) = a \otimes a$
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
- $\mu(a,b) = ab$
- $\Delta(a) = a \otimes a$
- $\eta(1) = 1_G$.
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
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Example: group algebra

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- $S(g) = g^{-1}$
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
- $\mu(a, b) = ab$
- $\Delta(a) = a \otimes a$
- $\eta(1) = 1_G$, $\epsilon g = 1$
- $S(g) = g^{-1}$

In this case, the adjoint map is given on basis elements by $ad(a \otimes b) = b^{-1}ab$. 
Two important properties

(A) \[ S(A) (B) = S(\cdot) \]

(B) \[ S(A) (B) = S(\cdot) \]
Two important properties

(A)

(B)
Woronowicz sol’n to YBE
Frobenius Algebra

\[
\begin{align*}
\mu & : xy \\
\eta & : \eta \\
\varepsilon & : \varepsilon \\
\beta & : \beta \\
\gamma & : \gamma \\
\Delta & : \Delta \\
\tau & : \tau \\
\end{align*}
\]

- Multiplication
- Unit
- Frobenius form
- Pairing
- Copairing
- Comultiplication
- Transposition

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Algebraic Deformations and Solutions to the YBE
Frobenius Algebra

Associativity

\[
\begin{array}{c}
\begin{array}{cc}
\triangle & = \\
\triangledown & = \\
\end{array}
\end{array}
\]

Coassociativity

\[
\begin{array}{c}
\begin{array}{cc}
\triangledown & = \\
\triangle & = \\
\end{array}
\end{array}
\]

Compatibility

\[
\begin{array}{c}
\begin{array}{cc}
\triangle & = \\
\triangledown & = \\
\end{array}
\end{array}
\]

Unit

\[
\begin{array}{c}
\begin{array}{cc}
\triangle & = \\
\triangledown & = \\
\end{array}
\end{array}
\]

Cancelation

\[
\begin{array}{c}
\begin{array}{cc}
\triangledown & = \\
\triangle & = \\
\end{array}
\end{array}
\]

Conversion

\[
\begin{array}{c}
\begin{array}{cc}
\triangle & = \\
\triangledown & = \\
\end{array}
\end{array}
\]
Sol to YBE in Frobenius alg.
Let $V$ be a vector space over $\mathbb{F}$ with a pairing $\beta : V \otimes V \to \mathbb{F}$. 
Duality pairing algebra

Let $V$ be a vector space over $\mathbb{F}$ with a pairing $\beta : V \otimes V \to \mathbb{F}$. $\beta$ is non-degenerate if there is a $\gamma : \mathbb{F} \to V \otimes V$ such that

$$(\beta \otimes |)(| \otimes \gamma) = | = (| \otimes \beta)(\gamma \otimes |)$$
Let $V$ be a vector space over $\mathbb{F}$ with a pairing $\beta : V \otimes V \rightarrow \mathbb{F}$.

$\beta$ is non-degenerate if there is a $\gamma : \mathbb{F} \rightarrow V \otimes V$ such that

$$(\beta \otimes |)(| \otimes \gamma) = | = (| \otimes \beta)(\gamma \otimes |)$$

Call these the **Switchback conditions**.
Let $V$ be a vector space over $\mathbb{F}$ with a pairing $\beta : V \otimes V \rightarrow \mathbb{F}$. $\beta$ is non-degenerate if there is a $\gamma : \mathbb{F} \rightarrow V \otimes V$ such that

$$((\beta \otimes |)(| \otimes \gamma) = | = (| \otimes \beta)(\gamma \otimes |)$$

Call these the **Switchback conditions**.
Kauffman/Penrose Pairings

Let $\beta = (0, iA, (iA)^{-1}, 0)$ and $\gamma(1) = iAx \otimes y + (iA)^{-1}y \otimes x$
Let $\beta = (0, iA, (iA)^{-1}, 0)$ and $\gamma(1) = iAx \otimes y + (iA)^{-1}y \otimes x$

[So $\gamma = \beta^t$ — the transpose of $\beta$.]

Kauffman/Penrose Pairings
Kauffman/Penrose Pairings

Let $\beta = (0, iA, (iA)^{-1}, 0)$ and $\gamma(1) = iAx \otimes y + (iA)^{-1}y \otimes x$
[So $\gamma = \beta^t$ — the transpose of $\beta$. In this situation we have the bracket solution to YBE:

$$\begin{align*}
\left( \begin{array}{c}
\downarrow \\
\downarrow
\end{array} \right) = A 
\left( \begin{array}{c}
\downarrow \\
\downarrow
\end{array} \right) + A^{-1}
\end{align*}$$
In all the cases, alg., coalg., Hopf alg., Frobenius alg., duality pairing algebra, there is a symmetric monoidal category, and the relations in the algebra are identities among the generating morphisms.
Commentary

In all the cases, alg., coalg., Hopf alg., Frobenius alg., duality pairing algebra, there is a symmetric monoidal category, and the relations in the algebra are identities among the generating morphisms. I call these identities Elaborate plans.
In all the cases, alg., coalg., Hopf alg., Frobenius alg., duality pairing algebra, there is a symmetric monoidal category, and the relations in the algebra are identities among the generating morphisms. I call these identities *Elaborate plans*. 1 and 2 dimensional cohomology is obtained by *infiltrating elaborate plans*.
In all the cases, alg., coalg., Hopf alg., Frobenius alg., duality pairing algebra, there is a symmetric monoidal category, and the relations in the algebra are identities among the generating morphisms. I call these identities *Elaborate plans*. 1 and 2 dimensional cohomology is obtained by *infiltrating elaborate plans*. In the next slides, I will exemplify:
1-chains and 1-coboundaries

\[ f: V \rightarrow V \]

\[ d^1 = - \]
\[ d^2 = - \]
\[ d^3 = - \]
2-coboundaries for Hopf Algebras

(A)

(B)
2-coboundaries for Hopf Algebras

\[ \eta_1 = \begin{array}{c}
\begin{array}{c}
\text{A}
\end{array}
\end{array}, \quad d^{2,1} \left( \begin{array}{c}
\text{A}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
\text{A}
\end{array} + \begin{array}{c}
\text{A}
\end{array} - \begin{array}{c}
\text{A}
\end{array} = 0, \quad d^{2,2} \left( \begin{array}{c}
\text{A}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
\text{A}
\end{array} - \begin{array}{c}
\text{A}
\end{array} = 0
\end{array}\]
As in the case for coalgebras, the coboundary condition from coassoc. is the upside-down version of the top equation.

\[ d^{2,1} \left( \begin{array}{c} \uparrow \\ \Downarrow \end{array} \right) = \begin{array}{c} \uparrow \\ \Downarrow \end{array} + \begin{array}{c} \uparrow \\ \Downarrow \end{array} - \begin{array}{c} \uparrow \\ \Downarrow \end{array} - \begin{array}{c} \uparrow \\ \Downarrow \end{array} \]
2-coboundaries for Frobenius alg.

\[ d^{2,1}(\begin{array}{c}
\uparrow \\
\downarrow
\end{array}, \begin{array}{c}
\downarrow \\
\uparrow
\end{array}) = \begin{array}{c}
\uparrow \\
\downarrow
\end{array} + \begin{array}{c}
\uparrow \\
\downarrow
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \]

\[ d^{2,2}_{(1)}(\begin{array}{c}
\uparrow \\
\downarrow
\end{array}, \begin{array}{c}
\downarrow \\
\uparrow
\end{array}) = \begin{array}{c}
\uparrow \\
\downarrow
\end{array} + \begin{array}{c}
\uparrow \\
\downarrow
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \]

\[ d^{2,2}_{(2)}(\begin{array}{c}
\uparrow \\
\downarrow
\end{array}, \begin{array}{c}
\downarrow \\
\uparrow
\end{array}) = \begin{array}{c}
\uparrow \\
\downarrow
\end{array} + \begin{array}{c}
\uparrow \\
\downarrow
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow
\end{array} - \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \]
As in the case for coalgebras, the coboundary condition from coassoc. is the upside-down version of the top equation.
Coboundary conditions in duality pairing algebras

\[ d^{1,1} = \quad \quad \quad d^{1,2} = \quad \quad \quad d^{2,1} = \quad \quad \quad d^{2,2} = \quad \quad \quad d^{3,1} = \quad \quad \quad d^{3,2} = \]

\[ (\xi_1, \xi_2) \]

\[ (\xi_1, \xi_2) \]

\[ (\xi_1, \xi_2) \]

\[ (\xi_1, \xi_2) \]
The diagrammatic proofs that the given maps provide solutions for the YBE, can be extended to give diagrammatic proofs that the extension cocycles give new solutions to the YBE.

In the case of the duality pairing algebras, the resulting invariant is still the Jones polynomial. The other solutions remain to be investigated.
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The diagrammatic proofs that the given maps provide solutions for the YBE, can be extended to give diagrammatic proofs that the extension cocycles give new solutions to the YBE. In the case of the duality pairing algebras, the resulting invariant is still the Jones polynomial. The other solutions remain to be investigated.
Thanks

Thanks to the conference organizers.
Thanks

Thanks to the conference organizers. Thanks to my collaborators.
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