Twisted Quandle Homology Theory and Construction of Cocycles

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Goals for this talk:

- Define twisted quandle cocycles
- Use cocycles to construct extensions
- Give structures of some quandles
- Define twisted cocycle invariants
- Use diagrams to prove algebraic results

$$\Lambda = \mathbb{Z}[T, T^{-1}]$$

$$C_n^{\mathsf{TR}}(X) = C_n^{\mathsf{TR}}(X; \Lambda)$$

— free Λ-module gen. $(x_1, \ldots, x_n)$ $x_j \in X$.

$$\partial = \partial_n^T : C_n^{\mathsf{TR}}(X) \to C_{n-1}^{\mathsf{TR}}(X)$$

def'd by

$$\begin{aligned} &\partial_n^T(x_1, x_2, \ldots, x_n) \\ &\quad = \sum_{i=1}^{n} (-1)^i \Big[T(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &\quad - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \Big] \end{aligned}$$

for $n \geq 2$ and $\partial_n^T = 0$ for $n \leq 1$.

$C_*^{\mathsf{TR}}(X)$ is a chain complex.

A is a Λ-module,

$$C^n_{\mathsf{TR}}(X;A) = \mathrm{Hom}_\Lambda(C_n^{\mathsf{TQ}}(X), A)$$

$\delta = \delta^n_{\mathsf{TR}} : C^n_{\mathsf{TR}}(X;A) \to C^{n+1}_{\mathsf{TR}}(X;A)$ def'd by $(\delta f)(c) = (-1)^n f(\partial c)$

Homology and cohomology groups: $H_n^{\mathsf{TR}}(X;A)$ and $H^n_{\mathsf{TR}}(X;A)$.

Degenerate chains: $C_n^{\mathsf{TD}}(X;A)$ generated by n-tuples $(x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \dots, n-1\}$ if $n \geq 2$.

$\partial_n^T(C_n^{\mathsf{TD}}(X;A)) \subset C_{n-1}^{\mathsf{TD}}(X;A)$

Corresponding cochain complex: $C^*_{\mathsf{TD}}(X;A) = \{C^n_{\mathsf{TD}}(X;A), \delta^n_T\}$

Def

$$C_n^{\mathsf{TQ}}(X;A) = C_n^{\mathsf{TR}}(X;A)/C_n^{\mathsf{TD}}(X;A)$$

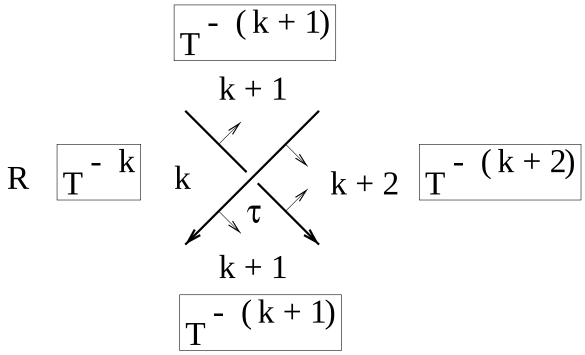
$$C_*^{\mathsf{TQ}}(X;A) = \{C_n^{\mathsf{TQ}}(X;A), \partial_n^T\}$$

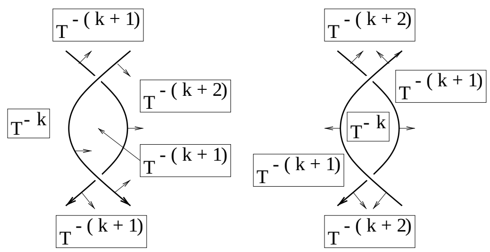
$$C^*_{\mathsf{TQ}}(X;A) = \{C^n_{\mathsf{TQ}}(X;A), \delta^n_T\}$$

Twisted homology and cohomology groups:

$$H_n^{\mathsf{TQ}}(X;A) = H_n(C_*^{\mathsf{TQ}}(X;A)),$$

$$H^n_{\mathsf{TQ}}(X;A) = H^n(C^*_{\mathsf{TQ}}(X;A)).$$


R
T - k
k
T - (k + 1)
k + 1
τ
k + 2
T - (k + 2)
k + 1
T - (k + 1)


T - (k + 1)
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T - (k + 1)
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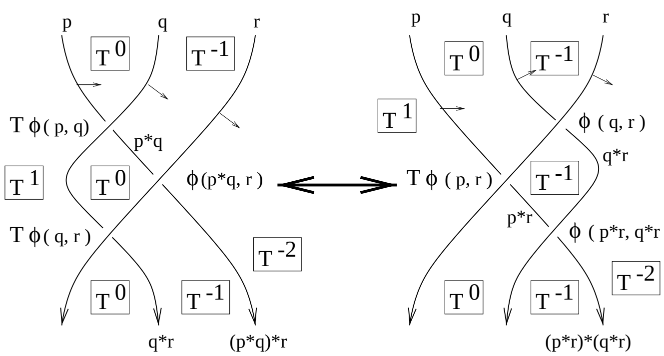

p
q
r
T 0
T -1
T φ(p, q)
p*q
T 1
T 0
φ(p*q, r)
T φ(q, r)
T -2
T 0
T -1
q*r
(p*q)*r
p
q
r
T 0
T -1
T 1
φ (q, r)
q*r
T φ (p, r)
T -1
p*r
φ (p*r, q*r)
T -2
T 0
T -1
(p*r)*(q*r)

The Λ-module A is a quandle with $a * b = Ta + (1-T)b$.

Let $\eta \in Z^1_{\mathrm{TQ}}(X; A)$. Then

$\delta(\eta)(x,y) = (1-T)\eta(y) + T\eta(x) - \eta(x * y) = 0$

So η is a quandle homom.

Consider a seseq of Λ-modules $0 \to N \xrightarrow{i} G \xrightarrow{p} A \to 0$

Let $s : A \to G$ be a set-thy section $ps = \mathrm{id}_A$ (with $s(0) = 0$).

$\exists \phi : X \times X \to A$ s.t.

$Ts\eta(x_1) + s\eta(x_2) = i\phi(x_1, x_2) + [Ts\eta(x_2) + s\eta(x_1 * x_2)]$

THM. ϕ is a 2-cocycle. If $\phi = \delta\xi$, then $s\eta$ extends to a quandle homom.

RMK. 3-cocycles can be defined in a similar manner. The proofs are facilitated by knot diagrams and the diagram of the tetrahedral move.

Let $\phi \in Z^2_{\mathsf{TQ}}(X; A)$. Let $AE(X, A, \phi)$ be def'd by

$$(a_1, x_1) * (a_2, x_2) = (a_1 * a_2 + \phi(x_1, x_2), x_1 * x_2)$$

—This is called *The Alexander extension* of X by (A, ϕ).

THM. Alex. extensions are "equivalent" iff the defining 2-cocycles are cohomologous.

Examples.

Recall, $R_n = \{0, \ldots, n-1\}$ with $a * b = 2b - a$ (mod n). (If $n = \infty$, just int. arithmetic.)

(1) The dihedral quandle R_{p^m} is an Alex. ext. of $R_{p^{m-1}}$ by R_p: $R_{p^m} = AE(R_{p^{m-1}}, \ R_p, \ \phi)$, $\exists \phi \in Z^2_{\mathrm{TQ}}(R_{p^{m-1}}; R_p)$.

Specifically, we have a 2-cocycle $\phi \in Z^2_{\mathrm{TQ}}(R_3; R_3)$

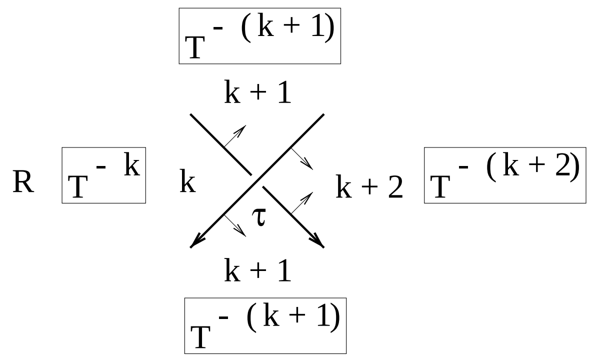
$$\phi = \chi_{0,2} + \chi_{1,2} + 2\chi_{1,0} + 2\chi_{2,0}.$$

(2) R_∞ is an Alex. ext of R_n by R_∞, for any positive integer n.

$$\phi(a,b) = \begin{cases} -1 & \text{if} \quad 2b < a, \\ 0 & \text{if} \quad 2b < n+a \ \text{ and } \ a \leq 2b, \\ 1 & \text{if} \quad n+a \leq 2b. \end{cases}$$

(3) Lots of other similar examples. These are based on the "carry digit" idea which can be used to construct group cocycles.

State-sum invariants (classical case here, knotted surface case is similar).



Alexander numbering: $\mathcal{L}(\tau) = k$

Given

— a diagram K,

— a finite quandle X,

— a finite Alexander quandle A.

— a twisted 2-cocycle ϕ

Define a *twisted (Boltzmann) weight*,

$$B_T(\tau,\mathcal{C}) = [\phi(x,y)^{\epsilon(\tau)}]^{T^{-\mathcal{L}(\tau)}},$$

Invariant:

$$\Phi(K) = \sum_{\mathcal{C}} \prod_{\tau} B_T(\tau,\mathcal{C}).$$

Well-defined up to an action of $\mathbb{Z} = \langle T \rangle$

PROP. Suppose $\phi \in Z^2_{\mathsf{TQ}}(X;N)$ is an obstruction 2-cocycle (as constructed above from seseq of Alex. quandles $0 \to N \to G \to A \to 0$). Then the state-sum invariant Φ defined via ϕ is a positive integer.