Non-orientable surface knots that have an arbitrarily large number of triple points in their projections

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Joint work with:
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Background

Quandles that have good involutions
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Quandles that have good involutions — introduced by S. Kamada
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Definition

The triple point number of a surface link is the smallest number of triple points among all diagrams of the link.
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Quandles that have good involutions — introduced by S. Kamada and studied by K. Oshiro and Kamada and Oshiro — were introduced to extend quandle cocycle invariants to non-orientable surfaces.
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Definition

The *triple point number* of a surface link is the smallest number of triple points among all diagrams of the link.
Triple points

Suppose that a surface is embedded in $M^3 \times I$. 

Perturb the surface slightly (if necessary). Project the surface generically into the 3-mfd $M$. The image will have branch points, double point arcs, and triple points. The triple points are analogous to crossing points. So the triple pt. # is analogous to the crossing number.
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The example given here is the first connected non-orientable surface with triple point bounds.
Main Result

Theorem

For any positive integer $N$, 

$t(F) > N$. 
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**Theorem**

*For any positive integer \( N \), there is a closed 3-manifold \( M \) and a non-orientable surface-knot \( F \) in \( M \times [0,1] \) such that \( t(F) > N \).*
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For any positive integer $N$, there is a closed 3-manifold $M$ and a non-orientable surface-knot $F$ in $M \times [0,1]$ such that $t(F) > N$. 
Quandles

Definition

A *quandle* is a set $X$ that has a binary operation $\triangleleft$ such that

I. $\forall x \in X \ x \triangleleft x = x$.
II. $\forall x, y \in X \ \exists! z \in X$ such that $z \triangleleft x = y$.
We write $z = y \triangleleft -1 x$.
III. $\forall x, y, z \in X \ (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

If $b \triangleleft a = b \triangleleft a^{-1}$, then $X$ is called *involutory*.

If there is an involution $\rho : X \to X$ such that $\rho(x \triangleleft y) = \rho(x) \triangleleft y$ and $x \triangleleft \rho(y) = x \triangleleft y^{-1}$, then $\rho$ is a *good involution*. 
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If there is an involution $\rho : X \rightarrow X$ such that $\rho(x \triangleleft y) = \rho(x) \triangleleft y$ and $x \triangleleft \rho(y) = x \triangleleft^{-1} y$, it is called *good*.
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Definition

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  \[ G_{(X,\rho)} = \langle x \in X : x \triangleleft y = y^{-1}xy, \ \rho(x) = x^{-1} \rangle. \]
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- \( X \) quan., an \( X \)-set is a set \( Y \) w/ a rt. action of \( G_X \).

- \( (X,\rho) \) sym. quand., an \( (X,\rho) \)-set is a set \( Y \) w/ a rt. action of \( G_{(X,\rho)} \).
\( y \cdot (x_1 x_2) = (y \cdot x_1) \cdot x_2, \)
\[ y \cdot (x_1 x_2) = (y \cdot x_1) \cdot x_2, \]
\[ y \cdot (x_1 \triangleleft x_2) = y \cdot (x_2^{-1} x_1 x_2), \]
\[ y \cdot (x_1 x_2) = (y \cdot x_1) \cdot x_2, \]
\[ y \cdot (x_1 \triangleleft x_2) = y \cdot (x_2^{-1} x_1 x_2), \quad \text{and} \]
\[ y \cdot (\rho(x_1)) = y \cdot (x_1^{-1}). \]
Symmetric Quandle homology 1.

$Y$ is an $(X, \rho)$-set.
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Chains: $C_n(X)_Y$ f. ab. g. gen by:
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[(y, x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)]
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$$[\left(\left(y, x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n\right)\right]$$

$$-(y \cdot x_i, x_1 \triangleleft x_i, x_2 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, \hat{x}_i, x_{i+1}, \ldots, x_n)$$
Sym. quan. hom. 2.

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$$(y, x_1, \ldots, x_n)$$

$$+(y \cdot x_i, x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, \rho(x_i), x_{i+1}, \ldots, x_n)$$
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\]
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subcomplexes.

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subcomplexes.

$$C_n^{Q,\rho}(X)_Y = C_n(X)_Y / (D_n^Q(X)_Y + D_n^\rho(X)_Y).$$
The local orientation can be reversed at the expense of applying the good involution.
Choose a region.

\[ + (y, x_1, x_2, x_3) \]
Choose a region. Get T,M,B normals outward.

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$$+(y, x_1, x_2, x_3)$$
Choose a region. Get T,M,B normals outward. Make the chain \((b, m, t)\) be the colors here. Adjust for sign. Eval. sym. 3-cocycle on this chain.
Lemma

Kamada, Kamada-Oshiro

Let $\theta : \mathbb{Z}(X^3) \rightarrow \mathbb{Z}$ be a sym. quan. 3-coc. s.t. $\theta(a, b, c) \in \{0, -1, 1\} \ \forall (a, b, c) \in X^3$. If $\theta([C_D]) = \alpha \in \mathbb{Z}$, then $t(F) \geq |\alpha|$. 
Lemma
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The quandle that we need: \( \tilde{R}_3 = QS_6 \)

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Had to use a computer ... 

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\[
A(x, y, z) = \\
\chi(x,y,z) - \chi(\rho(x),y,z) - \chi(x\triangleleft y,\rho(y),z) - \chi(x\triangleleft z,y\triangleleft z,\rho(z)) + \chi(\rho(x)\triangleleft y,\rho(y),z) + \chi(\rho(x)\triangleleft y,y\triangleleft z,\rho(z)) + \chi((x\triangleleft y)\triangleleft z,\rho(y)\triangleleft z,\rho(z)) - \chi((\rho(x)\triangleleft y)\triangleleft z,\rho(y)\triangleleft z,\rho(z)).
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\[ + \chi(\rho(x)\triangleleft y,\rho(y),z) + \chi(\rho(x)\triangleleft y, y\triangleleft z, \rho(z)) \]
\[ + \chi((x\triangleleft y)\triangleleft z, \rho(y)\triangleleft z, \rho(z)) - \chi((\rho(x)\triangleleft y)\triangleleft z, \rho(y)\triangleleft z, \rho(z)) \].
Lemma

Let $\tilde{R}_3$ be as above.
Lemma

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(i) \( H_3^{Q, \rho}(\tilde{R}_3, \mathbb{Z}) \cong \mathbb{Z} \).

(ii) The 3-chain \( c = (2, 1, 2) + (2, 0, 1) - (1, 0, 2) - (0, 2, 1) \in C_3^{Q, \rho}(\tilde{R}_3, \mathbb{Z}) \) represents a generator of \( H_3^{Q, \rho}(\tilde{R}_3, \mathbb{Z}) \).

(iii) Any 3-cycle with less than 4 basis terms (triples) is null-homologous.

(iv) \( \phi = A(0, 1, 0) + A(0, 1, 2) - A(0, 2, 1) \in \mathbb{Z}_3^{Q, \rho}(\tilde{R}_3, \mathbb{Z}) \) is dual to \([c] \), i.e. \( \phi([c]) = 1 \).
Lemma

Let $\tilde{R}_3$ be as above.

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$$c = (2, 1, 2) + (2, 0, 1) - (1, 0, 2) - (0, 2, 1) \in C^Q_3(\tilde{R}_{2n+1}, \mathbb{Z})$$

represents a generator of $H^Q_3(\tilde{R}_3, \mathbb{Z})$.

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(iv) $\phi = A(0, 1, 0) + A(0, 1, 2) - A(0, 2, 1) \in Z^3(\tilde{R}_3, \mathbb{Z})$ is dual to $[c]$, i.e. $\phi([c]) = 1$. 
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$$c = (2, 1, 2) + (2, 0, 1) - (1, 0, 2) - (0, 2, 1) \in C^3_{Q,\rho}((\tilde{R}_{2n+1}, \mathbb{Z})$$ represents a generator of $H^3_{Q,\rho}(\tilde{R}_3, \mathbb{Z})$.

(iii) Any 3-cycle with less than 4 basis terms (triples) is null-homologous.

(iv) $\phi = A(0, 1, 0) + A(0, 1, 2) - A(0, 2, 1) \in Z^3_{Q,\rho}(\tilde{R}_3, \mathbb{Z}))$ is dual to $[c]$,
Lemma

Let $\tilde{R}_3$ be as above.

(i) $H^Q_3(\tilde{R}_3, \mathbb{Z}) \cong \mathbb{Z}$.

(ii) The 3-chain

$$c = (2, 1, 2) + (2, 0, 1) - (1, 0, 2) - (0, 2, 1) \in C^Q_3(\tilde{R}_{2n+1}, \mathbb{Z})$$

represents a generator of $H^Q_3(\tilde{R}_3, \mathbb{Z})$.

(iii) Any 3-cycle with less than 4 basis terms (triples) is null-homologous.

(iv) $\phi = A(0, 1, 0) + A(0, 1, 2) - A(0, 2, 1) \in Z^3_{Q, \rho}(\tilde{R}_3, \mathbb{Z})$ is dual to $[c]$, i.e. $\phi([c]) = 1$. 
Ta Daaaaaa
Easier to parse
Thank you California.