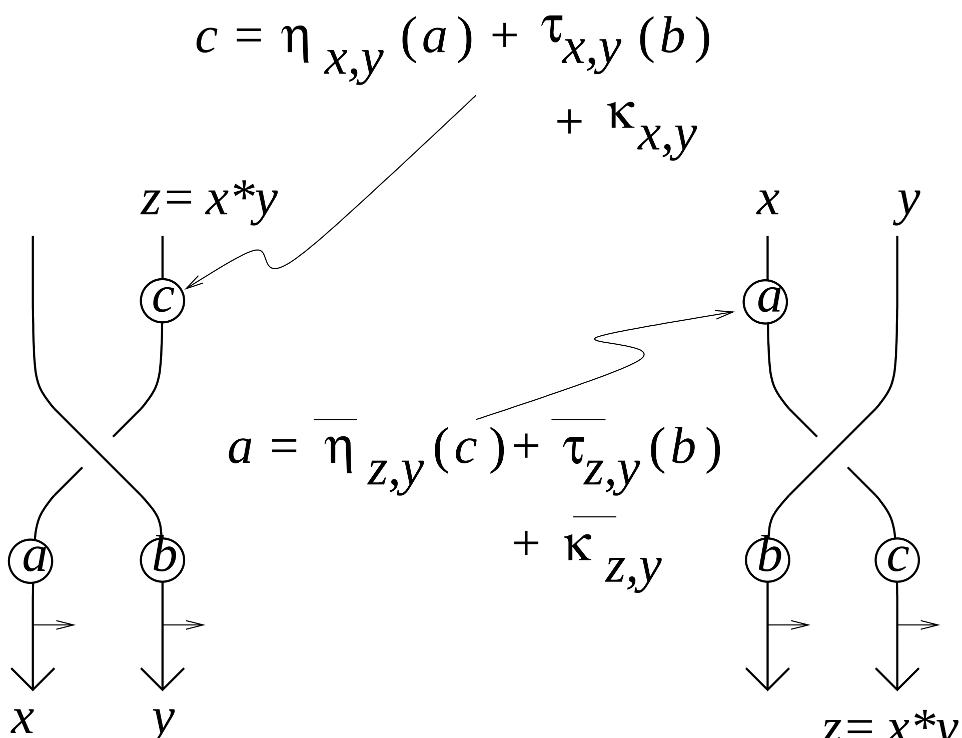
2. Quandle Modules

— X is a quandle.

— $\Omega(X)$ is the free $\mathbb{Z}$-algebra gen. by $\eta_{x,y}$, $\tau_{x,y}$ for $x, y \in X$ s.t. $\eta_{x,y}$ is invertible.



Define: $\overline{\eta_{z,y}} = \eta^{-1}_{z\bar{*}y,y}$

$\overline{\tau_{z,y}} = -\overline{\eta_{z,y}}\tau_{z\bar{*}y,y}$.

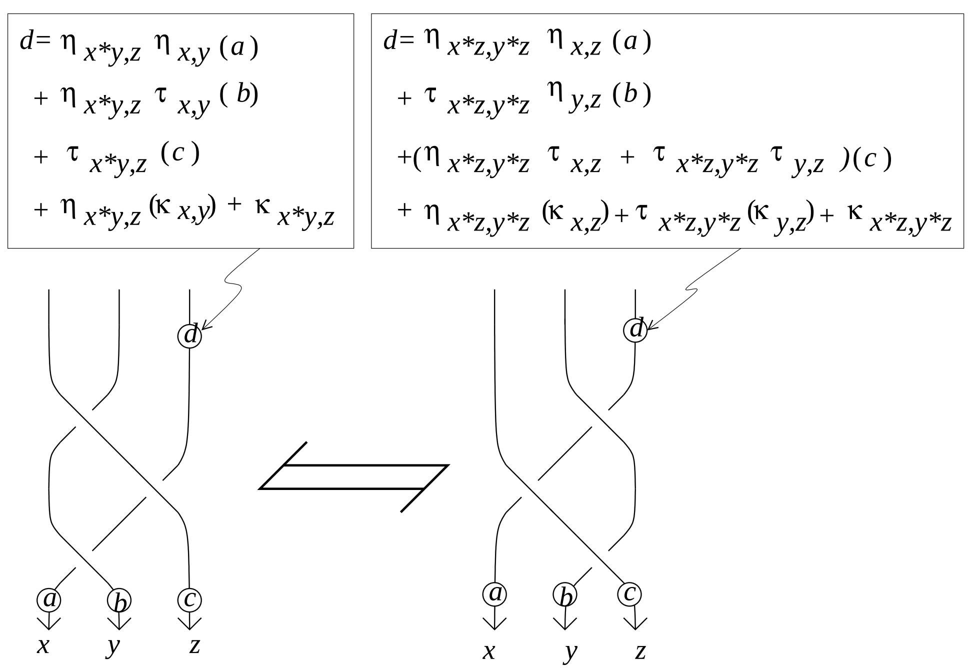
$\mathbb{Z}(X) = \Omega(X)/R$ where R is the ideal gen. by

1. $$\eta_{x*y,z}\eta_{x,y} - \eta_{x*z,y*z}\eta_{x,z}$$

2. $$\eta_{x*y,z}\tau_{x,y} - \tau_{x*z,y*z}\eta_{y,z}$$

3. $$\tau_{x*y,z} - \eta_{x*z,y*z}\tau_{x,z} - \tau_{x*z,y*z}\tau_{y,z}$$

4. $$\tau_{x,x} + \eta_{x,x} - 1$$



Examples: 1. $\Lambda = \mathbb{Z}[t, t^{-1}]$ — Laurent polynomials. Any Λ-module M is a $\mathbb{Z}(X)$-module for any quandle X,

$$\eta_{x,y}(a) = ta \quad \tau_{x,y}(b) = (1-t)(b) \quad \forall x, y \in X.$$

2. $G_X = \langle x \in X \mid x * y = yxy^{-1} \rangle$ — the *enveloping group* [AG] (or the *associated group* [FR].)

For any quandle X, any G_X-module M is a $\mathbb{Z}(X)$-module by $\eta_{x,y}(a) = ya$ and $\tau_{x,y}(b) = (1 - x * y)(b)$, where $x, y \in X$, $a, b \in M$.

$$\eta_{x,y} = \frac{\partial}{\partial x}(yxy^{-1}) \qquad \tau_{x,y} = \frac{\partial}{\partial y}(yxy^{-1})$$

Observation:

If w is a word in x, y s.t. $x * y = w$ defines a quandle, then

$$\eta_{x,y} = \frac{\partial}{\partial x}(w) \qquad \tau_{x,y} = \frac{\partial}{\partial y}(w)$$

defines a quandle module.

Pf.Chain Rule (Fox Deriv.)

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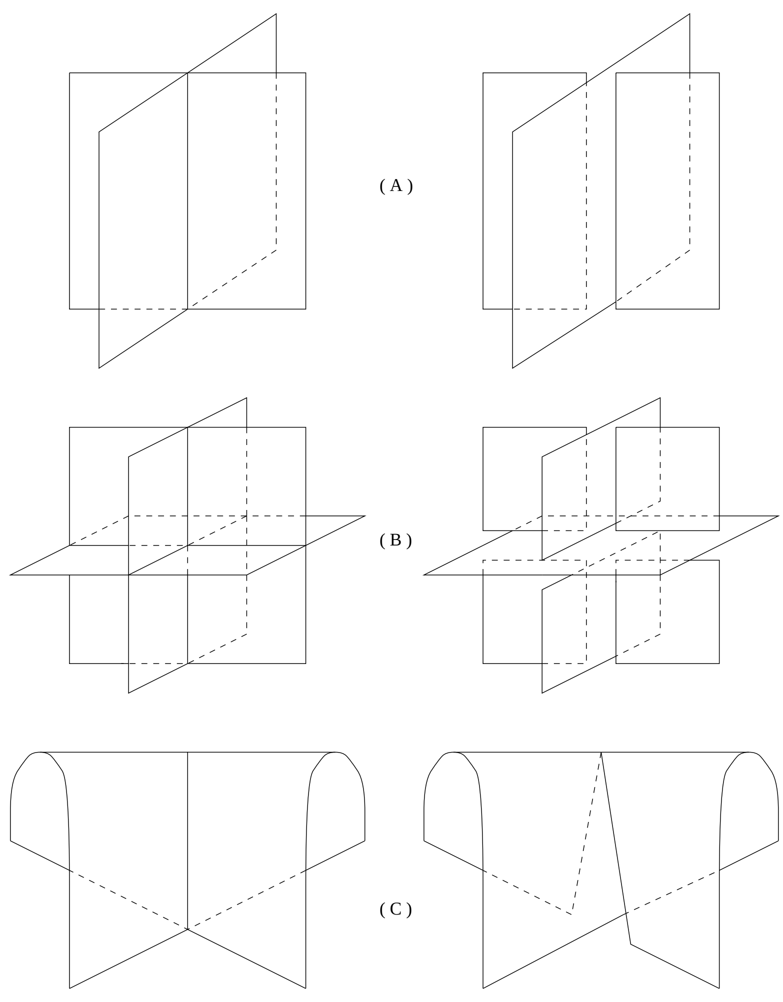
4. Assigning homology classes to colored diagrams

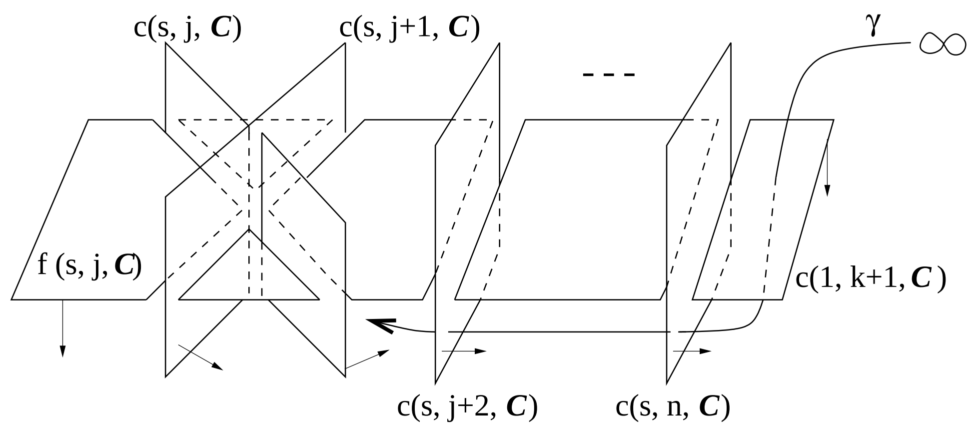
Knotted surface case (classical case is similar).

D is a diagram of a knotted surface K.



- Orientation Normals
- Quandle Colorings
- Source Region
- Target Region

- γ is an arc from ∞ to the target region of a given triple point r.
- a_i, $i = 1, \ldots, k$, the sheets of D that intersect γ (in this order).
- $\mathcal{C}$, a coloring of D by a quandle X.



- 3-chain

$$\mathcal{C}(D) = \sum_r \epsilon(r)(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)}$$

$$\cdots \mathcal{C}(a_k)^{\epsilon(a_k)})(x, y, z) \in C_3(X)$$

- The chain $\mathcal{C}(D)$ is a cycle. $[\mathcal{C}(D)] \in H_3(X)$ is well-defined (indep. of arc γ)

5. Cocycle Invariants

Def. Given

- A knotted surface diagram D;
- A coloring by a quandle X;
- A 3-cocycle κ .

Define

$$B(\mathcal{C}(D), X, \kappa) = \sum_{r} \epsilon(r)(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)}$$

$$\cdots \mathcal{C}(a_k)^{\epsilon(a_k)})\kappa_{x,y,z}$$

Define

$$\Phi_{\kappa}(D) = \{B(\mathcal{C}(D), X, \kappa)\}_{\mathcal{C}}$$

The union is over all colorings, and Φ denotes a multiset.

7. Example Computations (Knotted Surface Case)

Consider the collection of 3-colorable knots with fewer than 9 crossings. These are 3_1, 6_1, 7_4, 7_7, 8_5, 8_{10}, 8_{11}, 8_{15}, $8_{18} - - - 8_{21}$. Take the 2-twist-spin of each. Such a knotted sphere is also 3-coloarble.

9. Module Invariants

General Remarks:

Related to invariants of Bowen and Franks invariants.

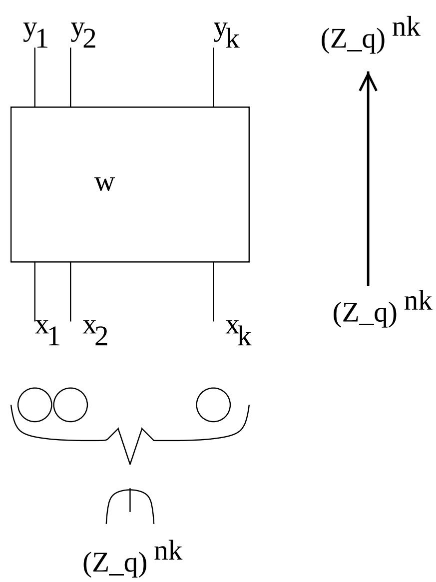
Computations not difficult to program (Mathematica has problems computing Smith Normal Forms when zero divisors are present).

Construction:

- w is a k-braid word with closure $\widehat{w}$.

- $X \subset \text{Conj}(\Sigma_n)$, $E = (\mathbb{Z}_q)^n \rtimes \Sigma_n$, and $\tilde{X} = \pi^{-1}(X)$.

- α is a dyn cocycle in the sense of AG. so that $\tilde{X} = (\mathbb{Z}_q)^n \times_\alpha X$. $\alpha_{x,y}(a,b) = y(a) + (1 - x * y)(b)$



$$0 \longrightarrow (Z\_q)^n \longrightarrow E \longrightarrow \sum_n \longrightarrow 0$$

THM.*Let* $L = \widehat{w}$*, Col*$_X(L)$ *be the set of colorings of* L *by a quandle* X*. For* $\mathcal{C} \in$ *Col*$_X(L)$*, let* $\vec{x}$ *be the color vector of bottom strings of* w *that is the restriction of* $\mathcal{C}$*. Then the family*

$$\tilde{\Phi}(X, \alpha\,; L) = \{(\mathbb{Z}_q)^{nk}/\mathrm{Im}(M(w, \vec{x}) - I)\}_{\mathcal{C} \in \mathit{Col}_X(L)}$$

of isomorphism classes of modules presented by the maps $(M(w, \vec{x}) - I)$*, is a link invariant.*

- All 7-colorable knots up to 9-crossings, except 9_{41}, have the module invariant $(\mathbb{Z}_D)^3 \oplus \mathbb{Z}^7$, for 7 trivial colorings, and $\mathbb{Z}^{10}$ for 42 non-trivial colorings. For 9_{41}, it is $(\mathbb{Z}_D)^6 \oplus \mathbb{Z}^7$ for 7 trivial colorings, and $\mathbb{Z}^{10}$ for 336 non-trivial colorings.

- All 11-colorable knots up to 9-crossings have the module invariant $(\mathbb{Z}_D)^5 \oplus \mathbb{Z}^{11}$, for 11 trivial colorings, and $\mathbb{Z}^{16}$ for 110 non-trivial colorings.

- All 13-colorable knots up to 9-crossings have the module invariant $(\mathbb{Z}_D)^6 \oplus \mathbb{Z}^{13}$, for 13 trivial colorings, and $\mathbb{Z}^{19}$ for 156 non-trivial colorings.

Knot	Tor	Rank	Col type	Knot	Tor
3_1	3	3	3 Trivial	9_{11}	33
	0	4	6 Non-trivial		0
6_1	9	3	3 Trivial	9_{15}	39
	0	4	6 Non-trivial		0
7_4	15	3	3 Trivial	9_{16}	39
	0	4	6 Non-trivial		2
7_7	21	3	3 Trivial	9_{17}	39
	0	4	6 Non-trivial		0
8_5	21	3	3 Trivial	9_{23}	45
	2	4	6 Non-Trivial		0
8_{10}	27	3	3 Trivial	9_{24}	45
	2	4	6 Non-trivial		2
8_{11}	27	3	3 Trivial	9_{28}	51
	0	4	6 Non-trivial		2
8_{15}	33	3	3 Trivial	9_{29}	51
	2	4	6 Non-trivial		0
8_{18}	3, 15	3	3 Trivial	9_{34}	69
	3	4	24 Non-trivial		0
8_{19}	3	3	3 Trivial	9_{35}	3, 9
	2	4	6 Non-Trivial		0
8_{20}	9	3	3 Trivial		2
	2	4	6 Non-trivial	9_{37}	3, 15
8_{21}	15	3	3 Trivial		2
	2	4	6 Non-trivial		0
9_1	9	3	3 Trivial	9_{38}	57
	0	4	6 Non-trivial		0
9_2	15	3	3 Trivial	9_{40}	5, 15
	0	4	6 Non-trivial		4
[illegible]	21	3	3 Trivial	9 [illegible]	3, 3

Knot	Tor	Rank	Col type	Kno
4_1	5, 5	5	5 Trivial	9_2
	0	7	20 Non-trivial	
5_1	5, 5	5	5 Trivial	9_{12}
	0	7	20 Non-trivial	
7_4	15, 15	5	5 Trivial	9_{23}
	0	7	20 Non-trivial	
8_7	25, 25	5	5 Non-trivial	9_{24}
	0	7	20 Trivial	
8_8	25, 25	5	5 Trivial	9_{31}
	0	7	20 Non-trivial	
8_{16}	35, 35	5	5 Trivial	9_{37}
	0	7	20 Non-trivial	
8_{18}	3, 3, 15, 15	5	5 Trivial	9_{39}
	2, 2	7	20 Non-trivial	
8_{21}	15, 15	5	5 Trivial	9_{40}
	0	7	20 Non-trivial	
				9_{48}