Diagrammatic Cohomology and Knot Invariants

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Joint work with:

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Plan

1. Statements
2. Type III and cocycle conditions
3. Applications of quandle cocycles
4. Other contexts
5. Infiltrations of Elaborate plans
6. Computation of some cohomology
7. Conclusion

Results

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Solutions to the YBE constructed.

A **quandle** $(Q, \triangleleft)$ is a set Q equipped with a binary operation

$$\triangleleft \to Q \times Q \to Q$$

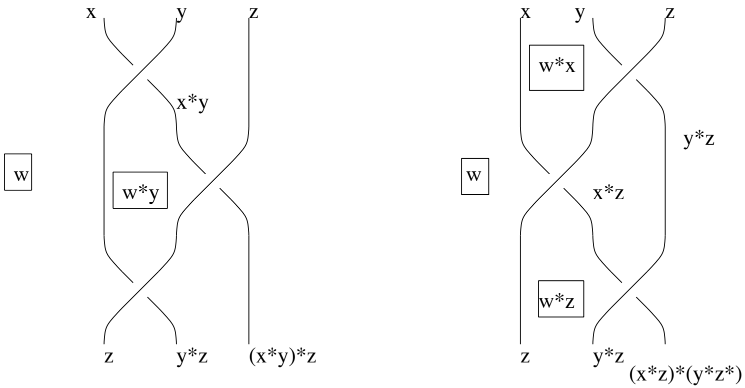
that satisfies:

(i) $x \triangleleft x = x$ for all $x \in Q$

(ii) For all $x, y \in Q$, there exists a unique $z \in Q$ such that $z \triangleleft x = y$

(iii) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ for all $x, y, z \in Q$

Type III move and cocycles



$$\phi\,(x,y) + \phi\,(x{*}y,z) + \phi\,(y,z) = \phi\,(y,z) + \phi\,(x,z) + \phi\,(x{*}z,y{*}z)$$

$$\theta\,(w,x,y) + \theta\,(w{*}y,x{*}y,z) + \theta\,(w,y,z) = \theta\,(w,x,z) + \theta\,(w{*}x,y,z) + \theta\,(w{*}z,x{*}z,y{*}z))$$

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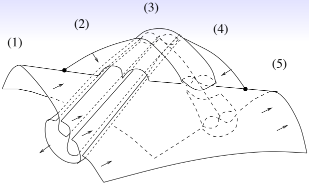
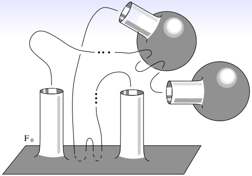
8. applications to Lefschetz fibrations (Zablow).

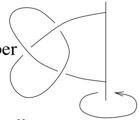
Non–invertibility, Triple point number



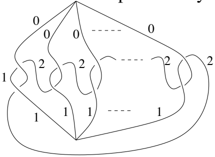
Ribbon Concordance



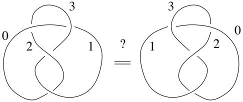
Sheet number



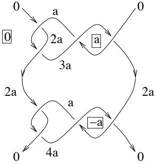
Graph chirality



Colored chirality

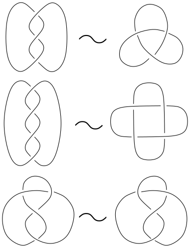


Tangle embeddings



Minimal # of type III moves

(i)
(ii)
(iii)



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- Solutions to YBE are found in Frob. algs;
- bracket solution to YBE obtained by $\cup\cap$ manipulations;
- underlying Frob. alg in Khovanov theory;
- 3-cocycles related to categorical internalization.

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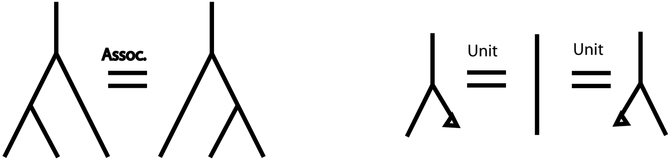
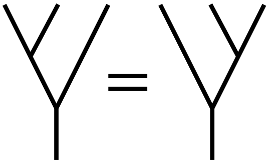
- Enrichment of KhoHo to get non-trivial invs for surfaces;
- deeper connections between classical and quantum invariants.

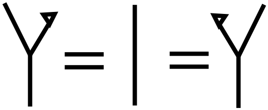
Diag Alg.



Coalgebra

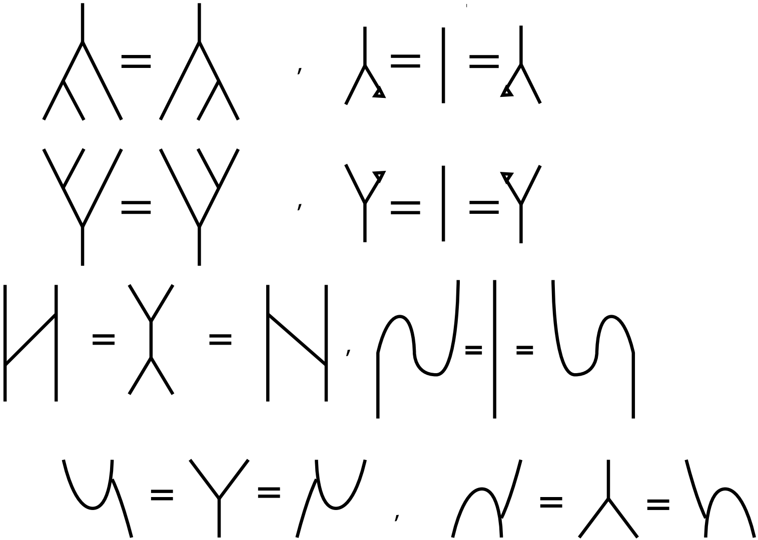


Co-assoc.

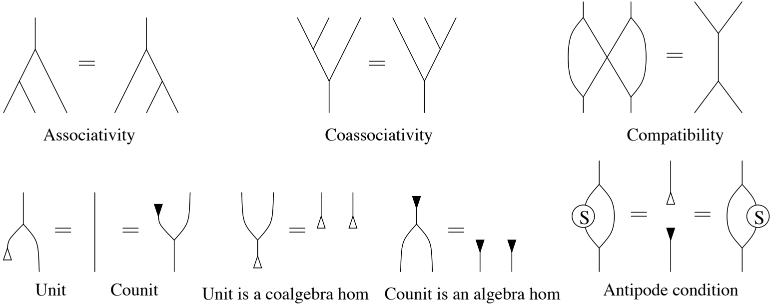


Co-unit

Frobenius Algebra

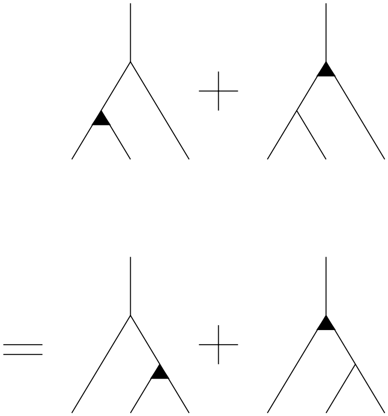


Hopf Algebra



Algebraic Infiltration

The cocycle cond. for assoc. algs. is given by:



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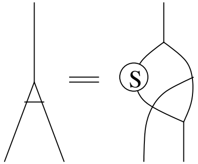
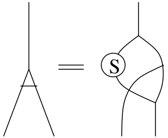
In this case, the adjoint map is given on basis elements by $ad(a \otimes b) = b^{-1}ab$.

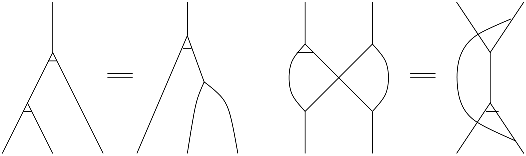
For example the adjoint map is given by:



Two important properties



(A)



(B)

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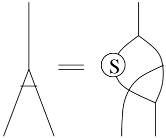

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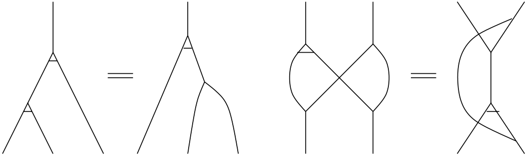
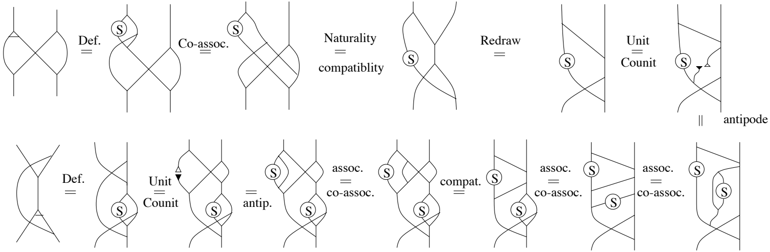
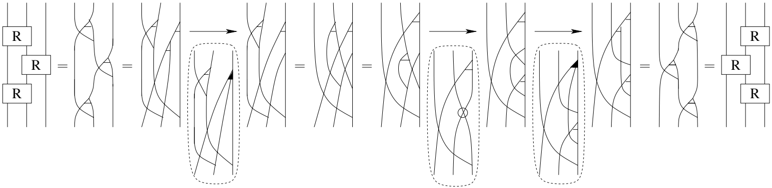
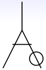
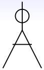
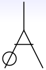


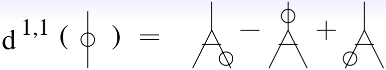
(B)

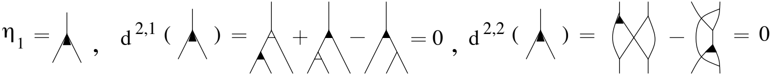


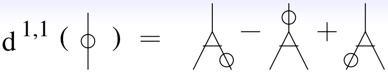
Woronowicz sol'n to YBE



$d^{1,1}\Big($  $\Big) =$  $-$  $+$ 


d 1,1 ( ) = − +


η 1 = , d 2,1 ( ) = + − = 0 , d 2,2 ( ) = − = 0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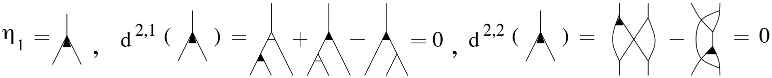
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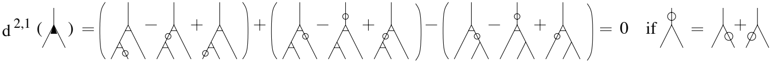
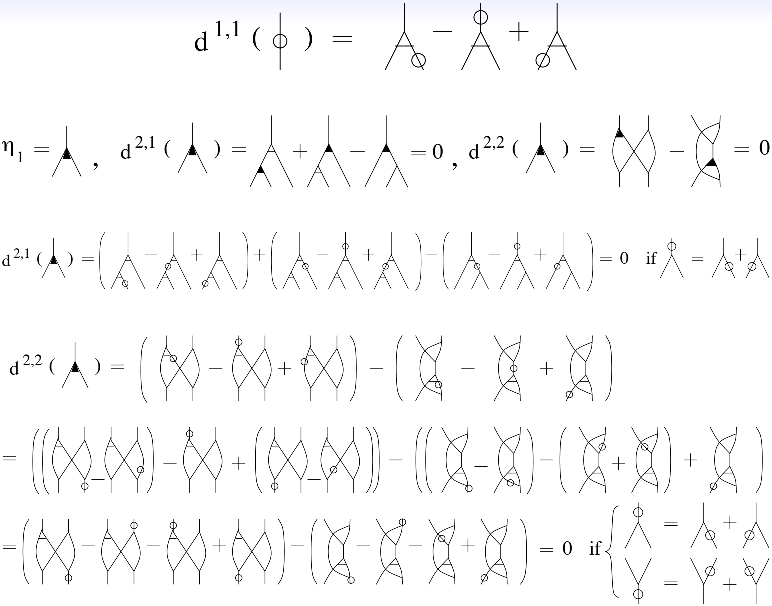

d 2,1 ( ) = − + + − + − − + = 0 if = +



d 1,1 ( ) =
η1 = , d 2,1 ( ) = = 0 , d 2,2 ( ) = = 0
d 2,1 ( ) = = 0 if =
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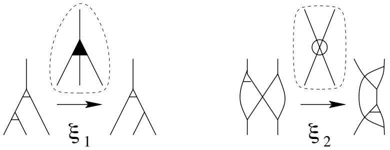
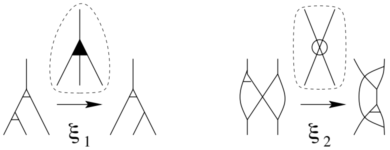
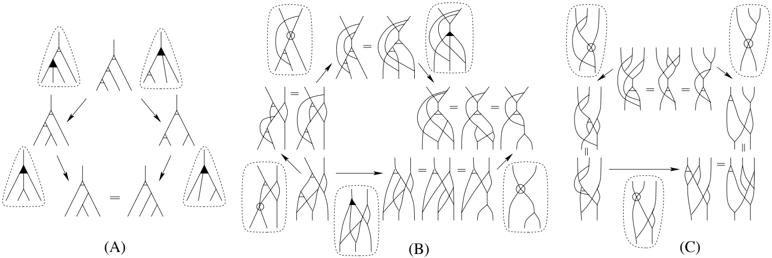
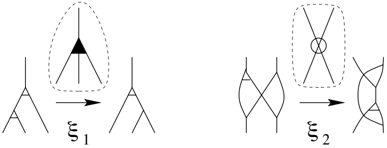
=
= = 0 if


ξ_1
ξ_2





(A) (B) (C)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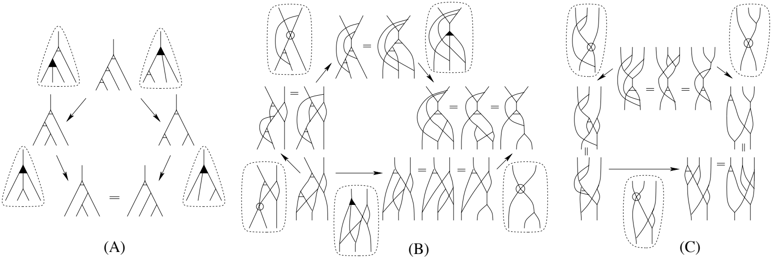
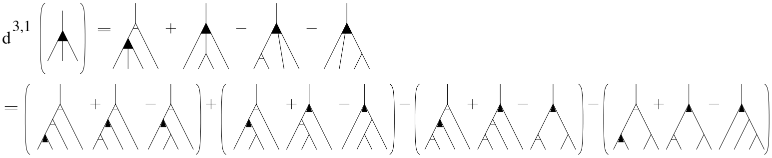
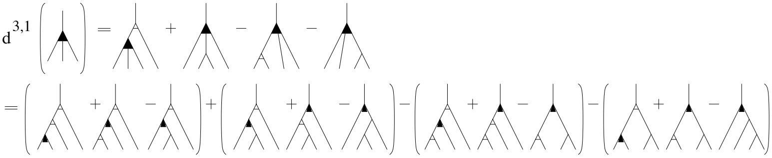
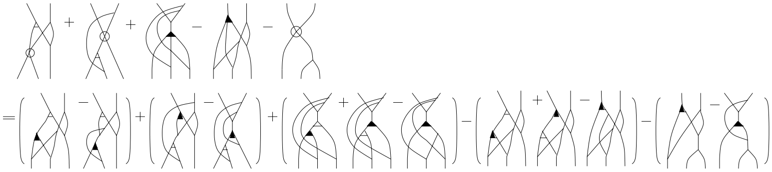
ξ_2

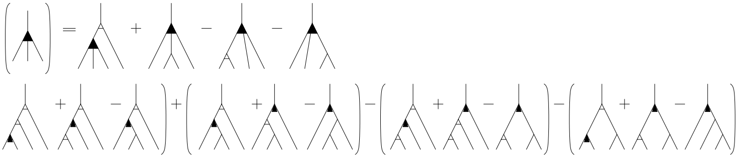

(A)
(B)
(C)

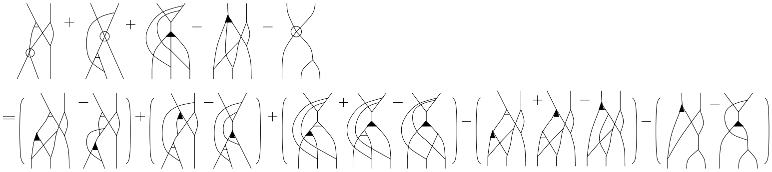

d 3,1

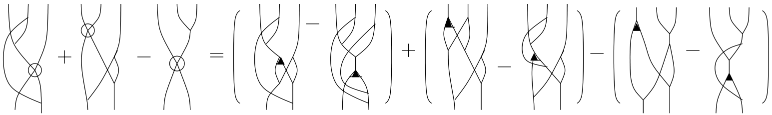


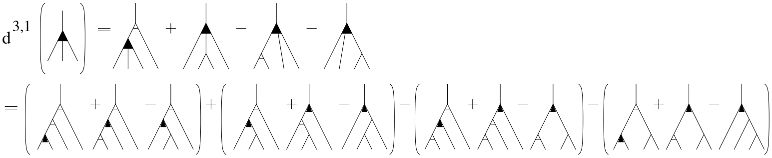


$d^{3,1}$





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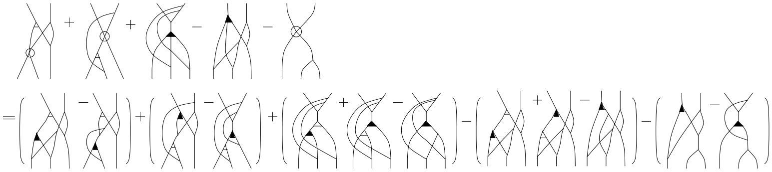

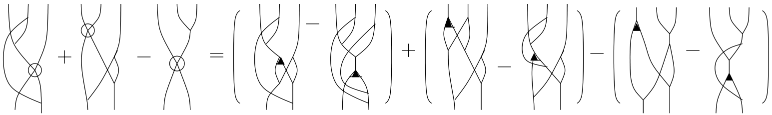

Example Computations

The 2-cocycle condition, the 3-cocycle condition, and the 3-coboundary condition, respectively, gives rise to the equations

$$a(x,y) + a(y^{-1}xy, z) - a(x, yz) = 0,$$

$$c(x,y,z) + c(x,yz,w) - c(y^{-1}xy, z, w) - c(x,y,zw)$$

$$c(x,y,z) = a(x,y) + a(y^{-1}xy, z) - a(x,yz).$$

Example Computations

The 2-cocycle condition, the 3-cocycle condition, and the 3-coboundary condition, respectively, gives rise to the equations

$$a(x, y) + a(y^{-1}xy, z) - a(x, yz) = 0,$$
$$c(x, y, z) + c(x, yz, w) - c(y^{-1}xy, z, w) - c(x, y, zw)$$
$$c(x, y, z) = a(x, y) + a(y^{-1}xy, z) - a(x, yz).$$

Theorem

For the symmetric group $G = S_3$ *on three letters, we have* $H^1_{\mathrm{ad}}(kG; kG) = 0$ *and* $H^2_{\mathrm{ad}}(kG; kG) \cong (kG)^{\otimes 3}$ *for* $k = \mathbb{C}$, $\mathbb{F}_2$ *and* $\mathbb{F}_3$.

Function algs. of a finite group

G is a finite group and $\mathbb{F}$ a field. The set $\mathbb{F}^G$ has a Hopf algebra structure using $\mathbb{F}^{G\times G} \cong \mathbb{F}^G \otimes \mathbb{F}^G$ with comultiplication defined through $\Delta : \mathbb{F}^G \to \mathbb{F}^{G\times G}$ by $\Delta(f)(u,v) = f(uv)$ and the antipode by $S(f)(x) = f(x^{-1})$.

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Lemma

$Z^2_{\mathrm{ad}}(\mathbb{F}^G; \mathbb{F}^G) = 0.$

Theorem

For any finite group G and a field $\mathbb{F}$, we have $H^n_{\mathrm{ad}}(\mathbb{F}^G; \mathbb{F}^G) = 0$ for $n = 1, 2$.

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6. Details on Frob algs in Masahico's talk.
7. Details of Cat. SD in Alissa's talk.

Dankeschöne!