Non-orientable surface knots that have an arbitrarily large number of triple points in their projections

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Plan

1. Background
2. Main Result
3. Quandles
4. Symmetric Quandle Homology
5. The Quandle QS_6
6. Cool Computations
7. Diagram of the Example

Background

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The *triple point number* of a surface link

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The *triple point number* of a surface link is the smallest number of triple points among all diagrams of the link.

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The triple points are analogous to crossing points. So the triple pt. # is analogous to the crossing number.

more background

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The example given here is the first connected non-orientable surface with triple point bounds.

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For any positive integer N, there is a closed 3-manifold M and a non-orientable surface-knot F in $M \times [0,1]$ such that $t(F) > N$.

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&y \cdot (\rho(x_1)) = y \cdot (x_1^{-1}).
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$$\big[(y, x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)$$

$$-(y \cdot x_i, x_1 \triangleleft x_i, x_2 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, \hat{x}_i, x_{i+1}, \ldots, x_n)\big]$$

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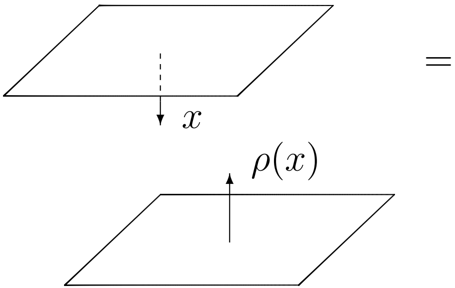
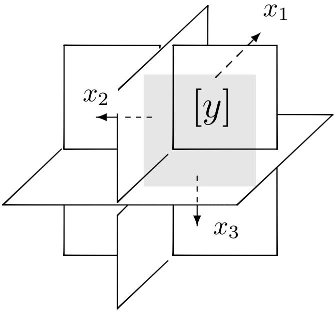
$C_n^{Q,\rho}(X)_Y = C_n(X)_Y/(D_n^Q(X)_Y + D_n^\rho(X)_Y)$.

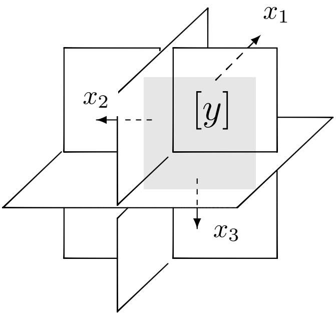


The local orientation can be reversed at the expense of applying the good involution.



$$+(y, x_1, x_2, x_3)$$

Choose a region.



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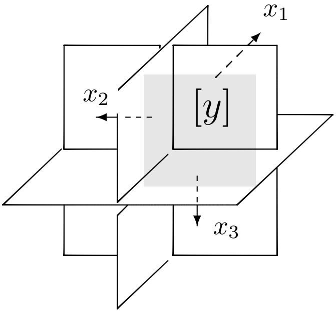
Choose a region. Get T,M,B normals outward.



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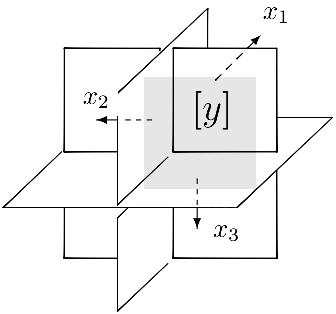
Make the chain (b, m, t) be the colors here.



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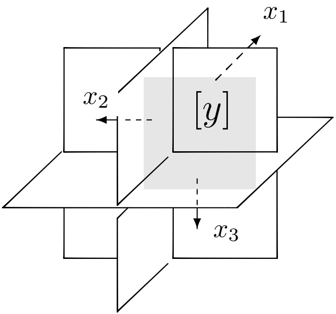

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Lemma

Kamada, Kamada-Oshiro

Let $\theta : \mathbb{Z}(X^3) \to \mathbb{Z}$ *be a sym. quan. 3-coc. s.t.* $\theta(a, b, c) \in \{0, -1, 1\}\ \forall (a, b, c) \in X^3$. *If* $\theta([C_D]) = \alpha \in \mathbb{Z}$, *then* $t(F) \geq |\alpha|$.

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The quandle that we need: $\tilde{R}_3 = QS_6$

$R \triangleleft C$	0	1	2	3	4	5
0	0	5	1	0	2	4
1	2	1	3	5	1	0
2	4	0	2	1	3	2
3	3	2	4	3	5	1
4	5	4	0	2	4	3
5	1	3	5	4	0	5

Had to use a computer ...

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$$+\chi_{(\rho(x)\triangleleft y,\rho(y),z)} + \chi_{(\rho(x)\triangleleft y,y\triangleleft z,\rho(z))}$$
$$+\chi_{((x\triangleleft y)\triangleleft z,\rho(y)\triangleleft z,\rho(z))} - \chi_{((\rho(x)\triangleleft y)\triangleleft z,\rho(y)\triangleleft z,\rho(z))}.$$

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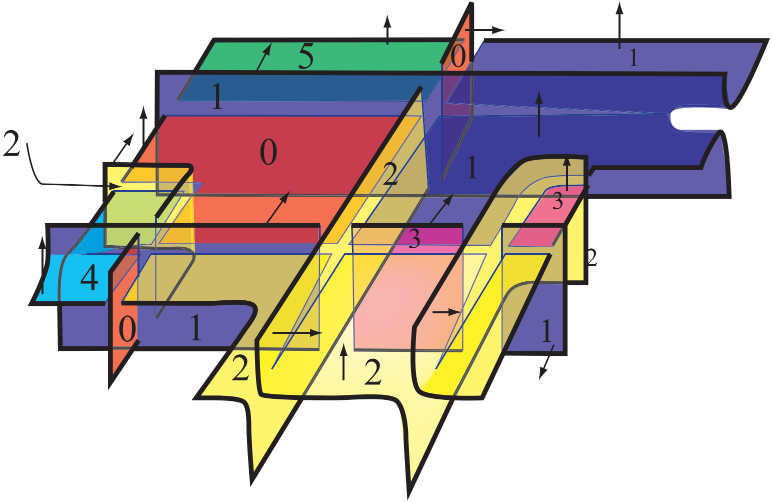
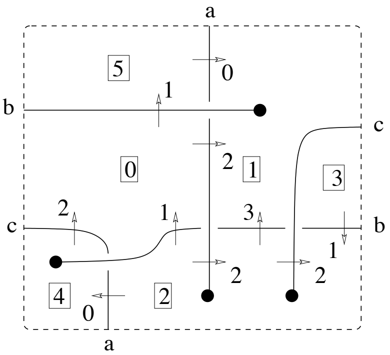
Ta Daaaaa



Easier to parse



Thank you California.